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STOCHASTIC LIFE CONTINGENCIES WITH SOLVENCY CONSIDERATIONS

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ABSTRACT

The extension of the theory of life contingencies to a stochastic interest environment and its application to solvency valuation are discussed. Although life contingencies are widely used in traditional actuarial valuations of life insurance contracts, certain complications arise in a stochastic interest environment that are not evident when using traditional deterministic interest assumptions. In particular, many insurance functions can no longer be expressed in a simple form, resulting in a loss of the intuitive appeal of these functions. In this paper, a stochastic interest environment is introduced and analyzed in terms of its effects on insurance functions. Although the model is less general than others introduced in the literature, it is sufficiently flexible to handle the volatility and certain autocorrelation aspects of interest series. Its main advantage is the simple form of the resulting insurance functions and, hence, its intuitive appeal.

To examine the performance of a block of business, the assets as well as the liabilities are considered. For liabilities of a block of business under a common stochastic interest environment, limit theorems for approximating the behavior of sums of policies are no longer readily available. Even if the mortality experiences of the policies are independent, the liabilities are not independent because of the common interest environment. By considering assets as well as liabilities, matching of cash flows reduces the volatility of surplus, defined to be assets in excess of liabilities. In fact, under an extreme type of matching, limit laws for sums of homogeneous policies can be described under more general interest environments than those described above.

OVERVIEW

This paper addresses the stochastic theory of the valuation of a risk-taking enterprise from a solvency perspective. For concreteness, the enterprise is assumed to be a life insurance company. In the development, the incorporation of elements of financial economics is emphasized where possible. However, it is not the intent of this paper to show how well-developed valuation theories in finance, such as the capital asset pricing model, arbitrage pricing model, and so on, can be applied in an insurance context as in Garven [15]. These models employ several assumptions such as frictionless trading, perfectly informed market participants, and the like, which have been questioned in the relatively efficient asset markets and are dubious in an insurance liabilities context. In particular, the lack of a broad secondary market for trading life insurance liabilities makes the straightforward application of these valuation models to the insurance problem highly suspect; compare Tilley [35] and Giaccotto [17].

The approach is to extend some of the traditional actuarial valuation techniques to incorporate ideas from modern financial economics. To this end, the paper comprises two parts, stochastic life contingencies and valuation from a solvency perspective. Part I, stochastic life contingencies, reviews and extends a literature that has appeared in actuarial circles since the mid-1970s, that is, that not only the time of decrement but also the valuation discount rate may be random. Stochastic life contingencies are useful in pricing, but their true value is in the ability to determine a value for an established contract at current and future times. Part II, solvency valuation, addresses broader questions concerning the use of stochastic interest ideas developed in Part I. Here, many of the financial interpretations not explicitly mentioned in Part I are provided. Further, at the expense of weaker results, much weaker assumptions on the interest environment are made.

An important goal of this study is to provide actuaries with at least a partial response to the criticism of other financial analysts that valuation models do not take into account the stochastic nature of interest rates. There are many different levels to a complete response to this criticism. A basic response, as noted by Hickman [18], is that life contingency models traditionally use deterministic interest discounting, and these models have been successful for centuries. Another response is that the stochastic variability of interest rates is not crucial in a world in which assets and liabilities are nearly "immunized." Here, immunization refers to an asset management system in which an asset portfolio is constructed so that the asset cash inflows occur at roughly the same time and in the same amount as benefit payment outflows. The objective is to reduce the risk of asset price fluctuations due to changes in interest rates. See Bierwag [2] for an introduction to this area. This response assumes, however, that total liabilities are nonstochastic or, at least, can be predicted quite accurately. When total liabilities arise from several policies with unrelated losses, the average loss can be estimated within a desirable level of accuracy. However, in an environment of stochastic liabilities, matching techniques may or may not be adequate. It is precisely the extent of this adequacy that I wish to quantify. Perhaps the most complete response is the introduction of a model that includes annual forecasts of liabilities, a stochastically changing term structure of interest rate and the stochastic relationships among different types of assets within a portfolio. I hope that this study provides a step towards constructing such a model.

I. STOCHASTIC LIFE CONTINGENCIES

1. Introduction to Life Contingencies with Stochastic Discounting

For every insurance contract, the uncertain timing of contingent events is a key feature in quantifying financial aspects of these aleatory agreements. This is particularly true in life contingencies, the financial study of contracts in which the benefit payment and premium structure are considered known at contract initiation. Under the traditional approach to life contingencies, as in Jordan [20], premiums and reserves have been calculated by deterministically discounting for the effect of interest and various decrements including mortality, disability, and so on. Under the modern approach the decrements are assumed to be stochastic. This approach is described in the text by Bowers et al. [3]; see Wolthius and van Hoek [41] for an alternative description. Thus, several summary measures of financial contracts can be examined, including the median, standard deviation, 95th percentile, and so on, in lieu of using only the mean discounted value. This flexibility allows, for example, the financial analyst to explicitly consider the extent of potential adverse deviations from the mean. In this paper, I allow not only the various decrements but also the force of interest to be stochastic. Stochastic interest models have been considered previously in several studies including those of Pollard [27], Boyle [4], Wilkie [40], Waters [38], Panjer and Bellhouse [26], Bellhouse and Panjer [1], Westcott [39], de Jong [11], Giaccotto [17], and Dhaene [12]. One goal of this study is to review the contributions of these papers and recast the results in the notation of Bowers et al. [3], the current standard notation used in the North American actuarial literature.

Following Pollard [27] and Boyle [6], to model a stochastic interest environment we use the sequence $\{\Delta_k\}$. Here, Δ_k is a capital Greek delta that represents the random force of interest in the k-th period. Section 5 argues that Δ_k can be interpreted as a one-period spot rate. It is convenient to model the force of interest as the random quantity, in lieu of the effective interest or discount rate, due to the linear nature of correlation and autoregressive models and the multiplicative nature of compound interest. In this paper attention is restricted to discrete time models, and for convenience, we refer to time intervals as years. Continuous time models can be formulated (compare Panjer and Bellhouse [26] and Martin-Lof [23]) but are more complex and of less interest in actuarial practice.

The following outlines the rest of this part of the paper. In Section 2, I consider the case in which $\{\Delta_k\}$ is identically and independently distributed (i.i.d.). With the interpretation of $\{\Delta_k\}$ as one-period spot rates and the assumption that $\{\Delta_k\}$ as i.i.d., the logarithm of the accumulation of a one dollar investment, $\Delta_1 + \Delta_2 + \ldots + \Delta_k$, follows a random walk. This is desirable from the viewpoint of financial economics theory because the random walk is a special case of a discrete time martingale. See Gerber [16] for an introduction to martingales from an actuarial perspective. In investment pricing theory the martingale structure does not permit riskless arbitrage. In Section 2, I focus on summary measures, or parameters, of general insurance and annuity policies. For the mean and variance of many basic policies such as whole life, n-year term, life annuity due, and so on, the notation used in Bowers et al. [3] extends to the more general model. This represents two important differences between this study and those cited above. First, previous papers dealt primarily with the whole life and life annuity due policies, leaving the extension to more complex policies as implicit. Second, previous studies dealt explicitly only with net single premiums, leaving the extension to reserves as implicit. In Section 3, the Section 2 results are extended to an autocorrelated interest environment. Autocorrelated models for interest rates have received a resurgence of popularity under the label of "mean-reverting" walks in the financial economics literature lately; compare de Bondt and Thaler [10] for a recent overview. By restricting the model to a simple moving average model of order one, tractable results are achieved. The proofs of all the propositions are in the Appendix.

2. *Single Policy—Independent Interest Case*

In this section the notation for a single generalized policy is introduced. The mean and variance for the net single premium, net level premium and reserves are developed. As in Waters [38] and Westcott [39], higher-order moments can be developed in a similar yet tedious fashion. As emphasized by de Jong [11], the variance is an important component of the error in forecasting expected present values. As shown later in the paper, the first two moments are sufficient to use the Tchebycheff inequality to get a crude bound on the entire distribution. Further, it is well-known, in the special case of the normal distribution, that the first two moments are sufficient to characterize the entire distribution.

For convenience, I first describe some notation to be used throughout the paper. Assume initially that $\{\Delta_k\}$ is an i.i.d. sequence such that, for positive constants δ and α, $E(e^{-\Delta})=e^{-\delta}$ and $E(e^{-2\Delta})=e^{-\alpha}$. Since

$$0 \leq \mathrm{Var}(e^{-\Delta}) = e^{-\alpha} - e^{-2\delta},$$

we know that $\alpha \leq 2\delta$. If $\alpha = 2\delta$, the force of interest is a degenerate random variable and the techniques in Bowers et al. [3] apply. At time 0, the random present value of \$1 payable at time k is

$$v_k = \prod_{s=1}^{k} \exp(-\Delta_s) = \exp\left(-\sum_{s=1}^{k} \Delta_s\right),$$

and thus, the logarithm of v_k is a random walk. (See Section 5 for more interpretations of $\{v_k\}$.) The following example is central to many discussions in the literature.

Example 1.1. Lognormal Distribution

Assume $\Delta_1 = \Delta \sim N(\mu,\sigma^2)$; that is, Δ is distributed normally with mean μ and variance σ^2. In this case, $\exp(-\Delta)$ is said to be lognormally distributed with parameters $-\mu$ and σ^2; that is, $\exp(-\Delta) \sim \log N(-\mu,\sigma^2)$. The moment generating function of Δ is $E(e^{\Delta t}) = \exp(\mu t + \sigma^2 t^2/2)$. Thus, with $t = -1$, we have

$$e^{-\delta} = E\, e^{-\Delta} = \exp(-\mu + \sigma^2/2) \text{ or } \delta = \mu - \sigma^2/2.$$

With $t = -2$, we have

$$e^{-\alpha} = \exp(-2\mu + 2\sigma^2) \text{ or } \alpha = 2(\mu - \sigma^2).$$

Finally, with $\{\Delta_k\}$ i.i.d.,

$$\sum_{s=1}^{k} \Delta_s \sim N(k\mu, k\sigma^2),$$

and thus

$$v_k \sim \log N(-k\mu, k\sigma^2).$$

Note that the i.i.d. normality assumption will rarely be satisfied in practice; compare the discussion in Section 3 below. However, it does serve as a useful benchmark. For example, in this example the force of interest used in expected value calculations can be interpreted as a mean force minus $\sigma^2/2$, that is, a price paid for volatility.‡

Assume initially that there is only one decrement and that, as in Bowers et al. [3], K is the curtate time of decrement. Use the notation $P(K=k)={}_{k|}q_x$ and $P(K>k)={}_{k+1}p_x$, $k=0, 1, 2, \ldots$, for the mass and survival function, respectively. Also assume initially that K is independent of $\{\Delta_k\}$. Two types of general contracts are considered, insurance and annuity contracts. Under the general insurance contract, a benefit b_{k+1} is payable at the end of the year of loss. The present value of this benefit is

$$Z_{k+1} = v_{k+1}\, b_{k+1}. \tag{2.1}$$

Under the general annuity contract, payments a_s are payable at the beginning of each year up to and including the year of loss. The present value of the benefits is

$$a(k) = \sum_{s=0}^{k} v_s\, a_s. \tag{2.2}$$

where $v_0=1$. In principle, both insurance and annuity payments, b_{k+1} and a_s, respectively, may be positive, negative or zero.

