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EQUIVALENCE OF RESERVE METHODOLOGIES

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ABSTRACT

The paper considers policies with annual premiums and discusses four types of life insurance reserves calculations: curtate, fully continuous, discounted continuous, and semicontinuous. It is shown that when appropriate corrections are made, each method gives the same reserve; this is as expected in view of the equality of the actual cash flows. The paper concludes with consideration of the methods used in making a practical year-end valuation.

1. INTRODUCTION

Actuaries can use several alternative reserving methodologies in valuing the same life insurance policy. The text by Tullis and Polkinghorn [4] lists in tables on pages 47 and 48 four types of reserves:

(1) Curtate
(2) Fully continuous
(3) Discounted continuous
(4) Semicontinuous.

Also indicated are five items that need to be considered in calculating a reserve:

(a) Refund of unearned premium on death
(b) Nondeduction of deferred premium on death
(c) Immediate payment of claims reserve
(d) Deferred premium asset
(e) Unearned premium liability.

In this paper only annual premium policies are considered, so (b) and (d) are not relevant.

The actuary also has a choice between using midterminal reserves and mean reserves. Thus a reserve can be calculated in many ways. However, the actual timing and amount of benefits and premiums are fixed by the terms of the policy. Thus if all appropriate corrections are taken into account, then the reserve calculated under all methods should be the same. Indeed reserves could be calculated directly, taking into account all benefits and premiums and the exact days on which they are paid with appropriate probabilities.

Some of the methods and reserve items listed above exist because in previous decades actuaries needed to simplify calculations and to use labor-saving grouping or "binning" techniques. Very common is the technique whereby policies are binned by policy year and then assumed to have the average policy issue date for that bin. The following examples assume a December 31, 1997 valuation date:

Issue Date	Policy Year at Dec. 31, 1997	Assumed Issue Date
November 18, 1997	1	June 30/July 1, 1997
January 19, 1997	1	June 30/July 1, 1997
August 3, 1995	3	June 30/July 1, 1995
October 9, 1993	5	June 30/July 1, 1993

If the valuation date is December 31, then the binning by policy year is equivalent to binning by calendar year of issue. For the above valuation date of December 31, 1997, all policies issued in calendar year 1996 will be allocated to the bin of policies in policy year 2 at valuation. Then they may for some purposes be assumed to have been issued on June 30, 1996 or July 1, 1996 with probability 0.5 for each of the two dates. For nonannual premiums, this may be an approximation that adequately allows for the mid-policy year premium. Half of the policies are assumed at valuation to be about to pay a premium, and half are assumed to have just paid a premium. This amounts to an approximation to the integration of quantities with use of a uniform distribution of issue dates. A more accurate technique would be to perform seriatim valuations using actual premium due dates.

2. TERMINAL RESERVES

Consider an annual premium whole life insurance policy with the following benefits at the moment of death a fraction s of a year since the last policy anniversary:

(a) \$1, plus

(b) a refund of unearned premium, calculated as

$$\bar{P}(\bar{A}_x)\bar{a}_{\overline{1-s}|}.$$

The average refund of unearned premium can be approximated as half the gross premium, but here it is assumed, following Boermeester [1] and Scher [3], that the above theoretically accurate refund is paid. It will be shown that if this refund is assumed, then each of the three types of reserves (1), (3), and (4) above can be adjusted to produce the same terminal reserve ${}_t\bar{V}(\bar{A}_x)$, for the annual premium policy defined above. Here "terminal reserve" means, as usual, that calculated immediately before the annual premium due date.

The policy considered in the fully continuous case (2) is different in that only benefit (a) above is paid.

A. Discontinued Continuous Type

In verifying that the terminal reserves all equal ${}_t\bar{V}(\bar{A}_x)$, we first study the discounted continuous type (Table 1).

Let us consider the amount of annual net valuation premium in the discounted continuous case for the policy with benefits (a) and (b) defined above. A direct approach to determination of this premium is the same as that used in deriving Equation (2.8) below in the curtate case; logically if the same benefits are paid, then the same annual premium must result.

A somewhat different argument is used here to derive the annual net valuation premium for benefits (a) and (b) in the discounted continuous case. Let us postulate that the annual premium under the discounted continuous case is $\bar{P}(\bar{A}_x)\bar{a}_{\overline{1}|}$. Then in each policy year (except the year of death), the value at the policy anniversary of that year's premium equals the discounted value $\bar{P}(\bar{A}_x)\bar{a}_{\overline{1}|}$ of the continuous premium for benefit (a) in the fully continuous case. At the policy anniversary preceding death, the discounted value of the fully continuous premium for benefit (a) is $\bar{P}(\bar{A}_x)\bar{a}_{\overline{s}|}$. But we have

$$\bar{P}(\bar{A}_x)\bar{a}_{\overline{1}|} - (1 + i)^{-s}\bar{P}(\bar{A}_x)\bar{a}_{\overline{1-s}|} = \bar{P}(\bar{A}_x)\bar{a}_{\overline{s}|}. \tag{2.1}$$

Thus we have equality at each policy anniversary of

- The discounted value of the year's premiums paid for benefit (a) in the fully continuous case
- The discounted value of the excess of the postulated discounted continuous premium over the required benefit (b).

Thus we have verified our postulated annual premium, $\bar{P}(\bar{A}_x)\bar{a}_{\overline{1}|}$, which can now be used as the valuation net premium for a policy with benefits (a) and (b) under the discounted continuous case.

The choice of refund benefit, $\bar{P}(\bar{A}_x)\bar{a}_{\overline{1-s}|}$ (Scher [3]), is confirmed by its leading to the desirable premium $\bar{P}(\bar{A}_x)\bar{a}_{\overline{1}|}$.