Summary measures for the insurance benefit are easy to evaluate because there is only one benefit payment. By the law of iterated expectations, we have

$$\begin{aligned} E\,(Z_{K+1}) &= E\,[E(v_{K+1}\, b_{K+1}|K = k)] \\ &= E\,[e^{-\delta(K+1)}\, b_{K+1}] = \sum_{k=0}^{\infty} e^{-\delta(k+1)}\, b_{k+1}\; {}_{k|}q_x. \end{aligned} \tag{2.3}$$

Similarly,

$$E\,(Z_{K+1}^2) = E\,[e^{-\alpha(K+1)}\, b_{K+1}^2]. \tag{2.4}$$

For example, in the case of whole life insurance, $b_{k+1}=1$ for each k. Then, from (2.3) and (2.4), we have

$$E\,Z_{K+1} = E\,[e^{-\delta(K+1)}] = {}^{\delta}A_x = A_x$$

and

$$\mathrm{Var}(Z_{K+1}) = {}^{\alpha}A_x - (A_x)^2.$$

Compared to the results in Bowers et al. [3], the results are the same except we use α in lieu of 2δ in the variance calculation. Indeed, this holds true more generally in instances where $b_{K+1}^2 = b_{K+1}$ as in Theorem 4.1 of Bowers et al. [3, p. 85]. As pointed out there, this is convenient for computations. Finally, note that, when a specific distribution for $\{\Delta_k\}$ is assumed, it is straightforward to calculate the distribution of Z_{K+1}. We have the following.

Example 1.1 (continued)

Assume that $\Delta \sim N(\mu, \sigma^2)$ and use $\Phi(y)$ for the distribution function of a standard normal random variable, that is, for $Y \sim N(0,1)$, we have $\Phi(y) = P(Y \le y)$. To calculate the distribution function of $Z_{K+1} = v_{K+1}\, b_{K+1}$, we have

$$\begin{aligned} P(Z_{K+1} \le y) &= E[P\,(v_{K+1}\, b_{K+1} \le y | K = k)] \\ &= \sum_{k=0}^{\infty} P(v_{k+1} \le y/b_{k+1})\ {}_{k|}q_x \\ &= \sum_{k=0}^{\infty} \Phi\ \{[\log(y/b_{k+1}) + (k+1)\mu] \\ &\qquad / \, [\sigma(k+1)^{1/2}]\}\ {}_{k|}q_x. \end{aligned} \tag{2.5}$$

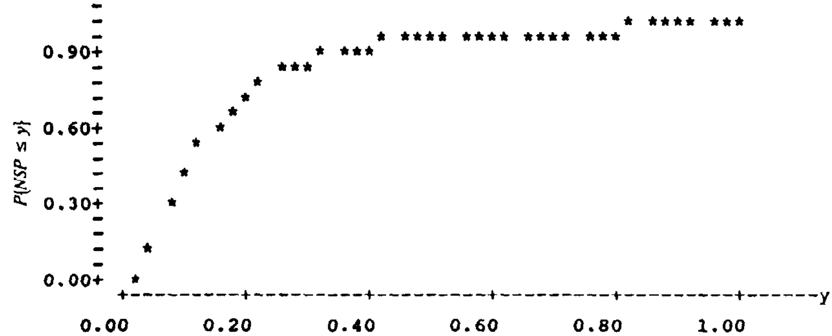
Consider the specific case of whole life insurance. The quantity in (2.5) is easy to compute. A graph can be found in Figure 1 for a life age 30, $\mu = 4.5\%$, $\sigma = 7\%$ and using the 1979–81 U.S. Male Life Tables for the mortality decrement (compare Bowers et al. [3, pp. 55–58]). It is instructive to approximate the median from this graph, which turns out to be roughly 0.12. With $\delta = \mu - \sigma^2/2 = 0.04255$, this can be compared to the mean A_{30}, which turns out to be 0.16744. The fact that the median is less that the mean is one indication that the distribution is skewed to the right.‡

Summary measures for annuity benefits are more complex because of the multiplicities of payments. Similarly to (2.3) and (2.4), with (2.2) we have

$$E[a(K)] = E\left(\sum_{s=0}^{K} e^{-\delta s}\, a_s\right) \tag{2.6}$$

FIGURE 1

DISTRIBUTION FUNCTION OF THE NET SINGLE PREMIUM FOR WHOLE LIFE INSURANCE



and

$$E[a(K)^2] = E\left\{\sum_{s=0}^{K} e^{-\alpha s}\, a_s^2 + 2\sum_{r=1}^{K}\sum_{s=0}^{r-1} e^{-\delta s}\, e^{-(\alpha-\delta)r}\, a_r\, a_s\right\}. \quad (2.7)$$

The derivation of (2.7) takes several lines of algebra.

For example, suppose $a_s = 1$ for each s. Then $a(K)$ is the random variable associated with a life annuity due. Here, after some algebra, from (2.6) and (2.7), we get

$$E\left(\sum_{s=0}^{K} v_s\right) = E\left(\sum_{s=0}^{K} e^{-\delta s}\right) = {}^{\delta}\ddot{a}_x = \ddot{a}_x \quad (2.8)$$

and

$$E\left(\sum_{s=0}^{K} v_s\right)^2 = {}^{\alpha}\ddot{a}_x + 2(\ddot{a}_x - {}^{\alpha}\ddot{a}_x)/[1 - e^{-(\alpha-\delta)}]. \quad (2.9)$$

Suppose in addition that $K = n-1$ with probability one. Then $a(K)$ is the random variable associated with an n-year certain annuity due and (2.8) and (2.9) reduce to

$$E\left(\sum_{s=0}^{n-1} v_s\right) = {}^{\alpha}\ddot{a}_{\overline{n|}} = \ddot{a}_{\overline{n|}}$$

and

$$E\left(\sum_{s=0}^{n-1} v_s\right)^2 = {}^{\alpha}\ddot{a}_{\overline{n}|} + 2(\ddot{a}_{\overline{n}|} - {}^{\alpha}\ddot{a}_{\overline{n}|}) / [1 - e^{-(\alpha-\delta)}].$$

The important point is that even in special cases, such as Example 1.1, there is a simple expression for the distribution of $\Sigma_{s=0}^{K} \Delta_s$ and thus v_K. This is not the case for $\Sigma_{k=1}^{n-1} v_k$. This is easy to see since $v_1, v_2, \ldots, v_{n-1}$ depend on Δ_1 and hence are dependent random variables.

Net level premiums can be constructed by using the equivalence principle, as in Bowers et al. [3, p. 162]. To this end, consider an insurance contract with premiums Pa_s payable at the beginning of each year, at times $s=0, 1, 2, \ldots$. Benefits b_{k+1} are payable at the end of the year of loss, at time $k+1$. At contract initiation, with Z_{k+1} and $a(k)$ defined in (2.1) and (2.2), respectively, let

$${}_0L(K,P) = Z_{K+1} - P\,a(K)$$

be the loss at time 0 for a generic premium level P. The net level premium P_N is defined to be the solution of $E[{}_0L(K,P)]=0$; that is, $P_N=E(Z_{K+1})/E[a(K)]$. This is straightforward to compute from (2.3) and (2.6). See Frees [14] for some alternative definitions of a net level premium. Waters [37] considers the loss at time zero for an endowment policy and remarks on the difficulty of calculating its distribution.

The extension to reserves is similar. Following Bowers et al. [3, Chapter 7], for duration k, define $J=K-k$ and E_k to be the expectation conditional of the event $\{K \geq k\}$. Let

$${}_kL(J,P) = v_{J+1}\, b_{k+J+1} - P \sum_{s=0}^{J} v_s\, a_{k+s} \tag{2.10}$$

be the loss at time k. The reserves are defined to be

$${}_kV = E_k[{}_kL(J,P)]$$

$$= E_k\,[e^{-\delta(J+1)}\, b_{k+J+1}] - P\,E_k\left(\sum_{s=0}^{J} e^{-\delta s}\, a_{k+s}\right). \tag{2.11}$$

Henceforth, when the context is clear, use E for E_k. To calculate the variance associated with the loss function, we have

$$E\,[{}_kL(J,P)]^2 = E\,(e^{-\alpha(J+1)}\,b_{k+J+1}^2) + P^2\,E\left(\sum_{s=0}^{J} v_s\,a_{k+s}\right)$$

$$-\;2P\,E\left(e^{-\delta(J+1)}\,b_{k+J+1}\sum_{s=0}^{J} e^{-(\alpha-\delta)s}\,a_{k+s}\right).$$

Here, the second term on the right-hand side is calculated similarly to (2.7).

3. Single Policy–Autocorrelated Interest Case

The assumption that the interest environment, represented by the sequence $\{\Delta_k\}$, is i.i.d. is a useful modification of the traditional assumption that $\{\Delta_k\}$ is deterministic. This modification permits volatility of interest rates in the model. In this section the results of Section 2 are extended by assuming that $\{\Delta_k\}$ can be represented as a moving average model of order one, that is, $MA(1)$. This model accounts for certain autocorrelation aspects of the sequence $\{\Delta_k\}$ and is particularly tractable in the calculation of insurance functions. I believe this tractability will help actuaries develop the proper intuition concerning the behavior of insurance functions in an autocorrelated environment. Although the $MA(1)$ model is not known to prohibit riskless arbitrage, financial economists have lately shown a willingness to investigate models that cannot be reduced to a martingale. The argument is that when examining the microstructure of investments, returns will follow a martingale *plus* some corrupting influences. It is posited that the corrupting influences account for the observed autocorrelations of returns. Cho and Frees [7] examine one such corrupting influence: the discreteness of prices. Cohen, Maier, Schwartz and Whitcomb [8] give additional background on the microstructure of security returns.

Previous studies that develop insurance functions in an autocorrelated environment include Pollard [27], Panjer and Bellhouse [26], Bellhouse and Panjer [1], Giaccotto [17], and Dhaene [12]. All these studies dealt only with autoregressive (AR) models except Giaccotto [17] and Dhaene [12]. Giaccotto and Dhaene considered ARIMA, autoregressive integrated moving average, models, which, although more general, lack the interpretability of this section.

There is not yet a consensus in the literature on the selection of a particular model to represent $\{\Delta_k\}$. As noted by the Institute of Actuaries' Maturity

Guarantees Working Party [28], the long-term nature of actuaries' concerns may engender model selection criteria substantially different than those of other financial analysts. There is a general, although not unanimous, agreement among financial data analysts that the ARIMA class of models is a good starting point based on the principle of parsimony. Statisticians tend to prefer AR models because simple transforms enable one to analyze a multiple linear regression model with AR errors easily. Probabilists tend to prefer AR models because of the nice duality between continuous and discrete time stochastic process models; compare Shiu and Beekman [33]. However, there is a duality between AR and MA models described in, for example, Miller and Wichern [24]. An important corollary of this result is that it is often difficult, if not impossible, to distinguish an AR(1) model with a small lag one autocorrelation from a MA(1) model. Some model extensions are discussed briefly at the end of this section.

Now consider the MA(1) model,

$$\Delta_k = \mu + \epsilon_k - \theta\, \epsilon_{k-1},\ k = 1, 2, \ldots \tag{3.1}$$

where $\{\epsilon_k\}_{k=0}^{\infty}$ is a mean zero, i.i.d. sequence with variance σ^2. The case $\theta=0$ reduces to the i.i.d. structure discussed in Section 2. Often θ is restricted to be in the interval $(-1,1)$ so that the model is invertible; that is, it can be expressed as an autoregressive model. The following simple proposition is a driving force behind this section. To simplify notation, define $M(t)=Ee^{t\epsilon}$ to be the moment generating function of ϵ, which is assumed to exist throughout the paper.

Proposition 1

Consider the MA(1) sequence defined in (3.1) and recall

$$v_k = \exp\left(-\sum_{s=1}^{k} \Delta_s\right).$$

Then, for $k=1, 2, \ldots$

$$E\,(v_k) = C_1\, e^{-k\delta_1}, \tag{3.2}$$

where $\delta_1=\mu-\log M(\theta-1)$ and $C_1=M(\theta)M(-1)/M(\theta-1)$.

Note, in the special case of $\theta=0$, that $\delta_1=\delta$ and $C_1=1$. The above proposition is useful because we can easily calculate net single premiums using the law of iterated expectations. As in (2.3) and (2.6), we have

$$E\,(Z_{K+1}) = C_1\, E\,[e^{-\delta_1(K+1)}\, b_{K+1}] \tag{3.3}$$

and

$$E\,[a(K)] = C_1\, E\left(\sum_{s=0}^{K} e^{-\delta_1 s}\, a_s\right). \tag{3.4}$$

Thus, even when $\theta \neq 0$ we have

$$P_N = E\,[e^{-\delta_1(K+1)}\, b_{k+1}]\;/\;E\left(\sum_{s=0}^{K} e^{-\delta_1 s}\, a_s\right).$$

To be precise, the above equation and (3.4) are approximate equalities. This is pointed out by Dufresne is the subsequent discussion. Net premiums are calculated as in Section 2, except we use δ_1 in lieu of δ. Thus, it is of interest to compare δ_1 and δ, and this can be done in the context of the following example.

Example 3.1

Use the conditions of Example 1.1 and assume $\epsilon \sim N(0,\sigma^2)$. In the case $\theta = 0$, from Section 2 we have $\delta = \mu - \sigma^2/2$. With $\theta \neq 0$ and by the moment generating function properties of the normal distribution,

$$M(\theta - 1) = \exp[\sigma^2(1 - \theta)^2/2].$$

Thus, $\delta_1 = \mu - \sigma^2(1-\theta)^2/2$. The increase in the force of interest by assuming an autocorrelated interest environment is

$$\delta_1 - \delta = (1 - \theta/2)\,\theta\sigma^2.$$

To interpret this, recall that the lag 1 autocorrelation for the MA(1) model defined in (3.1) is $\rho_1 = -\theta/(1+\theta^2)$. Thus, for a positively autocorrelated environment with $\rho_1 > 0$, this yields $\theta < 0$ and $\delta_1 < \delta$. This indicates that the actuary should use a higher interest assumption than in the corresponding i.i.d. environment.‡

Second moment ideas are similar but more complex. The main ideas are summarized below.

Proposition 2

Under the assumptions of Proposition 1, for $k = 1, 2, \ldots$

$$E\,(v_k^2) = C_2\, e^{-k\alpha_1}, \tag{3.5}$$

and for $s < r$,

$$E\,(v_s v_r) = C_3\, e^{-s\alpha_1}\, e^{-(r-s)\delta_1} \tag{3.6}$$

where $\alpha_1 = 2\mu - \log M(2\theta - 2)$,

$$C_2 = M(2\theta)M(-2)/M(2\theta - 2),$$

and

$$C_3 = M(\theta - 2)M(2\theta)M(-1)/\{M(\theta - 1)M(2\theta - 2)\}.$$

As before, if $\theta = 0$, then $\alpha_1 = \alpha$ and $C_2 = C_3 = 1$.

Reserve considerations are similar, but, unlike net premiums, the constants do not vanish. As in (2.11), from Proposition 1, one can check that

$$\begin{aligned} {}_kV_1 &= E[{}_kL(J,P)] \\ &= C_1 \left\{ E\left(e^{-\delta_1(J+1)} b_{k+J+1}\right) - P\, E\left(\sum_{s=0}^{J} e^{-\delta_1 s} a_{k+s}\right)\right\}. \end{aligned} \tag{3.7}$$

Similar to (2.10), we have

$$\begin{aligned} E_k[L(J,P)]^2 = E\,(v_{J+1}\, b_{K+J+1})^2 + P^2 E\left(\sum_{s=0}^{J} v_s\, a_{k+s}\right)^2 \\ -2P\, E\left(v_{J+1}\, b_{k+J+1} \sum_{s=0}^{J} v_s\, a_{k+s}\right) \end{aligned} \tag{3.8}$$

where

$$E(v_{J+1}\, b_{k+J+1})^2 = C_2\, E[e^{-\alpha_1(J+1)}\, b^2_{k+J+1}], \tag{3.9}$$

$$\begin{aligned} E\left(\sum_{s=0}^{J} v_s\, a_{k+s}\right)^2 = E\Bigg\{ C_2 \sum_{s=0}^{J} e^{-\alpha_1 s}\, a^2_{k+s} \\ + 2\, C_3 \sum_{r=1}^{J} \sum_{s=0}^{r-1} e^{-\delta_1 s}\, e^{-(\alpha_1-\delta_1)r}\, a_{k+r}\, a_{k+s}\Bigg\} \end{aligned} \tag{3.10}$$

and

$$E\left(v_{J+1}\, b_{k+J+1} \sum_{s=0}^{J} v_s\, a_{k+s}\right) = C_3\, E\left\{ b_{k+J+1} \sum_{s=0}^{J} e^{-\alpha_1 s}\, e^{-(J+1-s)\delta_1}\, a_{k+s}\right\} \tag{3.11}$$

A special case is a fully discrete whole life policy issued to (x) at duration k. In this case, from (2.9),

$${}_kV_x = C_1\{A_{x+k} - P_x\, \ddot{a}_{x+k}\} \tag{3.12}$$

and

$$E({}_kL)^2 = C_2^{\alpha_1} A_{x+k}$$
$$+ P_x^2 \{C_2^{\alpha_1} \ddot{a}_{x+k} + 2\, C_3(\ddot{a}_{x+k} - {}^{\alpha_1}\ddot{a}_{x+k})/[1 - e^{-(\alpha_1 - \delta_1)}]\}$$
$$- 2\, C_3\, P_x(A_{x+k} - {}^{\alpha_1}A_{x+k})/[1 - e^{-(\alpha_1-\delta_1)}]. \tag{3.13}$$

The notation ${}^{\alpha_1}\ddot{a}$ means use the force of interest α_1 in calculating $\ddot{a}$ and similarly for A. If no force of interest is specified, use δ_1. Some numerical examples of (3.12) and (3.13) appear in Section 4.

Thus, it is of interest to establish relationships between the pairs (C_1, δ_1) and $(1,\delta)$ under general conditions. Naturally, for specific distributions such as in Example 3.1, C_1 and δ_1 can be computed exactly. More generally, we have the following.

Proposition 3

Consider the $MA(1)$ in (3.1) and assume $\sigma^2>0$. Then, if $-1\le\theta<0$,

$$\delta_1 < \delta,\ \alpha_1 < \alpha \quad \text{and} \quad C_i < 1, \quad i = 1, 2, 3. \tag{3.14}$$

If $0<\theta\le1$, then (3.14) holds with the inequalities reversed. Further, if $\theta=0$, then the inequalities in (3.14) are equalities.

The interpretation is that in the case of reserves we have offsetting factors. For example, in a positively autocorrelated interest environment, $\theta<0$, and thus $C_1<1$ and $\delta_1<\delta$. Now, in general, a lower interest factor (δ_1) means that the reserve is higher. However, this is slightly offset in the calculation of ${}_kV_1$ in (3.8), because we multiply by C_1, a factor less than one.

We close this section with the following.

Example 3.2. Bond Index Returns

Consider the annual returns of the Salomon Brothers Bond Index for the period 1926–85, inclusive. The data can be found in Ibbotson and Sinquefield [19]. For each return R_i, $i=1, \ldots, 60$, let $D_i=\log(1+R_i)$ be the corresponding force of interest. For this series, the average force of interest is $\bar{D}=0.04676$, and the standard deviation is $s_D=0.07363$. Using maximum likelihood (with a normal model), the estimated force of interest is

$$\hat{\delta} = \bar{D} - s_D^2/2 = 0.04405,$$

and the second moment parameter estimate is

$$\hat{\alpha} = 2(\bar{D} - s_D^2) = 0.08268.$$

An examination of the histogram in Figure 2 indicates that the assumption of normality may be acceptable, although one or two observations could be considered to be too far away to be generated by a normal distribution.