In determining the reserve corresponding to the refund, we consider the values of continuous payments of $\bar{P}(\bar{A}_x)$ per annum:

TABLE 1

ADJUSTMENTS TO NET PREMIUMS

Type	Net Premium				Refer To
	Basic	Refund on Death	Immediate Payment of Claims	Corrected	
(1) Curtate	$\frac{A_x}{\ddot{a}_x}$	$+\frac{\bar{P}(\bar{A}_x)(\bar{A}_x - A_x)}{\delta \ddot{a}_x}$	$+\frac{\bar{A}_x - A_x}{\ddot{a}_x}$	$= \bar{P}(\bar{A}_x)\bar{a}_{\overline{1}\vert}$	Equation (2.8)
(2) Fully Continuous	$\frac{\bar{A}_x}{\bar{a}_x}$	$+0$	$+0$	$= \bar{P}(\bar{A}_x)$	
(3) Discounted Continuous	$\frac{\bar{A}_x}{\bar{a}_x}\bar{a}_{\overline{1}\vert}$	$+0$	$+0$	$= \bar{P}(\bar{A}_x)\bar{a}_{\overline{1}\vert}$	
(4) Semicontinuous	$\frac{\bar{A}_x}{\ddot{a}_x}$	$+\frac{\bar{P}(\bar{A}_x)(\bar{A}_x - A_x)}{\delta \ddot{a}_x}$	$+0$	$= \bar{P}(\bar{A}_x)\bar{a}_{\overline{1}\vert}$	Equation (2.4)

Origin for Discounting	Commencement of Payments	Termination of Payments	Discounted Value
Age $x+t$	Age $x+t$	∞	$\frac{\bar{P}(\bar{A}_x)}{\delta}$
Age $x+t$	Death	∞	$\frac{\bar{P}(\bar{A}_x)}{\delta}\bar{A}_{x+t}$
Age $x+t$	End of year of death	∞	$\frac{\bar{P}(\bar{A}_x)}{\delta}A_{x+t}$
Age $x+t$	Death	End of year of death	$\frac{\bar{P}(\bar{A}_x)}{\delta}(\bar{A}_{x+t}-A_{x+t})$

Thus the value of this premium refund feature is $\bar{P}(\bar{A}_x)(\bar{A}_{x+t}-A_{x+t})/\delta$, as indicated in the above table (see also Scher [3]).

Then the terminal reserve, ${}_tV_x^{DC}$, under the discounted continuous method is the reserve considering the value of the main \$1 benefit, the annual premiums and the refund benefit:

$$\begin{aligned}
{}_tV_x^{DC} &= \bar{A}_{x+t} - \ddot{a}_{x+t}\bar{P}(\bar{A}_x)\bar{a}_{\overline{1}|} + \frac{\bar{P}(\bar{A}_x)}{\delta}[\bar{A}_{x+t} - A_{x+t}] \\
&= \bar{A}_{x+t} - \ddot{a}_{x+t}\bar{P}(\bar{A}_x)\bar{a}_{\overline{1}|} + \frac{\bar{P}(\bar{A}_x)}{\delta}[1 - \delta\bar{a}_{x+t} - 1 + d\ddot{a}_{x+t}] \\
&= \bar{A}_{x+t} - \bar{P}(\bar{A}_x)\bar{a}_{x+t} \\
&= {}_t\bar{V}(\bar{A}_x). \qquad (2.2)
\end{aligned}$$

Thus with the "correct" choice of refund of unearned premium on death, the terminal reserve at a policy anniversary under the discounted continuous method equals that under the fully continuous method.

The relevant abbreviations used in this paper are:

DC = discounted continuous
SC = semicontinuous
PR = (unearned) premium refund
IM = immediate payment of claims
CU = curtate
ACC = accurate.

B. Semicontinuous Type

For a whole life insurance, the semicontinuous annual premium is given by $P(\bar{A}_x) = \bar{A}_x/\ddot{a}_x$. Assume that, again, the refund of unearned premium on death a period s after the policy anniversary is $\bar{P}(\bar{A}_x)\bar{a}_{\overline{1-s|}}$. The net annual premium required to pay for the refund benefit is, using the analysis of Table 2 and of Scher [3],

$$P(\bar{A}^{PR}) = \frac{1}{\delta}\,\frac{\bar{P}(\bar{A}_x)(\bar{A}_x - A_x)}{\ddot{a}_x}. \tag{2.3}$$

Hence the total annual premium is

$$\begin{aligned}
P_x^{SC} &= \frac{\bar{A}_x}{\ddot{a}_x^{(m)}} + P(\bar{A}^{PR}) \\
&= \bar{A}_x\left[\frac{1}{\ddot{a}_x}\right] + \frac{\bar{P}(\bar{A}_x)(\bar{A}_x - A_x)}{\delta\ddot{a}_x} \\
&= \bar{A}_x\left[\frac{\delta\bar{a}_x + 1 - \delta\bar{a}_x - 1 + d\ddot{a}_x}{\delta\ddot{a}_x\bar{a}_x}\right] \\
&= \bar{P}(\bar{A}_x)\bar{a}_{\overline{1|}}.
\end{aligned} \tag{2.4}$$

Thus the total premium under the semicontinuous method is the same as that under the discounted continuous method. This is reasonable since the benefits are the same and equality of the present value of the premiums is satisfied in view of the premium refund feature. Thus the addition to the reserve in respect of the premium refund feature is

$$\begin{aligned}
V(\bar{A}^{PR}) &= \frac{1}{\delta}\,\bar{P}(\bar{A}_x)(\bar{A}_{x+t} - A_{x+t}) - P(\bar{A}^{PR})\ddot{a}_{x+t} \\
&= \frac{1}{\delta}\,\bar{P}(\bar{A}_x)\left[\bar{A}_{x+t} - A_{x+t} - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}(\bar{A}_x - A_x)\right] \\
&= \frac{1}{\delta}\,\bar{P}(\bar{A}_x)\,[{}_tV(\bar{A}_x) - {}_tV_x]
\end{aligned} \tag{2.5}$$

The total terminal reserve under the semicontinuous case is then given by

$$
\begin{aligned}
{}_tV_x^{SC} &= \bar{A}_{x+t} - P(\bar{A}_x)\ddot{a}_{x+t} + \frac{1}{\delta}\bar{P}(\bar{A}_x)\,[{}_tV(\bar{A}_x) - {}_tV_x] \\
&= \bar{A}_{x+t} - \frac{\bar{A}_x}{\ddot{a}_x}\ddot{a}_{x+t} + \frac{\bar{A}_x}{\delta\bar{a}_x}\left[\bar{A}_{x+t} - \frac{\bar{A}_x}{\ddot{a}_x}\ddot{a}_{x+t} - A_{x+t} + \frac{A_x}{\ddot{a}_x}\ddot{a}_{x+t}\right] \\
&= \bar{A}_{x+t} - \frac{\bar{A}_x}{\bar{a}_x}\bar{a}_{x+t} + \bar{A}_x\left[-\frac{\ddot{a}_{x+t}}{\ddot{a}_x} + \frac{1}{\delta\bar{a}_x} - \frac{\ddot{a}_{x+t}}{\delta\bar{a}_x\ddot{a}_x}\right. \\
&\quad \left. + \frac{\ddot{a}_{x+t}}{\ddot{a}_x} - \frac{1 - d\ddot{a}_{x+t}}{\delta\bar{a}_x} + \frac{\ddot{a}_{x+t}}{\delta\bar{a}_x\ddot{a}_x} - \frac{d\ddot{a}_{x+t}}{\delta\bar{a}_x}\right] \\
&= {}_t\bar{V}(\bar{A}_x)
\end{aligned}
\tag{2.6}
$$

That the terminal reserve under the semicontinuous case equals that under the fully continuous case is intuitively reasonable. The benefits have equal value $\bar{A}_x$, and the annual premiums are set up so that, including the refund benefit, they are level and equal in present value.

C. Curtate Type

The curtate premium is the standard $P_x = A_x/\ddot{a}_x$. Again assume that the refund of unearned premium on death is made of amount $\bar{P}(\bar{A}_x)\bar{a}_{\overline{1-s}|}$. The premium for the refund benefit is again $P(\bar{A}^{PR})$, given by Equation (2.3). For the curtate type only, since A_x is used rather than $\bar{A}_x$, an additional premium is required for the immediate rather than end of year payment of claims, of amount

$$P_x^{IM} = \frac{\bar{A}_x - A_x}{\ddot{a}_x}. \tag{2.7}$$

Thus, taking into account also the premium for the refund of unearned premium on death, the total net annual premium is:

$$
\begin{aligned}
P_x^{CU} &= \frac{A_x}{\ddot{a}_x} + P(\bar{A}^{PR}) + \frac{\bar{A}_x - A_x}{\ddot{a}_x} \\
&= \frac{\bar{A}_x}{\ddot{a}_x} + \frac{\bar{P}(\bar{A}_x)(\bar{A}_x - A_x)}{\delta\ddot{a}_x} \\
&= \bar{P}(\bar{A}_x)\bar{a}_{\overline{1}|},
\end{aligned}
\tag{2.8}
$$

where use has been made of Equations (2.3) and (2.4). There is an associated additional reserve:

$$
\begin{aligned}
{}_tV_x^{IM} &= \bar{A}_{x+t} - A_{x+t} - P_x^{IM}\,\ddot{a}_{x+t} \\
&= {}_tV(\bar{A}_x) - {}_tV_x. \qquad (2.9)
\end{aligned}
$$

Hence the total curtate reserve is

$$
\begin{aligned}
{}_tV_x^{CU} &= A_{x+t} - P_x\,\ddot{a}_{x+t} + \frac{\bar{P}(\bar{A}_x)}{\delta}\,[{}_tV(\bar{A}_x) - {}_tV_x] \\
&\quad + [{}_tV(\bar{A}_x) - {}_tV_x] \\
&= {}_tV(\bar{A}_x) + \frac{\bar{P}(\bar{A}_x)}{\delta}\,[{}_tV(\bar{A}_x) - {}_tV_x] \\
&= {}_t\bar{V}(\bar{A}_x), \qquad (2.10)
\end{aligned}
$$

where the curtate has been reduced to the semicontinuous case and then Equation (2.6) has been used.

D. Summary Tables and Numerical Examples

Table 1 summarizes the calculation of the annual premium when there is immediate payment of claims and a refund of unearned premium $\bar{P}(\bar{A}_x)\bar{a}_{\overline{1-s}|}$ on death. As would be expected, the premium produced under all the methods considered is the same: $\bar{P}(\bar{A}_x)\bar{a}_{\overline{1}|}$ annually in advance, or $\bar{P}(\bar{A}_x)$ continuously. This equality is seen to be a consequence of the equality of the cash flows and assumptions under all the methods.

Table 2 summarizes the calculation of the terminal reserve. Again, under all four methods the terminal reserve taking account of the immediate payment of claims and of the refund benefit is ${}_t\bar{V}(\bar{A}_x)$.

"Basic" net premiums and "basic" terminal reserves are mentioned in Tables 1 and 2, respectively. These are for "basic" policies, which differ between the four methods as follows:

(1) Curtate: benefit of $1 at the end of the year of death
(2) Fully continuous: benefit of $1 immediately on death
(3) Discounted continuous: benefit of $1 and premium refund of $\bar{P}(\bar{A})\bar{a}_{\overline{1-s}|}$, both immediately on death
(4) Semicontinuous: benefit of $1 at the end of the year of death.

Tables 3 and 4 give a numerical example of the calculation of the terminal reserve at duration 10 of a whole life policy issued at age 40. Mortality is

TABLE 2

ADJUSTMENTS TO TERMINAL RESERVES

Type	Terminal Reserve				Refer to
	Basic	Refund on Death	Immediate Payment of Claims	Corrected	
(1) Curtate	$A_{x+t} - P_x \ddot{a}_{x+t}$	$+ \frac{1}{\delta} \bar{P}(\bar{A}_x)[{}_tV(\bar{A}_x) - {}_tV_x]$	$+{}_tV(\bar{A}_x) - {}_tV_x$	$= {}_t\bar{V}(\bar{A}_x)$	Equation (2.9)
(2) Fully Continuous	$\bar{A}_{x+t} - \bar{P}(\bar{A}_x)\bar{a}_{x+t}$	$+0$	$+0$	$= {}_t\bar{V}(\bar{A}_x)$	Equation (2.2)
(3) Discounted Continuous	$\bar{A}_{x+t} - \ddot{a}_{x+t}\bar{P}(\bar{A}_x)\bar{a}_{\overline{1}\rceil}$	$+ \frac{\bar{P}(\bar{A}_x)}{\delta}(\bar{A}_{x+t} - A_{x+t})$	$+0$	$= {}_t\bar{V}(\bar{A}_x)$	Equation (2.2)
(4) Semicontinuous	$\bar{A}_{x+t} - P(\bar{A}_x)\ddot{a}_{x+t}$	$+ \frac{1}{\delta} \bar{P}(\bar{A}_x)[{}_tV(\bar{A}_x) - {}_tV_x]$	$+0$	$= {}_t\bar{V}(\bar{A}_x)$	Equation (2.6)

that of the Illustrative Life Table of Bowers et al. [2, p. 72]. Thus the q_x are calculated by integration of the Makeham law

$$\mu_x = A + B\,c^x \tag{2.11}$$

with constants $A=0.00078$, $B=0.00005$ and $c=10^{0.04}$. For the current purpose and to facilitate calculation of the continuous functions, a step form of the force of mortality was used. Thus the q_x, which agree with those of Bowers et al. [2], were used to calculate a force of mortality assumed constant within each year of age. Thus the force of mortality used to calculate the $\bar{A}_x$ and μ_x was slightly different from that given by (2.11) and indeed is a form of weighted average of (2.11) within each year of age.