FIGURE 2

HISTOGRAM OF THE SALOMON BROTHERS BOND INDEX FOR 1926–1985
(DATA ARE IN NATURAL LOGARITHMS)

N = 60

Midpoint	Count	
-0.10	1	*
-0.05	6	******
0.00	14	**************
0.05	24	************************
0.10	8	********
0.15	5	*****
0.20	0	
0.25	1	*
0.30	0	
0.35	1	*

The time series plot in Figure 3 shows the temporal aspects of these data. Some summary statistics are $r_1 = 0.144$ and $r_2 = 0.078$, the first- and second-order autocorrelations, respectively. After considerable examination of the data, it was decided that the MA(1) and AR(1) models were the best fitting models of the ARIMA class. For the MA(1) model, the estimated parameters were

$$\hat{\mu} = 0.04731,\ \hat{\theta} = -0.1465, \text{ and } \hat{\sigma} = 0.07346.$$

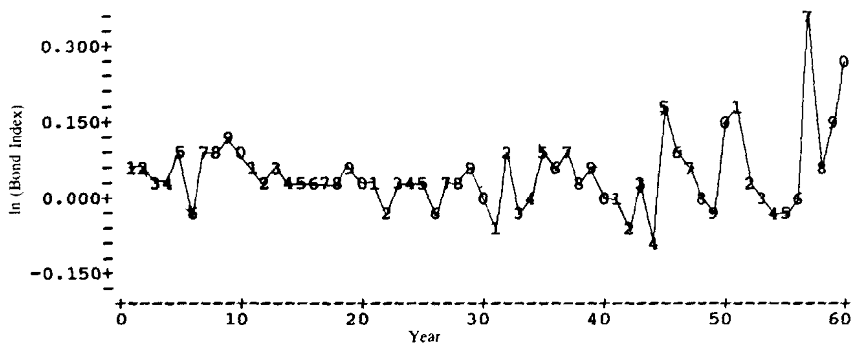
Hence, under a normal model,

$$\hat{\delta}_1 = 0.04376 \text{ and } \hat{\alpha}_1 = 0.08043.$$

While the Box-Pierce statistic (compare Miller and Wichern [24, page 391, equation (10.27)]) indicated model adequacy, in each case the autocorrelation parameter estimates were only about one standard deviation away from zero. This low significance was somewhat disappointing, and a careful inspection of the time series in Figure 3 provides some insights. Note that, although the series is stationary in the mean, it appears to be more volatile in the later years than in the early years. Indeed, the largest observation is in 1982, which corresponds to an annual return of 42.5 percent! To get an idea of the effect on the model of this large outlying observation, I arbitrarily truncated the observation to 22.5 percent (still 2.5 standard deviations above the average) and reran the MA(1) model. The absolute t-statistic for $\hat{\theta}$ jumped from 1.07 to 1.60, an increase of 60 percent. This illustrates the large effect of one observation on the model fitting exercise. It also reminds us that the task of modeling interest rates for valuation purposes is by no means complete. Models that allow the volatility parameter to change over time may be a useful next step. See Tsay [36] for a recent overview of this developing methodology.

FIGURE 3

TIME SERIES PLOT OF THE SALOMON BROTHERS BOND INDEX FOR 1926–1985 (DATA ARE IN NATURAL LOGARITHMS)



II. SOLVENCY VALUATION

In this part I address various aspects of how to value a life insurance company when solvency considerations drive the choice of valuation techniques. In the development, Section 4 is a direct extension of Section 3 to the case of several policies, with two important differences. First, a more general interest environment is considered, and second, the notion of a vector of cash flows, in lieu of discounting everything back to an arbitrary valuation date, is actively used. I interpret Section 4 to be a discussion of valuation models in which all assets are valued at market. This type of valuation is useful in considering the liquidation or sale of a block of business, that is, the liabilities and assets of a group of contracts, of a company or a division of a company. In Section 5 begins the real discussion of matching asset and liability cash flows. Here, certain portions of the asset portfolio are not valued at market. Bringing assets explicitly into the picture allows us to give a financial interpretation to the discounting mechanism. The introduction of assets also permits us to discuss various central limit theorems for surplus in Section 6.

4. Liabilities for a Block of Business

Now consider the case in which several policies share a common interest environment. The policies constitute a block of business that is supported by the same pool of assets. As such, the policies are not necessarily identical and may be of different duration, age at issue, benefit structure, etc. Conditional on the interest environment, the events of loss are assumed to be mutually independent. As noted by Waters [38], if a common interest environment is assumed for the policies, central limit theorems to approximate the distribution of the sum of losses are no longer available. I present two useful alternatives, both available for any sequence $\{\Delta_k\}$. First, I propose a technique for calculating the variance of the sum by using expected cash flows. Second, I establish an approximation to the distribution of the sum by the distribution of a simpler random variable. These results quantify, at least in one sense, the folklore opinion that interest variability dominates mortality variability. Another useful corollary of the second result is that, under the i.i.d. assumption, the entire distribution of the sum can be recursively calculated. This idea seems to be new even for certain annuities.

Specifically, assume there are n policies in this block of business. For the i-th policy, the age at issue is x_i, the duration is k_i, and the random time until loss is J_i. Assume $J_1, \ldots, J_n$ are independent. The benefit structure is $\{b_{i,s}\}_{s=1}^{\infty}$ and the present value of future benefits is

$$Z_i(J_i) = v_{J_i+1}\, b_{i,k_i+J_i+1}.$$

Premiums payable at the beginning of each year are $\{P_i a_{i,s}\}$ and the present value of such premiums is

$$P_i\, a_i(J_i) = P_i \sum_{s=0}^{J_i} v_s\, a_{i,k_i+s}.$$

Thus, each policy incurs the random loss

$$L_i\,(J_i,P_i) = Z_i\,(J_i) - P_i\,a_i\,(J_i) \tag{4.1}$$

and the sum of such losses is denoted by

$$S_L = \sum_{i=1}^{n} L_i\,(J_i,P_i). \tag{4.2}$$

Since

$$E(S_L) = \sum_{i=1}^{n} E[L_i\,(J_i,P)],$$

it is straightforward to calculate the expected loss by using, for example, (3.7) in the MA(1) environment.

In order to apply curve fitting techniques to the distribution, it is crucial to also calculate $\mathrm{Var}(S_L)$. For example, Tchebycheff's inequality guarantees that the probability that S_L is less than $E(S_L)+3\sqrt{\mathrm{Var}\,S_L}$ is a conservative 11.1 percent. With a normal approximation of S_L, this probability is close to 0.0001. In an actuarial context, Waters [38] provides a discussion of curve fitting using the Pearson family of curves when higher moments of S_L are known. In the present context, computation of $\mathrm{Var}(S_L)$ is tractable via examination of projected cash flows arising from liabilities. To this end, consider the i-th policy with flow of cash at time point $s+1$ defined by

$$F_{i,s+1} = \begin{cases} -P_i\,a_{i,k_i+s+1} & \text{if } J_i > s \\ b_{i,k_i+s+1} & \text{if } J_i = s \\ 0 & \text{if } J_i < s. \end{cases} \tag{4.3}$$

With the convention $F_{i,0} = -P_i\,a_{i,k_i}$, the loss in (2.10) or (4.1) can be expressed as the sum of discounted net cash flows, that is,

$$L_i\,(J_i,\,P_i) = \sum_{s=0}^{\infty} v_s\,F_{i,s} \tag{4.4}$$

Thus, S_L is a sum of discounted benefit payments in excess of premium income.

The projected (expected) cash flow at time $s+1$ is

$$f_{i,s+1} = E\,(F_{i,s+1}) = b_{i,k_i+s+1}\ {}_{s|}q_{x_i+k_i} - P_i\, a_{i,k_i+s+1}\ {}_{s+1}p_{x_i+k_i}.$$

With the convention $f_{i,0} = -Pa_{i,k_i}$, the reserve is

$$E\,[L_i(J_i, P_i)] = \sum_{s=0}^{\infty} E\,(v_s\, f_{i,s}).$$

Now consider the block of business. Define

$$F_s = \sum_{i=1}^{n} F_{i,s}$$

to be the total random cash flow at time point s and let $f_s = E(F_s)$ be its projected (expected) value. Define $\mathcal{I}$ to be the collection of interest information generated by $\{\Delta_k\}$. Note that in this section I no longer require $\{\Delta_k\}$ to be i.i.d. or even stationary. The calculation of $E(S_L^2)$ is summarized in the following.

Proposition 4

Assume $J_1, \ldots, J_n$ to be independent of $\mathcal{I}$. Then,

$$E\,(S_L^2) = \sum_{i=1}^{n} E[\mathrm{Var}(L_i|\mathcal{I})] + E\left(\sum_{s=0}^{\infty} v_s\, f_s\right)^2 \tag{4.5}$$

where

$$E\,[\mathrm{Var}(L_i|\mathcal{I})] = E[L_i\,(J_i, P_i)^2] - E\left(\sum_{s=0}^{\infty} v_s\, f_{i,s}\right)^2. \tag{4.6}$$

Note that computation of the second term in (4.5),

$$E\left(\sum_{s=0}^{\infty} v_s f_s\right)^2 = \sum_{s=0}^{\infty} E(v_s^2)\, f_s^2 + 2\sum_{r<s} E(v_r\, v_s)\, f_r\, f_s \tag{4.7}$$

is straightforward in, for example, an MA(1) environment from Proposition 2.

Example 4.1. Block of Whole Life Policies

To illustrate the application of Proposition 4, consider a block of whole life business. For simplicity, policies are categorized into three groups of size N so the total size is $n=3N$. Assume, for each category, that ages at issue are $x=30$, 30, 40 and durations are $k=5$, 10, 5, respectively. Also assume the MA(1) environment of Example 3.2, and thus $\delta_1=0.04376$ and $\alpha_1=0.08043$. The mortality decrements are the 1979–81 U.S. Male Life Tables.