TABLE 3

NUMERICAL EXAMPLES OF NET PREMIUMS

Type	Net Premium per Thousand (Issue Age 40)			
	Basic	Refund on Death	Immediate Payment of Claims	Corrected
(1) Curtate	10.8882	0.0649	0.3259	11.2789
(2) Fully Continuous	11.6107	+0	+0	11.6107
(3) Discounted Continuous	11.2789	+0	+0	11.2789
(4) Semicontinuous	11.2140	0.0649	+0	11.2789

Relevant Values:

$1000A_{40} = 161.3242$
$1000\bar{A}_{40} = 166.1528$
$\ddot{a}_{40} = 14.81661$
$\bar{a}_{40} = 14.3103$
$1000\bar{P}(\bar{A}_{40}) = 11.6107$
$i = 0.06$
$\delta = 0.0582689$

3. RESERVES AT DECEMBER 31 VALUATION

A. Accurate Reserve

Figure 1 illustrates the reserve of an annual premium policy. An accurate calculation could be made at any date of the discounted value of benefits less the discounted value of future premiums, where timings are treated exactly to the day. The solid line illustrates the path of an exact reserve so calculated. It shows jumps when premiums are payable, and it has a slight curve in the period between premiums. Whether the path rises or falls between premiums depends on the relative importance of interest and the cost of insurance.

TABLE 4

NUMERICAL EXAMPLES OF TERMINAL RESERVES

	Terminal Reserve per Thousand (Issue Age 40; Duration 10)			
Type	Basic	Refund on Death	Immediate Payment of Claims	Corrected
(1) Curtate	104.5974	0.6259	3.1411	108.3644
(2) Fully Continuous	108.3644	0	0	108.3644
(3) Discounted Continuous	106.8770	1.4874	0	108.3644
(4) Semicontinuous	107.7385	0.6259	0	108.3644

Relevant Values:

$1000A_{50} = 249.0475$

$1000\bar{A}_{50} = 256.5122$

$\ddot{a}_{50} = 13.2668$

$\bar{a}_{50} = 12.7596$

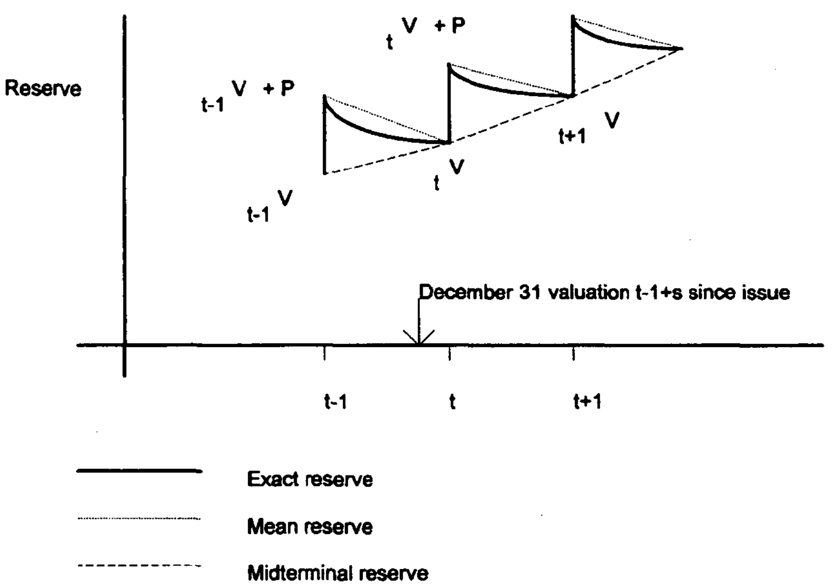
$1000\bar{P}(\bar{A}_{40}) = 11.2139$

$1000\,{}_{10}\bar{V}(\bar{A}_{40}) = 108.3644$

$1000\,{}_{10}V(\bar{A}_{40}) = 107.7385$

FIGURE 1

RESERVE AS A FUNCTION OF TIME



B. Interpolated Reserve

Generally the valuation will be at a date, typically December 31, between premium payment dates. In other words, as shown on Figure 1, we are required to calculate the reserve at a point on the curved path of the accurate reserve V_x^{ACC}. Assume that December 31 falls at time $t-1+u$ since policy issue at age x. In other words, valuation is a fractional year u after the start of the t-th policy year. Then the accurate reserve is given by:

$$ {}_{t-1+u}V_x^{ACC} = \bar{A}_{x+t+u} - \frac{D_{x+t}\,P^{CU}}{D_{x+t-1+u}}\,\ddot{a}_{x+t} \tag{3.1}$$

To ease calculations, it has been traditional to perform a linear interpolation:

$$\begin{aligned} {}_{t-1+u}V_x^{ACC} &\doteq (1 - u)\bar{A}_{x+t-1} + u\,\bar{A}_{x+t} \\ &\quad - P^{CU}\,[(1 - u)(\ddot{a}_{x+t-1} - 1) + u\ddot{a}_{x+t}] \\ &= (1 - u)[{}_{t-1}\bar{V}(\bar{A}_x) + P^{CU}] + u\,{}_t\bar{V}(\bar{A}_x) \qquad (3.2) \\ &= MV_{t-1+u} \end{aligned}$$

where

$$ MV_{t-1+u} = (1 - u)[{}_{t-1}\bar{V}(\bar{A}_x) + P^{CU}] + u\,{}_t\bar{V}(\bar{A}_x) \tag{3.3}$$

is a mean reserve not necessarily with equal weights of one-half. This formula (3.2) corresponds to following the interpolation defined by the dotted line in Figure 1.

Alternatively the interpolated reserve can be expressed as:

$$ {}_{t-1+u}V_x^{ACC} \doteq MTV_{t-1+u} + (1 - u)P^{CU} \tag{3.4}$$

where

$$ MTV_{t-1+u} = (1 - u)\,{}_{t-1}\bar{V}(\bar{A}_x) + u\,{}_t\bar{V}(\bar{A}_x) \tag{3.5}$$

is a (weighted) midterminal reserve. The term $(1-u)P^{CU}$ is then identified as an unearned premium reserve addition to the midterminal reserve.

The assumption can be made of uniform distribution of policy anniversaries within the calendar year. Then at December 31, 1999, for example, policies then in their t-th policy year were issued between January 1, 1999$-t+1$ and December 31, 1999$-t+1$. If we make the assumption of uniform distribution of policy anniversaries within the calendar year, then we can approximate all the policies as having a June 30/July 1, 1999$-t+1$ issue date. Then $u=1/2$ and we recover the familiar expressions for the mean and midterminal reserves

$$MV_{t-1+1/2} = \frac{1}{2}\,[{}_{t-1}\bar{V}(\bar{A}_x) + P^{CU}] + \frac{1}{2}\,{}_t\bar{V}(\bar{A}_x) \tag{3.6}$$

$$MTV_{t-1+1/2} = \frac{1}{2}\,{}_{t-1}\bar{V}(\bar{A}_x) + \frac{1}{2}\,{}_t\bar{V}(\bar{A}_x) \tag{3.7}$$

4. MODAL PREMIUMS

The intent of this paper was to illustrate the interrelationship of the various reserving methodologies and to highlight the steps used in deriving reserve values in practice. The discussion has been in terms of annual premiums. There is nothing in the early stages of the above analysis to prevent us from counting time in intervals $1/m$ rather than of one year. The annual premium $P^{CU} = \bar{P}(\bar{A}_x)\bar{a}_{\overline{1}|}$ would be replaced by $\bar{P}(\bar{A}_x)\bar{a}_{\overline{1/m}|}$, and, for example, Equation (2.6) would be modified to give, for k integer,

$$\begin{aligned} {}_{t-1+k/m}V_x^{SC} &= \bar{A}_{x+t-1+k/m} - P^{(m)}\,(\bar{A}_x)\,\ddot{a}^{(m)}_{x+t-1+k/m} \\ &\quad + \frac{1}{\delta}\bar{P}\,(\bar{A}_x)\,[{}_{t-1+k/m}V\,(\bar{A}_x) - {}_{t-1+k/m}V_x] \\ &= {}_{t-1+k/m}\bar{V}(\bar{A}_x). \end{aligned} \tag{4.1}$$

Then the interpolation Equation (3.2) could be set up for $0 < w < 1$ and $0 \le k < m$ as

$$\begin{aligned} {}_{t-1+(k+w)/m}V_x^{ACC} &\doteq (1 - w)[{}_{t-1+k/m}\bar{V}(\bar{A}_x) \\ &\quad + \bar{P}(\bar{A}_x)\bar{a}_{\overline{1/m}|}] + w\;{}_{t-1+(k+1)/m}\bar{V}(\bar{A}_x). \end{aligned} \tag{4.2}$$

However, in practice, it is more common to interpolate between the annual values ${}_{t-1}\bar{V}(\bar{A}_x)$ and ${}_t\bar{V}(\bar{A}_x)$, even if premiums are modal. Thus we are led into discussion in a future paper of the treatment of deferred premiums and the nondeduction of deferred premium on death.

5. CONCLUSION

The accurate reserve held for a policy depends only on the future cash flows, their probabilities and the assumptions used. The various reserving methods lead to the same reserve if the appropriate adjustments are made.

The algebraic demonstrations of this paper and of Scher [3] can provide greater clarity when considering this intuitively reasonable equivalence of the reserving models.

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DISCUSSION OF PRECEDING PAPER

ELIAS S.W. SHIU AND SERENA TIONG:

Dr. Sharp is to be thanked for this paper, clarifying the equivalence of various reserve methodologies. The purpose of this discussion is to supplement this fine exposition using the notion of *apportionable annuity-due* and *apportionable premium* presented in the textbook *Actuarial Mathematics* [1]. We consider the case in which premiums are payable m times a year.

1. Integer Functions

For a real number t, let $\lfloor t \rfloor$ denote the *floor* of t, which is the greatest integer less than or equal to t, and let $\lceil t \rceil$ denote the *ceiling* of t, which is the least integer greater than or equal to t. If $T=T(x)$ denotes the random variable of the future lifetime of a life now aged x [1, p. 46], then $\lfloor T \rfloor$ is K, the curtate-future-lifetime of (x) [1, p. 48], and $\lceil T \rceil$ is the time until the end of the year of death of (x). Because $12T$ is the time, measured in months, until the death of (x), we see that $\lceil 12T \rceil$ is the time, measured in months, until the end of the month of death of (x), and hence $\lceil 12T \rceil/12$ is the time, measured in years, until the end of the month of death of (x). Similarly, $\lceil 52T \rceil/52$ gives the time, measured in years, until the end of the week of death of (x), and so on. See Figure 1. Thus we have, for each positive integer m,

$$A_x^{(m)} = \mathrm{E}[v^{\lceil mT \rceil/m}], \qquad \text{(D.1.1)}$$

$$\ddot{a}_x^{(m)} = \mathrm{E}[\ddot{a}^{(m)}_{\overline{\lceil mT \rceil/m}|}], \qquad \text{(D.1.2)}$$

and

$$a_x^{(m)} = \mathrm{E}[a^{(m)}_{\overline{\lfloor mT \rfloor/m}|}]. \qquad \text{(D.1.3)}$$

In Exercise 5.14 of *Actuarial Mathematics* [1], $\lceil mT \rceil/m$ is denoted as $K+J_m$.

For two positive numbers s and t, we define

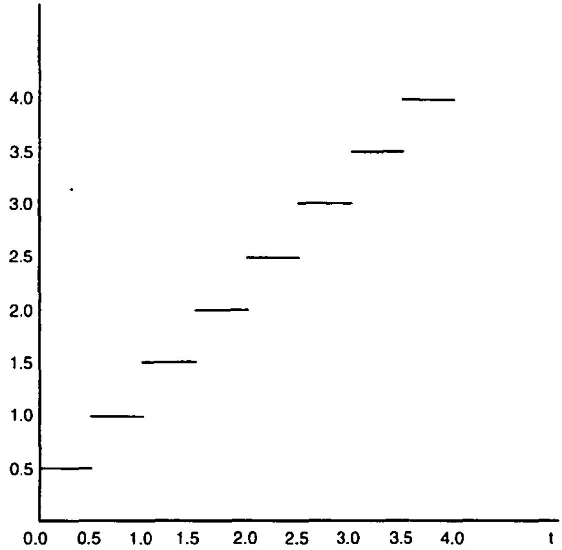
$$t \bmod s = t - s\lfloor t/s \rfloor \qquad \text{(D.1.4)}$$

and

$$t \text{ pad } s = s\lceil t/s \rceil - t. \qquad \text{(D.1.5)}$$

See Figure 2. The quantity "t mod s" is the (non-negative) remainder when t is divided by s, while "t pad s" is the least non-negative addition to t so