The reserve, or $E(S_L)$, calculation follows directly from (3.12). Thus,

$$E(S_L) = N\{C_1[(A_{35} - P_{30}\,\ddot{a}_{35}) + (A_{40} - P_{30}\,\ddot{a}_{40}) + (A_{45} - P_{40}\,\ddot{a}_{45})]\}$$
$$= N(0.18458).$$

Here, for C_1, I used a normal approximation in Example 3.1, which resulted in

$$C_1 = \exp(\hat{\sigma}^2\,\hat{\theta}^2/2)\exp(\hat{\sigma}^2/2)/\exp[(\hat{\sigma}^2(\hat{\theta} - 1)^2/2] \cong 0.99919.$$

Similarly, it turns out that $C_2 \cong 0.99677$ and $C_3 \cong 0.99757$. By using (3.13), similar calculations establish that

$$\sum_{i=1}^{n} E\,(L_i)^2 = N\,(0.14918).$$

From (4.4), the projected cash flow at time $s+1$ is

$$f_{i,s+1} = {}_{s|}q_{x_i+k_i} - P_i\ {}_{s+1}p_{x_i+k_i}.$$

Thus, as in (4.7), straightforward calculations yielded

$$\sum_{i=1}^{n} E[\mathrm{Var}(L_i|\mathcal{F})] = \sum_{i=1}^{n}\left\{E\left(L_i^2\right) - E\left(\sum_{s=0}^{\infty} v_s\, f_{i,s}\right)^2\right\}$$
$$= N(0.14918) - N(0.01538) = N(0.13380).$$

Finally, with (4.7), further tedious calculations establish

$$E(S_L^2) = N(0.13380) + N^2(0.04268).$$

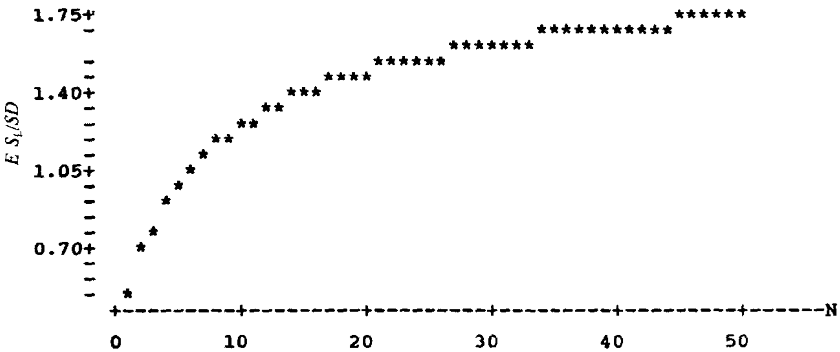
Thus,

$$\mathrm{Var}\,S_L = N^2(0.00861) + N(0.13380).$$

In Figure 4 is a graph of $E[S_L/\sqrt{\mathrm{Var}(S_L)}]$ compared to N. Because the limiting value of the ratio does not tend to zero, this is one indication that the usual limit laws for sums of policies do not hold.‡

FIGURE 4

PLOT OF THE EXPECTED LIABILITIES AS A PROPORTION OF THE CORRESPONDING STANDARD DEVIATION COMPARED TO THE SAMPLE SIZE IN EXAMPLE 4.1 (THE LIMITING VALUE IS 1.989)



Because the central limit theorem is no longer available, it is desirable to have other approximations for the distribution of S_L. I assume that the block of business is homogeneous. Thus, for simplicity, the following result is stated only for identical policies (losses).

Proposition 5

Assume $J_1, \ldots, J_n$ to be independent of $\mathcal{I}$. Assume $L_i(J,P)=L(J,P)$, defined in (2.10), are identical loss functions and define $Y=E[L(J,P)|\mathcal{I}]$. If $E[L(J,P)^2]<\infty$, then

$$\underset{n \to \infty}{\text{limit in distribution}} \ (S_L/n) = Y.$$

In the Appendix, I actually establish the order of the limiting approximation. It is important that the result holds by using any random sequence $\{\Delta_k\}$. This includes not only the autocorrelated sequences used as examples but also applications to interest rate scenarios. Proposition 5 is useful because often the distribution of S_L is complex and can be approximated by the distribution of nY, which is simpler to compute. For any sequence $\{\Delta_k\}$ and loss function L, the distribution of Y can be approximated via simulation. As demonstrated below, in certain important special cases, the distribution of Y can be computed exactly.

Example 4.2. n-*year Annuities*

Consider a modified immediate life annuity with C dollars payable at the end of each year, up to n years, and F dollars payable at the end of n years where payments are made if the annuitant (x) is alive. Assuming we are at contract initiation and the policy is paid up, from (2.2) the loss associated with the policy is

$$L = C \sum_{k=1}^{n-1} v_k \, \mathrm{I}(K \geq k) + F \, v_n \, \mathrm{I}(K \geq n).$$

Now, define

$$Y_{x:\overline{n|}} = Y = E(L|\mathcal{I}) = C \sum_{k=1}^{n-1} v_k \, {}_kp_x + F \, v_n \, {}_np_x.$$

Assuming that $\{\Delta_k\}$ is i.i.d., we have

$$\begin{aligned} Y_{x:\overline{n|}} &= C \sum_{k=1}^{n-1} \prod_{s=1}^{k} [(v_s/v_{s-1})({}_sp_x/{}_{s-1}p_x)] + F \, v_n \, {}_np_x \\ &= C \sum_{k=1}^{n-1} \prod_{s=1}^{k} [\exp(-\Delta_s) \, p_{x+s-1}] + F \exp[-(\Delta_1 + \ldots + \Delta_n)] \, {}_np_x \\ &= \exp(-\Delta_1) p_x \left\{ C + C \sum_{k=1}^{n-2} \prod_{s=1}^{k} \exp(-\Delta_{s+1}) \, p_{x+s} \right. \\ &\qquad \left. + F \exp[-(\Delta_2 + \ldots + \Delta_n)] \, {}_{n-1}p_{x+1} \right\} \\ &= \exp(-\Delta_1) \, p_x \, (C + Y^*_{x+1:\overline{n-1|}}), \end{aligned} \tag{4.8}$$

where $Y^*_{x+1:\overline{n-1|}}$ is independent Δ_1 and has the same distribution as $Y_{x+1:\overline{n-1|}}$. This suggests an efficient way to *recursively* compute the distribution of $Y_{x:\overline{n|}}$. Let $G_{x,n}$ and $g_{x,n}$ be the distribution function and probability density function, respectively, of $Y_{x:\overline{n|}}$. Then from (4.8), we have

$$G_{x,n}(y) = \int G_{x+1,n-1}(Fy/u - C) \, g_{x,1}(u) \, du \tag{4.9}$$

and

$$g_{x,n}(y) = \int (F/u)\, g_{x+1,n-1}(Fy/u - C)\, g_{x,1}(u)\, du. \tag{4.10}$$

As in Example 1.1, the normal distribution is the important benchmark choice of distributions, and thus we take $\Delta \sim N(\mu,\sigma^2)$. In this case, to start either recursion (4.9) or (4.10), we have

$$Y_{x:\overline{1|}} = F \exp(-\Delta_1)\, p_x \sim \text{lognormal}[-\mu + \log(F\, p_x),\, \sigma^2]$$

and thus

$$g_{x,1}(y) = (2\pi\, y^2\, \sigma^2)^{-1/2} \exp\{-[\log y - (-\mu + \log F\, p_x)]^2/(2\sigma^2)\}.$$

Equations (4.9) and (4.10) become greatly simplified in the case of an annuity certain in lieu of a life annuity. For the annuity certain case, take $p_x = 1$ for all x and drop the x variable in Equations (4.9) and (4.10). In this case, the annuity reduces to an ordinary bond without call provisions, and indeed, the notation C is for coupons, while F is for face value. To get an idea of the distribution of the present value of a bond, consider a \$1000 $(=F)$ 10-year bond with \$50 $(=C)$ coupons payable annually. Assume $\Delta \sim N(0.05,\sigma^2)$ where $\sigma = 0.01$, 0.05, 0.10 and recall that $\delta = 0.05 - \sigma^2/2$ is the mean force of interest. In Figure 5 are graphs that emphasize the effect of the volatility parameter σ on the distribution of $Y_{\overline{10|}}$. The distribution was approximated by simulation techniques. See Kahn [21] for an early discussion of the use of simulation techniques to value insurance benefits under a stochastic interest environment. Note that for each graph, the means are close (they turn out to be 1005, 972 and 961) while the standard deviations are dramatically different (260, 126 and 25). Extensive literature in financial economics provides *pricing* strategies for a bond, but the notion of the distribution of the present value of a bond has not been emphasized in that literature. The following section includes some interpretations of the sequence $\{\Delta_k\}$ that are more traditional in financial economics.

5. Matching Stochastic Assets and Liabilities

There is a widespread belief that the valuation actuary must examine the portfolio of assets that support the liabilities of a block of business (compare Tullis and Polkinghorn [37, Chapter 8]). Although this is particularly true for interest-sensitive products, similar arguments can be made for all lines of business. In this section I consider a simple portfolio of assets that support

FIGURE 5

COMPARISON OF DISTRIBUTIONS FOR SIGMA = 0.1, 0.05 AND 0.01, RESPECTIVELY. EACH DOTPLOT IS GENERATED FROM 2,000 SIMULATIONS. THE UNITS ON THE AXIS WERE KEPT THE SAME FOR COMPARISON PURPOSES.

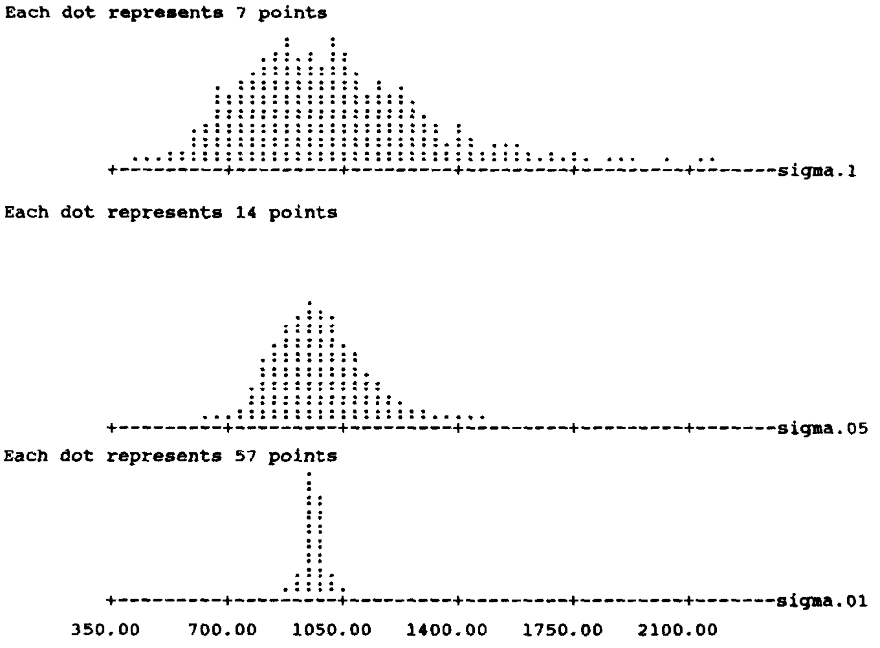
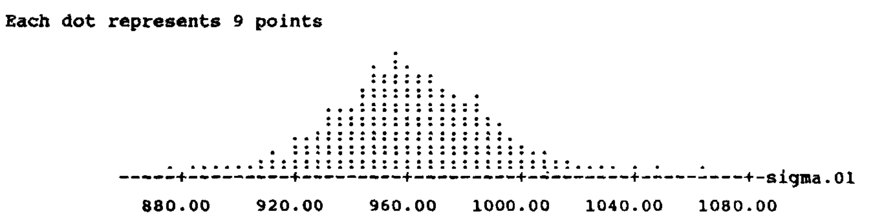