FIGURE 1
THE GRAPH OF THE FUNCTION $\lceil 2t \rceil/2$



that the result is divisible by s. The term *mod*, short for *modulo*, is standard mathematical usage. In defining *pad*, we are "borrowing from computer science, in which the term *padding* means the adding of blanks or nonsignificant characters to the end of a block or record in order to bring it up to a certain fixed size" [4, p. 572]. Note that Graham, Knuth and Patashnik [2, p. 83] use the term "mumble" for our "pad," and they write: "But of course we'd need a better name than 'mumble.' If sufficient applications come along, an appropriate name will probably suggest itself."

Note that, if t is not divisible by s, that is, if

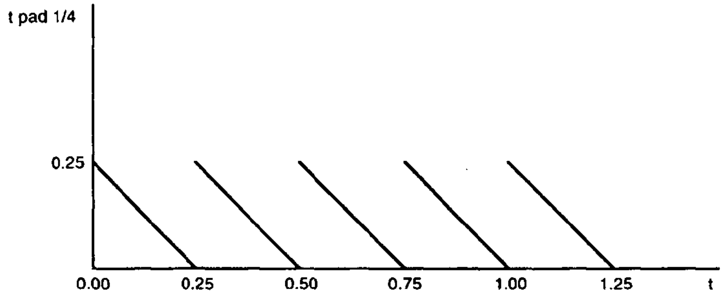
$$t \bmod s \neq 0,$$

then

$$t \bmod s + t \text{ pad } s = s. \qquad \text{(D.1.6)}$$

FIGURE 2

THE GRAPH OF THE FUNCTION t pad 1/4



2. *Apportionable Annuity-Due*

Let s be a positive number, not necessarily an integer; we define

$$\ddot{a}_{\overline{s}|}^{(m)} = \frac{1 - v^s}{d^{(m)}}. \tag{D.2.1}$$

This definition extends the usual definition for $\ddot{a}_{\overline{s}|}^{(m)}$, where s is a positive integer; with $m=1$, (D.2.1) can be found in Exercise 5.32.a of *Actuarial Mathematics* [1]. Then

$$\mathrm{E}[\ddot{a}_{\overline{T}|}^{(m)}] = \ddot{a}_x^{\{m\}} \tag{D.2.2}$$

is the single premium for an apportionable life annuity-due of 1 per year payable in installments of $1/m$ at the beginning of each m-th of a year while (x) survives (compare [1, Section 5.9]). It follows from (D.2.1) that

$$\ddot{a}_{\overline{\lceil mT \rceil/m}|}^{(m)} - \ddot{a}_{\overline{T}|}^{(m)} = v^T \ddot{a}_{\overline{(\lceil mT \rceil/m)-T}|}^{(m)}. \tag{D.2.3}$$

Taking expectations and applying (D.1.2) and (D.2.2) yields

$$\ddot{a}_x^{(m)} - \ddot{a}_x^{\{m\}} = \mathrm{E}[v^T \ddot{a}_{\overline{(\lceil mT \rceil/m)-T}|}^{(m)}]. \tag{D.2.4}$$

The amount of refund at T, the time of death of (x), is

$$\ddot{a}_{\overline{(\lceil mT \rceil/m)-T}|}^{(m)}. \tag{D.2.5}$$

From (D.1.5), we have

$$\frac{\lceil mT \rceil}{m} - T = T \operatorname{pad} \frac{1}{m}; \tag{D.2.6}$$

it is the time between death and the next payment date.

It follows from (D.2.1) that

$$\ddot{a}_{\overline{s}|}^{(m)} = \frac{\delta}{d^{(m)}} \bar{a}_{\overline{s}|} \tag{D.2.7}$$

$$= \frac{1}{m\bar{a}_{\overline{1/m}|}} \bar{a}_{\overline{s}|}. \tag{D.2.8}$$

Hence expression (D.2.5), the amount of refund at the time of death, can be rewritten as

$$\frac{1}{m\bar{a}_{\overline{1/m}|}} \bar{a}_{\overline{(\lceil mT \rceil/m)-T}|}, \tag{D.2.9}$$

which is [1, (5.9.5)].

3. Apportionable Premium

Apportionable premiums are treated in Section 6.5 of *Actuarial Mathematics* [1]. Consider the *equivalence principle* [1, p. 162]:

$$\text{E[present value of net premiums]} = \text{E[present value of benefits]}. \tag{D.3.1}$$

The provision for premium refund can be accounted for on the left-hand side of (D.3.1) or on its right-hand side. In the former approach, we have the equation

$$P^{\{m\}}(\bar{A}_x)\, \ddot{a}_x^{\{m\}} = \bar{A}_x, \tag{D.3.2}$$

while, in the latter,

$$P^{\{m\}}(\bar{A}_x)\, \ddot{a}_x^{(m)} = \bar{A}_x + \bar{A}_x^{PR,m}, \tag{D.3.3}$$

where $\bar{A}_x^{PR,m}$ denotes the single premium for the premium-refund benefit. The notation $\bar{A}_x^{PR,m}$ is due to Scher [3]; with $m=1$, it is written as $\bar{A}_x^{PR}$ in Scher [3] and in *Actuarial Mathematics* [1], and as $\bar{A}^{PR}$ in the paper. Because the amount of premium refund at the time of death is

$$P^{\{m\}}(\bar{A}_x)\, \ddot{a}^{(m)}_{\overline{(\lceil mT \rceil/m)-T|}}, \tag{D.3.4}$$

we have

$$\bar{A}_x^{PR,m} = P^{\{m\}}(\bar{A}_x)\ \mathrm{E}[v^T\, \ddot{a}^{(m)}_{\overline{(\lceil mT \rceil/m)-T|}}]. \tag{D.3.5}$$

Applying (D.2.4) to (D.3.5) yields

$$\bar{A}_x^{PR,m} = P^{\{m\}}(\bar{A}_x)[\ddot{a}_x^{(m)} - \ddot{a}_x^{\{m\}}], \tag{D.3.6}$$

verifying that equations (D.3.2) and (D.3.3) are equivalent.