FIGURE 6

MORE DETAILED VERSION OF THE DOTPLOT FOR THE SIGMA = 0.01 CASE



the liabilities of the block of business described in Section 4. The portfolio comprises two parts, those assets subject to net reinvestment risks and those assets subject to asset price as well as net reinvestment risk. The assets subject to only net reinvestment risk can be thought of as high-quality bonds without call provisions. This asset subportfolio yields an income stream of c_k at time k. Here, use c for coupon income associated with assets even though the asset stream consists of coupons plus bond maturities. Also, make the simplifying assumption that these assets are default-free. All other assets are assumed to be valued on a market basis with current asset value A_0. Assume that the fund earns interest governed by the stochastic environment $\{\Delta_k\}$. Further, make the simplifying assumption that the net reinvestment rate associated with amortized securities is also governed by $\{\Delta_k\}$. With these assumptions, the present value of assets is

$$S_A = A_0 + \sum_{k=1}^{\infty} v_k\, c_k. \tag{5.1}$$

Define the surplus as the excess of assets over liabilities, that is,

$$S = S_A - S_L \tag{5.2}$$

where S_L is defined in (4.2).

In this paper, vectors of asset- and liability-generated cash flows are compared and summarized by being discounted back to an arbitrary origin date. Assets are determined through efficient markets, and hence their worth is assumed to be known at the valuation date. The secondary market for liabilities is relatively inefficient, and hence the vector of liabilities is modeled stochastically. (A recent exception is the sale of a block of Prudential's policyholder's loans; compare Shante et al. [31].) The discounting factors can be interpreted in the more general framework of the term structure of interest rates, as follows (see also Bierwag [2] for additional background information).

At time 0, the valuation date, let $h_0(0,t)$ be the t-period spot rate. That is, the price of a t-year \$1 pure discount bond is $\exp[-th_0(0,t)]$. Equivalently, we think of a fund of \$1 at time 0 being worth $\exp[th_0(0,t)]$ in t years. The set of spot rates, $h_0(0,1)$, $h_0(0,2)$, ... is known as the term structure of interest rates. It is well-known, disallowing arbitrage opportunities, that the term structure also determines $h_0(s,t)$. Disallowing arbitrage opportunities is equivalent to all investment strategies yielding equivalent returns for all s,t. Interpret $\exp[(t-s)h_0(s,t)]$ as the value at time t of an initial fund of \$1 invested at time s. Here, the valuation is done at time 0, and at this time, the term structure is considered known. The term structure is generally calculated from the yield curve which, in turn, is calculated via least squares regression. See Siegel and Nelson [25] for a recent overview of these techniques.

At subsequent valuation dates, $j=1, 2, \ldots$, asset and liability values are subject to entirely new term structures $[h_j(0,1), h_j(0,2), \ldots]$. A variety of models have been proposed for developing relationships between the current and subsequent term structures (compare Bierwag [2, Chapter 11]). Perhaps the most well-known model is from the immunization study of Fisher and Weil [13]. They posited that the instantaneous movement of the term structure does not depend on the spot period, that is, $h_j(0,t)=h_{j-1}(0,t)+\delta_j$, where δ_j does not depend on t. This model for changing term structures may be appropriate for some valuations. However, for solvency valuations, I adopt the principle that required surplus should not depend on future investment strategies except within broad classes of assets. In this paper only two classes of assets, based on asset price risk characteristics, are used, although this classification scheme can and should be eventually refined. For many valuations, it is appropriate to recognize term structure and other effects. However, if one considers a statutory valuation, it seems that uniform standards with broad classes of assets should apply. While term effects of the current portfolio can be somewhat recognized with the two classes of assets, recognizing term effects of reinvestments made in the future would not allow enough structure in the model to hope that one valuation actuary could independently affirm the work of another. Thus, further restrictions are required. In this paper the one-period spot rates, $h_s(s,s+1)=\Delta_{s+1s}$, $s=0, 1, 2, \ldots$ are used for discounting asset and liability flows. Use of these rates can be justified under a risk-neutral expectations model in financial economics (see Cox, Ingersoll, and Ross [9, Appendix]). This model admits an $AR(1)$ model, among others, as a model for rates.

The key point is that, in a stochastic interest environment, it is the distribution of S and not S_L that is important. The notion of matching assets and liabilities is the main point of the immunization literature. See Boyle [5], [6] and Shiu [32] for some contributions to this literature from an actuarial perspective. The primary innovation of this paper is that the liabilities are considered to be stochastic in lieu of deterministic. The main result of this section is summarized in Proposition 6 below. As in Section 4, note that there is no specific assumption regarding the distribution of $\{\Delta_k\}$ here.

Proposition 6

Consider the surplus S defined in (5.2) and assume $E(S^2)$ is finite. Then

$$\mathrm{Var}(S) = \mathrm{Var}(S_L) + \mathrm{Var}(S_A) - 2\,\mathrm{Cov}[E(S_L|\mathcal{I}), S_A]. \tag{5.3}$$

Further, Var S is minimized by choosing $\{c_k\}_{k=1}^{\infty}$ so that

$$E(S_L|\mathcal{I}) = S_A + S_0 \tag{5.4}$$

where S_0, the initial surplus, is an arbitrary constant. In this case,

$$\mathrm{Var}\,(S) = \mathrm{Var}(S_L) - \mathrm{Var}(S_A). \tag{5.5}$$

The relation (5.4) suggests a new index of matching, $M = \mathrm{Var}[S_A - E(S_L|\mathcal{I})]$. When $M=0$, liabilities are "fully matched" on a projected basis. The condition that $M=0$ is more restrictive than the usual notion of matching duration moments. However, the condition also yields more informative properties; see Section 6. A sufficient condition for (5.4) is that for each period the asset income, c_k, equals f_k, the expected benefit outflow, conditional on the interest information. It is not hard to check that this is also a necessary condition under mild assumptions on the distribution of $\{\Delta_k\}$, for example, $\{\Delta_k\}$ is i.i.d. normal. In the case of full matching, Var(S) is easy to compute since, by (4.5),

$$\mathrm{Var}(S) = \sum_{i=1}^{n} E[\mathrm{Var}(L_i|\mathcal{I})].$$

In the special case of i.i.d. policies, we see that the variance of S is proportional to n in lieu of n^2. This suggests that a central limit theorem approximation may be available. Further, it would suggest a way to calculate the asymptotic distribution of S even when full matching is not achieved. This line of thought is pursued further in Section 6. From an investment manager's perspective, this asset position may be neither feasible nor desirable (compare, Leibowitz and Weinberger [22]).

The above model for the variance of surplus is simplistic from both an asset and a liability perspective. Further refinements of the liabilities, or loss random variables, are discussed in the following section, so here I discuss some of the drawbacks from the asset side. As mentioned above, the risk of asset default is ignored in the above analysis. This could be addressed by creating special subcategories of assets and using estimates of default probabilities for each subcategory, the probabilities presumably being interest-sensitive. A deeper problem is that all interest-sensitive assets are assumed to be valued at market, and hence all the stochastic characteristics of the asset are summarized by a single quantity at the valuation date. The main advantage of this approach is that it relies on efficient asset markets for pricing and thus avoids a host of problems that arise in developing consistent asset-pricing models. The main disadvantage of this approach is that it treats all interest-sensitive products equally. For example, once market values are established, there is no real recognition of the many different risk characteristics of a high-quality bond with a mild call provision as compared to a mortgage-backed security, which is heavily influenced by prevailing interest rates. Presumably, future enhancement of valuation models will involve projecting asset vectors of cash streams for subcategories of assets, conditional on an interest environment. The sum of these projected vectors over asset subcategories would be compared with the corresponding projected vector liabilities. The difference could then be discounted back to an arbitrary valuation date and the resulting random variable summarized via means, variances, or percentiles.

A third drawback is that examination of only the random variable S ignores the event of a shortfall of cash flows. Because of the experience suffered by the savings and loan industry in the U.S. in the 1980s, one can argue that consideration of this event is not merely an academic exercise. The event of shortfall could reasonably be ignored if the valuation is for a line of business, and it is assumed that other lines of business have available funds that could be lent in the event of a shortfall. Analogously, at a company level, it may be presumed that a parent company or other lender would be available to provide surplus relief on a short-term basis. Alternatively, the probability of shortfall could be quantified by using classical risk theory ideas, as follows.

Formally define the event of shortfall of cash flows to occur if surplus at time k is less than some predetermined threshold level, say T_k, for each k. To define surplus at time k, let $i_k = \exp(\Delta_k) - 1 = v_{k-1}/v_k - 1$ be the random interest rate for year k. Interpret $1+i_k$ to be the value at time k of \$1 invested at time $k-1$. Surplus at time k, S_k, is defined recursively by

$$S_k = S_{k-1}(1 + i_k) + c_k - \sum_{i=1}^{n} F_{i,k},\ k = 1, 2, \ldots$$

where

$$S_0 = A_0 - \sum_{i=1}^{n} F_{i,0}.$$

With this notation, the probability of shortfall is

$$1 - P\,(S_k \geq T_k, \text{ for each } k = 0, 1, 2, \ldots).$$

This, of course, is just the individual model version of the classical ruin problem in risk theory; compare Bowers et al. [3, Chapter 12]. Collective model ruin problems with stochastic interest have been discussed by Schnieper [29].

6. Central Limit Theorems for Surplus

In this section, limit approximations for surplus suggested in Section 5 are more fully developed. To simplify the discussion, only the fully matched case is presented explicitly, although some extensions to the general case are indicated. To extend the arena of potential applications, this section considers both the multidecrement model and the situation in which loss random variables may depend on the interest environment. As in Sections 4 and 5, no particular assumptions about the interest environment, such as i.i.d., are made in this section.