Putting $s=T$ in (D.2.8) and taking expectations, we have

$$\ddot{a}_x^{\{m\}} = \frac{1}{m\bar{a}_{\overline{1/m|}}}\, \bar{a}_x. \tag{D.3.7}$$

Substituting (D.3.7) into (D.3.2) yields

$$P^{\{m\}}(\bar{A}_x) = m\bar{a}_{\overline{1/m|}}\, \bar{P}(\bar{A}_x). \tag{D.3.8}$$

With $m=1$, the right-hand side of (D.3.8) simplifies as

$$\bar{a}_{\overline{1|}}\, \bar{P}(\bar{A}_x), \tag{D.3.9}$$

which is the "corrected" net premium in the paper. In other words, the "corrected" net premium in the paper is the apportionable premium $P^{\{1\}}(\bar{A}_x)$ in *Actuarial Mathematics* [1]. Substituting (D.3.8) and (D.2.8) (with $s=(\lceil mT \rceil/m)-T$) into (D.3.4) shows that the amount of premium refund can also be written as

$$\bar{P}(\bar{A}_x)\, \bar{a}_{\overline{(\lceil mT \rceil/m)-T|}}, \tag{D.3.10}$$

which, with $m=1$, is the "unearned premium"

$$\bar{P}(\bar{A}_x)\bar{a}_{\overline{1-s|}} \tag{D.3.11}$$

in the paper.

It follows from (D.3.3) that

$$\begin{aligned} \bar{A}_x^{PR,m} &= P^{\{m\}}(\bar{A}_x)\ddot{a}_x^{(m)} - \bar{A}_x \\ &= [P^{\{m\}}(\bar{A}_x) - P^{(m)}(\bar{A}_x)]\ddot{a}_x^{(m)}. \end{aligned} \tag{D.3.12}$$

Hence the net level annual premium for the premium-refund benefit, payable m times per year, is

$$P^{(m)}(\bar{A}_x^{PR,m}) = \frac{\bar{A}_x^{PR,m}}{\ddot{a}_x^{(m)}}$$

$$= P^{(m)}(\bar{A}_x) - P^{(m)}(\bar{A}_x), \qquad \text{(D.3.13)}$$

which generalizes (6.5.9) of *Actuarial Mathematics* [1] and (2.4) in the paper.

Putting $s=T$ in (D.2.7) and taking expectations, we have

$$\ddot{a}_x^{(m)} = \frac{\delta}{d^{(m)}}\,\bar{a}_x, \qquad \text{(D.3.14)}$$

which is [1, (5.9.7)]. Applying (D.3.14) to (D.3.2) and rearranging yields

$$P^{(m)}(\bar{A}_x) = \frac{d^{(m)}}{\delta}\,\bar{P}(\bar{A}_x). \qquad \text{(D.3.15)}$$

Substituting (D.3.15) into the right-hand side of (D.3.13), we obtain

$$P^{(m)}(\bar{A}_x^{PR,m}) = \frac{d^{(m)}}{\delta}\,\bar{P}(\bar{A}_x) - P^{(m)}(\bar{A}_x)$$

$$= \bar{P}(\bar{A}_x)\left[\frac{d^{(m)}}{\delta} - \frac{\bar{a}_x}{\ddot{a}_x^{(m)}}\right]$$

$$= \bar{P}(\bar{A}_x)\,\frac{\bar{A}_x - A_x^{(m)}}{\delta\ddot{a}_x^{(m)}}, \qquad \text{(D.3.16)}$$

which, for $m=1$, is (2.3) in the paper; see also Exercise 6.17 of *Actuarial Mathematics* [1].

4. *Reserves*

It follows from (D.3.14), (D.3.15), and a derivation similar to the one on page 207 of *Actuarial Mathematics* [1] that, if t is a positive number divisible by $1/m$, that is, if

$$t \bmod \frac{1}{m} = 0,$$

then

$${}_tV^{(m)}(\bar{A}_x) = {}_t\bar{V}(\bar{A}_x). \qquad \text{(D.4.1)}$$

Also, extending the proof on page 208 of *Actuarial Mathematics* [1], we have

$$ {}_tV^{(m)}(\bar{A}_x^{PR,m}) = {}_t\bar{V}(\bar{A}_x) - {}_tV^{(m)}(\bar{A}_x) $$
$$ = {}_tV^{\{m\}}(\bar{A}_x) - {}_tV^{(m)}(\bar{A}_x). \qquad \text{(D.4.2)} $$

5. Interpolation

Let s be a positive number not divisible by $1/m$; suppose that we are to estimate the reserve at time s, ${}_sV^{\{m\}}(\bar{A}_x)$, by *linear interpolation*. The reserve at time $\lceil ms \rceil/m$ is

$$ {}_{\lceil ms \rceil/m}V^{\{m\}}(\bar{A}_x) = {}_{\lceil ms \rceil/m}\bar{V}(\bar{A}_x) \qquad \text{(D.5.1)} $$

by (D.4.1). The reserve at a moment after time $\lfloor ms \rfloor/m$, that is, after the payment of $(1/m)P^{\{m\}}(\bar{A}_x)$, is

$$ {}_{\lfloor ms \rfloor/m}V^{\{m\}}(\bar{A}_x) + \frac{1}{m}P^{\{m\}}(\bar{A}_x) = {}_{\lfloor ms \rfloor/m}\bar{V}(\bar{A}_x) + \bar{P}(\bar{A}_x)\bar{a}_{\overline{1/m}|} \qquad \text{(D.5.2)} $$

by (D.4.1) and (D.3.8). Then

$$ {}_sV^{\{m\}}(\bar{A}_x) \approx m\left\{\left(s \text{ pad } \frac{1}{m}\right)[{}_{\lfloor ms \rfloor/m}\bar{V}(\bar{A}_x) + \bar{P}(\bar{A}_x)\bar{a}_{\overline{1/m}|}] \right. $$
$$ \left. + \left(s \text{ mod } \frac{1}{m}\right) {}_{\lceil ms \rceil/m}\bar{V}(\bar{A}_x)\right\} $$
$$ = (\lceil ms \rceil - ms)[{}_{\lfloor ms \rfloor/m}\bar{V}(\bar{A}_x) + \bar{P}(\bar{A}_x)\bar{a}_{\overline{1/m}|}] $$
$$ + (ms - \lfloor ms \rfloor){}_{\lceil ms \rceil/m}\bar{V}(\bar{A}_x), \qquad \text{(D.5.3)} $$

which is (4.2) in the paper. An alternative linear-interpolation formula is:

$$ {}_sV^{\{m\}}(\bar{A}_x) \approx (s \text{ pad } 1){}_{\lfloor s \rfloor}V^{\{m\}}(\bar{A}_x) + (s \text{ mod } 1){}_{\lceil s \rceil}V^{\{m\}}(\bar{A}_x) $$
$$ + \left(s \text{ pad } \frac{1}{m}\right) P^{\{m\}}(\bar{A}_x), \qquad \text{(D.5.4)} $$

which is Exercise 7.24.b of *Actuarial Mathematics* [1].