Specifically, there are two important ways in which the interest environment, $\mathcal{I}$, can affect each loss random variable. First, cash flows, either through the benefit amount or premium payment, may be determined by $\mathcal{I}$ while the time of loss random variable remains unaffected. Examples of this are the variable annuity or fully variable life insurance policies described in, for example, Bowers et al. [3, pp. 465–67]. Second, the time of loss random variable may be affected by $\mathcal{I}$. An example of this is a whole life policy with the lapse rate influenced by $\mathcal{I}$. To incorporate the latter example in the analysis, I consider the multidecrement model; compare Bowers et al. [3, Chapter 9]. For the i-th policy, let $J_i^{(j)}$ be the cause of loss due to the j-th cause, $j = 1, \ldots, m$, where m is a fixed, known number. Define $J_i = min\,[J_i^{(1)}, \ldots, J_i^{(m)}]$ to be the random time of policy cessation. Let $b_{i,s}^{(j)}$ be the benefit amount payable for loss due to the j-th cause, $i = 1, \ldots, m$. Similar to (4.3), the flow of cash at time point $s+1$ is defined by

$$F_{i,s+1}(\mathcal{I}) = \begin{cases} -P_i\, a_{i,k_i+s+1}\,(\mathcal{I}) & \text{if } J_i\,(\mathcal{I}) > s \\ b_{i,k_i+s+1}^{(j)}\,(\mathcal{I}) & \text{if } J_i^{(j)}\,(\mathcal{I}) = s,\ j = 1, \ldots, m \\ 0 & \text{if } J_i\,(\mathcal{I}) < s. \end{cases} \quad (6.1)$$

Here, the notation $(\mathcal{I})$ is added to emphasize the fact that *a, b, J,* and *F* may all depend on the interest environment $\mathcal{I}$. The argument $(\mathcal{I})$ is henceforth omitted in *F* to simplify notation.

With $\{F_{i,s}\}$ as in (6.1), define L_i as in (4.4). Let

$$f_{i,s} = E(F_{i,s}|\mathcal{I})$$

be the projected cash flow for a given interest environment $\mathcal{I}$. Let S_{FM} be the surplus arising from *n* homogeneous policies under full matching, that is,

$$S_{FM} = \sum_{i=1}^{n} \sum_{s=0}^{\infty} v_s(f_{i,s} - F_{i,s}) = S_A - S_L. \tag{6.2}$$

The following result quantifies the probability of achieving a specified required surplus level, $K_n = K\, n^{1/2}$, which may depend on $\mathcal{I}$.

Proposition 7

Conditional on $\mathcal{I}$, assume $\{L_i\}_{i=1}^{n}$ are i.i.d. loss functions with $E(L^2) < \infty$. Then

$$\lim_{n\to\infty} P(S_{FM} + Kn^{1/2} \geq 0) = \lim_{n\to\infty} P(S_L \leq S_A + Kn^{1/2})$$

$$= E\,\Phi\{K/[\mathrm{Var}(L|\mathcal{I})]^{1/2}\}. \tag{6.3}$$

Recall that Φ is the standard normal distribution function. Applications of Proposition 7 may involve the unmatched surplus in (5.2),

$$S = S_{FM} + \sum_{s=0}^{\infty} v_s(c_s - f_s)$$

where

$$f_s = \sum_{i=1}^{n} f_{i,s} = nf_{1,s} \text{ and } c_0 = A_0.$$

From Proposition 7, the probability that *S* exceeds a required surplus level, K_1, may be approximated by using

$$P(S + K_1 \geq 0) \cong E\left\{\Phi\left\{n^{-1/2}\left[K_1 + \sum_{s=0}^{\infty} v_s\,(c_s - f_s)\right] \Big/ [\mathrm{Var}(L|\mathcal{I})]^{1/2}\right\}\right\} \tag{6.4}$$

The right-hand sides of (6.3) and (6.4) are taken over various paths of interest rates. These quantities are straightforward to evaluate explicitly by using interest rate scenarios. For example, for $j=1, \ldots, m$, let

$$\mathcal{I}_j = (1, v_{1j}, v_{2j}, \ldots)$$

represent the j-th interest rate scenario, which occurs with probability p_j. Then, the right-hand side of (6.4) is

$$\sum_{j=1}^{m} p_j \, \Phi\left\{ n^{-1/2} \left[K_1 + \sum_{s=0}^{\infty} v_{sj} \, (c_s - f_{sj}) \right] / \, [\mathrm{Var}(L|\mathcal{I}_j)]^{1/2} \right\}.$$

Note that the expected cash flows, f_{sj}, may depend on the interest environment. If instead one uses a stochastic model for $\{\Delta_s\}$ such as in Section 2 or 3, these quantities can be evaluated numerically via Monte Carlo or simulation methods.

Example 6.1. Block of Whole Life Policies

Consider a block of $n=100$ whole life policies, each issued to a life aged $x=30$. Use the fully discrete model with benefit $=\$1$ payable at the end of the year of death and premium P_{30} payable at the beginning of each year. Under the $MA(1)$ environment of Examples 3.2 and 4.1, it turns out that $P_{30}=0.00816$. Assuming the insurer has the luxury of purchasing assets that fully match *expected* benefit flows, how much extra initial surplus is required to assure that the block will support itself with a reasonably high probability? From Proposition 7 and Table 1, one answer is that $K_n=\$2.50$ will purchase protection at the 96.18 percent level; that is, $P(S_L \leq S_A+2.5) \cong 96.18\%$. Other values of K_n are also provided in Table 1, which was computed by using 100 simulation trials. More trials could have easily been used, but the estimated standard error indicated that 100 trials gives accurate results to three decimal places.

Table 1 underscores the impact of full matching, or immunization, concepts on solvency probabilities. There is little movement in solvency probabilities between the $MA(1)$ environment of Example 3.2, the corresponding i.i.d. estimates ($\theta=0$) and the deterministic interest environment ($\sigma=\theta=0$). The latter environment is the one presented in Bowers et al. [3]. For comparison purposes, to see the effect of a much more volatile environment, 0.1 was added to the standard deviation in the $MA(1)$ model, and the result is reported in the bottom of Table 1. Although this causes the largest shift in solvency probabilities, the shift was not as large as one might have conjectured. The solvency probabilities are stable under the different values, especially when compared to the unmatched distributions in Example 4.2. It is possible that these disparities are due to product differences or the central limit theorem approximation in Example 6.1. However, the simulation suggests that a great deal of the stability can be attributed to the matching of assets and liabilities.

III. SUMMARY AND CONCLUSIONS

As remarked by Hickman [18], "Interest rate variation and resulting risk is a fact of business life." To enhance their credibility with managers and other financial analysts, actuaries should explicitly allow for interest rate variability in their modeling endeavors and their resulting recommendations.

This paper is split into two parts. In the first part, stochastic life contingencies, interest effects as well as decrements are assumed to be stochastic. By assuming one-period spot rates are independent or follow a simple moving average model, volatility and autocorrelation effects of the interest environment can be introduced into the model. Although more general assumptions have appeared in the literature, these assumptions allow the actuary to use the traditional insurance functions in a number of cases of importance with only a change in the interpretation of the force of interest. The appropriate model for interest rates has been much debated in the literature, but no real consensus has been achieved. The long-term nature of actuaries' concerns may engender model selection criteria substantially different than those of other financial analysts.

In the second part, valuation of a block of business is discussed in the context of stochastic life contingencies. When the policies share a common interest environment, the associated losses are no longer independent and the usual limiting distribution results for sums of independent random variables no longer hold. When all assets are valued at market and their value is assumed known at valuation date, the variance of the losses is calculated by using expected cash flows and the distribution is approximated by using simpler random variables. These results quantify the folklore opinion that interest variability dominates mortality variability. To compare cash flows in different periods, one-period spot rates are used in lieu of the more general term structure. This is done so that valuation models do not depend on investment strategies and any concomitant arbitrage possibilities. Bringing assets into the valuation model allows for some matching of the interest rate risk and a resulting reduction of volatility of surplus. Brought to the logical

TABLE 1

SOLVENCY PROBABILITIES WITH ESTIMATED STANDARD ERRORS FOR A BLOCK OF 100 WHOLE LIFE POLICIES EACH ISSUED AT AGE 30. μ, σ AND θ ARE PARAMETERS IN THE $MA(1)$ MODEL. NUMBER OF SIMULATIONS IS 100.

K_n	Mean Solvency Probability	Simulation Standard Error
$\mu = 0.04731$; $\sigma = 0.07346$; $\theta = -0.1465$		
0.0000	0.5000	0.0000
0.5000	0.6385	0.0001
1.0000	0.7608	0.0002
1.5000	0.8561	0.0003
2.0000	0.9218	0.0002
2.5000	0.9618	0.0002
3.0000	0.9832	0.0001
5.0000	0.9998	0.0000
$\mu = 0.04676$; $\sigma = 0.07363$; $\theta = 0.0$		
0.0000	0.5000	0.0000
0.5000	0.6378	0.0002
1.0000	0.7596	0.0003
1.5000	0.8549	0.0003
2.0000	0.9208	0.0003
2.5000	0.9610	0.0002
3.0000	0.9828	0.0001
5.0000	0.9998	0.0000
$\mu = 0.04676$; $\sigma = 0.0$; $\theta = 0.0$		
0.0000	0.5000	0.0000
0.5000	0.6399	0.0000
1.0000	0.7631	0.0000
1.5000	0.8587	0.0000
2.0000	0.9240	0.0000
2.5000	0.9633	0.0000
3.0000	0.9842	0.0000
5.0000	0.9998	0.0000
$\mu = 0.04731$; $\sigma = 0.17346$; $\theta = -0.1465$		
0.0000	0.5000	0.0000
0.5000	0.6275	0.0004
1.0000	0.7423	0.0007
1.5000	0.8353	0.0008
2.0000	0.9032	0.0007
2.5000	0.9478	0.0006
3.0000	0.9743	0.0004
5.0000	0.9994	0.0000

extreme of "full matching" of assets and projected liabilities, limiting distributions can be established to approximate the behavior of the surplus. Under a fully matched environment, surplus requirements are not sensitive to the choice of the interest model, at least for the simple block of whole life policies examined.

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APPENDIX

The appendix contains the proofs of the results of Sections 3 through 6.