6. Endowment Insurance

For two real numbers s and t, let $s \wedge t$ denote the minimum of s and t. Replacing $\lceil mT \rceil/m$ by $(\lceil mT \rceil/m) \wedge n$ and T by $T \wedge n$, we can extend the analysis above from whole life insurance to n-year endowment insurance. For example, in place of (D.1.1), (D.1.2), and (D.1.3), we have

$$A_{x:\overline{n}|}^{(m)} = \mathrm{E}[v^{(\lceil mT \rceil/m) \wedge n}], \qquad \text{(D.6.1)}$$

$$\ddot{a}_{x:\overline{n}|}^{(m)} = \mathrm{E}[\ddot{a}_{\overline{(\lceil mT \rceil/m) \wedge n}|}^{(m)}], \qquad \text{(D.6.2)}$$

and

$$a_{x:\overline{n}|}^{(m)} = \mathrm{E}[a_{\overline{(\lfloor mT \rfloor/m) \wedge n}|}^{(m)}], \qquad \text{(D.6.3)}$$

respectively.

7. Complete Annuities-Immediate

Parallel to the notion of the apportionable annuity-due is that of the *complete annuity-immediate*; see [1, Section 5.9]. Let t be a positive number, not necessarily an integer; we define

$$a_{\overline{t}|}^{(m)} = \frac{1 - v^t}{i^{(m)}} \qquad \text{(D.7.1)}$$

and

$$s_{\overline{t}|}^{(m)} = \frac{(1 + i)^t - 1}{i^{(m)}}. \qquad \text{(D.7.2)}$$

With $m=1$, (D.7.1) can be found in Example 5.13.b and Exercise 5.31.a of *Actuarial Mathematics* [1]. Then

$$\mathring{a}_x^{(m)} = \mathrm{E}[a_{\overline{T}|}^{(m)}]. \qquad \text{(D.7.3)}$$

It follows from (D.7.1) and (D.7.2) that

$$a_{\overline{T}|}^{(m)} - a_{\overline{\lfloor mT \rfloor/m}|}^{(m)} = v^T s_{\overline{T-(\lfloor mT \rfloor/m)}|}^{(m)}. \qquad \text{(D.7.4)}$$

Taking expectations and applying (D.7.3) and (D.1.3) yields

$$\mathring{a}_x^{(m)} - a_x^{(m)} = \mathrm{E}[v^T s_{\overline{T-(\lfloor mT \rfloor/m)}|}^{(m)}]. \qquad \text{(D.7.5)}$$

The adjustment payment at time T is $s_{\overline{T-\lfloor mT \rfloor/m}|}^{(m)}$. From (D.1.4), we have

$$T - \frac{\lfloor mT \rfloor}{m} = T \bmod \frac{1}{m}; \qquad \text{(D.7.6)}$$

it is the time between the last payment date before death and the date of death.

Let n be a positive number divisible by $1/m$. Replacing T by $T \wedge n$ and $\lceil mT \rceil/m$ by $(\lceil mT \rceil/m) \wedge n$ in (D.7.3) and (D.7.4), we have

$$\mathring{a}^{(m)}_{x:\overline{n}|} = \mathrm{E}[a^{(m)}_{\overline{T \wedge n}|}] \tag{D.7.7}$$

and

$$a^{(m)}_{\overline{T \wedge n}|} - a^{(m)}_{\overline{(\lfloor mT \rfloor/m) \wedge n}|} = v^{T \wedge n} s^{(m)}_{\overline{(T \wedge n) - (\lfloor mT \rfloor) \wedge n}|}. \tag{D.7.8}$$

Observe that

$$(T \wedge n) - [(\lfloor mT \rfloor/m) \wedge n] = \begin{cases} T - \lfloor mT \rfloor/m & \text{if } T < n \\ n - n = 0 & \text{if } T \geq n \end{cases}.$$

Let $I(\cdot)$ denote the indicator function,

$$I(A) = \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{if } A \text{ is false.} \end{cases}$$

Since $s^{(m)}_{\overline{0}|} = 0$, the right-hand side of (D.7.8) can be simplified as

$$v^T s^{(m)}_{\overline{T - (\lfloor mT \rfloor/m)}|} I(T < n).$$

Hence it follows from (D.7.8), (D.7.7), and (D.6.3) that

$$\mathring{a}^{(m)}_{x:\overline{n}|} - a^{(m)}_{x:\overline{n}|} = \mathrm{E}[v^T s^{(m)}_{\overline{T - (\lfloor mT \rfloor/m)}|} I(T < n)]. \tag{D.7.9}$$

Also, note that

$$\frac{\ddot{a}^{\{m\}}_{x:\overline{n}|}}{\mathring{a}^{(m)}_{x:\overline{n}|}} = \frac{i^{(m)}}{d^{(m)}}$$

$$= (1 + i)^{1/m}. \tag{D.7.10}$$

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(AUTHOR'S REVIEW OF DISCUSSION)

KEITH SHARP:

Dr. Shiu and Ms. Tiong are to be thanked for the useful extensions and alternative viewpoints given in their discussion. They were unaware of my unpublished paper on the case of modal premiums,* and there is some overlap. Their analysis, however, adds a thorough theoretical base to the development. The use of the pad and modulus terminology and the corresponding notation provides an elegant structure for their analysis. The apportionable annuity-due is indeed the concept corresponding to the "refund of unearned premium" practice. Clarification of the relationship between the underlying mathematics and the approximations used in practice will be a valuable outcome of this work.

*Sharp, K.P. "Reserves for Policies With Modal Premiums," *Research Report 95-15*. Waterloo, Ont.: University of Waterloo Institute of Insurance and Pension Research, 1995.