Proof of Proposition 1

Define the partial sum, $T_k = \epsilon_k + \epsilon_{k-1} + \ldots + \epsilon_1$, and from (3.1), note that,

$$\sum_{s=1}^{k} \Delta_s = k\,\mu + \epsilon_k - \theta\,\epsilon_0 + (1-\theta)\,T_{k-1}. \tag{A.1}$$

By the i.i.d. property of $\{\epsilon_k\}$, we have

$$\begin{aligned} E(v_k) &= E\{\exp[-k\mu - \epsilon_k + \theta\,\epsilon_0 - (1-\theta)\,T_{k-1}]\} \\ &= E[\exp(-\epsilon_k + \theta\,\epsilon_0)\,e^{-k\mu}]\left\{E\,\{\exp[-(1-\theta)\,\epsilon]\}\right\}^{k-1} \\ &= M(-1)\,M(\theta)\,e^{-k\mu}\,[M(\theta-1)]^{k-1}. \end{aligned}$$

This is sufficient for (3.2). ‡

Proof of Proposition 2

The proof of (3.5) is similar to the proof of Proposition 1 and is omitted. To prove (3.6), from (A.1) we have

$$\begin{aligned} E(v_s v_r) &= E\left[\exp\left(-\sum_{j=1}^{s}\Delta_j - \sum_{j=1}^{r}\Delta_j\right)\right] \\ &= E\Big\{\exp\{-[s\mu + \epsilon_s - \theta\,\epsilon_0 + (1-\theta)\,T_{s-1}] \\ &\quad - [r\mu + \epsilon_r - \theta\,\epsilon_0 + (1-\theta)\,T_{r-1}]\}\Big\} \end{aligned}$$

$$= E\{\exp[-2(s-1)\mu - 2(1-\theta)\,T_{s-1}]\}$$
$$E\{\exp[-(r-s-1)\mu - (1-\theta)\,(T_{r-1} - T_s)]\}$$
$$E\{\exp[-\epsilon_s - (1-\theta)\,\epsilon_s + 2\theta\epsilon_0 - \epsilon_r - 3\mu]\}$$
$$= e^{-(s-1)\alpha_1}\, e^{-(r-s-1)\delta_1}\, e^{-3\mu}\, E\{\exp[(\theta-2)\,\epsilon_2 + 2\theta\epsilon_1 - \epsilon_3]\}.$$

Now, since $e^{-\delta_1} = e^{-\mu} E(e^{-(1-\theta)\epsilon})$ and $e^{-\alpha_1} = e^{-2\mu} E(e^{-2(1-\theta)\epsilon})$, we get the result by direct substitution. ‡

Proof of Proposition 3

We first investigate the orders for the constants C_1, C_2, C_3. First, note that if $\theta=0$, straightforward calculations yield $C_1=C_2=C_3=1$.

Now, if $\theta>0$, we have $\mathrm{Cov}[\exp(\theta\epsilon), \exp(-\epsilon)]<0$. This immediately yields

$$M(\theta-1) = E\{\exp[(\theta-1)\epsilon]\} < E[\exp(\theta\,\epsilon)]\,E[\exp(-\epsilon)]$$
$$= M(\theta)\,M(-1) \tag{A.2}$$

and thus $1<C_1$. If $\theta<0$, then the inequality in (A.2) is reversed, and thus $1>C_1$. Similar arguments establish that $1<C_2$ if and only if $\theta>0$.

To establish $1<C_3$, we show

$$E\{\exp[(\theta-1)\,\epsilon]\}\,E\{\exp[2(\theta-1)\,\epsilon]\}$$
$$< E\{\exp[(\theta-2)\,\epsilon]\}\,E\{\exp(2\theta\,\epsilon)\}\,E\{\exp(-\epsilon)\}. \tag{A.3}$$

Similar to (A.2) for $0<\theta<1$, $\mathrm{Cov}[\exp(\theta-2)\epsilon, \exp(\theta\epsilon)]<0$ and thus

$$E\{\exp[2(\theta-1)\,\epsilon]\} < E\{\exp(\theta\,\epsilon)\}\,E\{\exp[(\theta-2)\,\epsilon]\}.$$

Putting this with (A.2), and the fact that $E\{\exp(2\theta\epsilon)\}>\{E[\exp(\theta\epsilon)]\}^2$, is sufficient for (A.3). The case $-1<\theta<0$ is similar.

As noted before, it is trivial that $\delta_1=\delta$ and $\alpha_1=\alpha$ for the case $\theta=0$. Further, without loss of generality, assume $\mu=0$. For $0<\theta\le 1$, we have $\mathrm{Cov}\{\exp[(\theta-1)\epsilon], \exp(-\theta\,\epsilon)\}>0$ and thus

$$e^{-\delta} = E\{\exp(-\epsilon)\} > E\{\exp[(\theta-1)\,\epsilon]\}\,E\{\exp(-\theta\,\epsilon)\}$$
$$= e^{-\delta_1}\,E\{\exp(-\theta\,\epsilon)\} > e^{-\delta_1}$$

since $E\{\exp(-\theta\epsilon)\}>\exp[E(-\theta\epsilon)]=1$ by Jensen's inequality. Thus $\delta<\delta_1$. The case of $-1\le\theta<0$ is similar except that it uses the reverse inequality in (A.2). The proof for α, α_1 is similar and is omitted. ‡

Proof of Proposition 4

From (4.4) and the independence of $J_1, \ldots, J_n$ and $\mathcal{J}$ note that

$$E(L_i|\mathcal{J}) = \sum_{s=0}^{\infty} v_s f_{i,s} \tag{A.4}$$

and

$$\mathrm{Var}(L_i|\mathcal{J}) = E\left[\left(\sum_{s=0}^{\infty} v_s F_{i,s}\right)^2 \middle| \mathcal{J}\right] - \left(\sum_{s=0}^{\infty} v_s f_{i,s}\right)^2. \tag{A.5}$$

This is sufficient for (4.6).

Now, from some basic identities,

$$\begin{aligned} E(S_L^2) &= [E(S_L)]^2 + \mathrm{Var}(S_L) \\ &= [E(S_L)]^2 + E[\mathrm{Var}(S_L|\mathcal{J})] + \mathrm{Var}[E(S_L|\mathcal{J})] \\ &= E\left[\mathrm{Var}\left(\sum_{i=1}^{n} L_i|\mathcal{J}\right)\right] + E\left[E\left(S_L|\mathcal{J}\right)\right]^2 \\ &= \sum_{i=1}^{n} E\left[\mathrm{Var}\left(L_i|\mathcal{J}\right)\right] + E\left(\sum_{s=0}^{\infty} v_s f_s\right)^2. \end{aligned}$$

The fourth equality is true since, conditional on $\mathcal{J}$, the losses $L_1, \ldots, L_n$ are independent. This is sufficient for the result. ‡

For the proof of Proposition 5, we intend to show

Theorem A.1

Under the conditions of Proposition 5, there exists a positive constant C so that

$$n\, E\, (S_L/n - Y)^2 < C, \text{ for all } n. \tag{A.6}$$

Remarks: An immediate consequence of (A.6) is that

$$\lim_{n \to \infty} E\, (S_L/n - Y)^2 = 0. \tag{A.7}$$

Further, it is well-known that convergence in mean square implies convergence in distribution (see, for example, Serfling [30, page 10]), and thus (A.7) implies Proposition 5. The advantage of (A.6) over Proposition 5 is that not only do we know the limiting distribution of S_L/n but also how quickly S_L/n approaches the limiting distribution.

Proof of Theorem A.1

To prove (A.6), we have

$$n\, E\, (S_L/n - Y)^2 = n\, E\{E[(S_L/n - Y)^2|\mathcal{I}]\}$$
$$= n\, E\{\text{Var}\,(S_L/n|\mathcal{I})\}$$
$$= E\{\text{Var}\,[L(J,P)|\mathcal{I}]\} < \infty$$

since $E[L(J,P)^2] < \infty$. This is sufficient for (A.6) and hence the result. ‡

Proof of Proposition 6

Begin with the standard relationship

$$\text{Var}(S) = \text{Var}[E(S|\mathcal{I})] + E[\text{Var}(S|\mathcal{I})] \tag{A.8}$$

Now, note that S_A is a constant, given $\mathcal{I}$. Thus, $E(S|\mathcal{I}) = E(S_L|\mathcal{I}) - S_A$ and $\text{Var}(S|\mathcal{I}) = \text{Var}(S_L|\mathcal{I})$. Putting this in (A.8) yields

$$\text{Var}(S) = \text{Var}[E(S_L|\mathcal{I}) - S_A] + E[\text{Var}(S_L|\mathcal{I})]. \tag{A.9}$$

Now, the second term on the right-hand side of (A.9) does not depend on $\{c_k\}$. Thus, to minimize Var(S) over choices of $\{c_k\}$, we minimize the first term on the right-hand side of (A.9). Clearly, (5.4) is such a choice. Now, from (A.9),

$$\text{Var}(S) = \text{Var}[E(S_L|\mathcal{I})] + \text{Var}(S_A) - 2\,\text{Cov}[E(S_L|\mathcal{I}), S_A] + E\,\text{Var}(S_L|\mathcal{I})$$

which is sufficient for (5.3). Equation (5.5) follows from direct substitution using (5.4). ‡

Proof of Proposition 7

Conditional on $\mathcal{I}$, by the usual central limit theorem,

$$\lim_{n\to\infty} P(n^{-1/2}\, S_{FM} \le K|\mathcal{I}) = \Phi\{K/[\text{Var}(L|\mathcal{I})^{1/2}]\}.$$

Taking expectations of both sides and applying the Bounded Convergence Theorem yields the result. ‡

DISCUSSION OF PRECEDING PAPER

DANIEL DUFRESNE:

I found Dr. Frees' paper very interesting. After making some general comments, I first describe the technique of time reversal, which is very useful when discount rates are random, and then address some specific aspects of the paper.

Dr. Frees correctly emphasizes the tractability afforded by i.i.d. or moving average (MA) rates of interest. These processes allow explicit formulas for moments or, at the very least, simple algorithms to compute them. Any lack of fit with actual data, as compared with ARIMA models, may well be more than compensated by the simplifications they permit. See [3], [4], [7] and [8] for applications of random rates of return to pension funding.

Dr. Frees is also right in pointing out that most (if not all) previous authors have not dealt with reserves, but only with level payment insurances or annuities; in this respect I am as guilty as the others (see [5] and [6]). I try to earn forgiveness below.

For the most part I agree with the author that discrete functions have more practical use than continuous ones. Nevertheless,

(a) Continuous functions do arise when payments are made very often, or when claims are paid at the moment of death;

(b) Continuous functions give a different intuitive understanding of a problem, which can sometimes lead to the solution of the discrete counterpart.

An example of the second point above can be found in [6], where all the moments of $\tilde{\ddot{a}}_x$ are derived using the same idea that had previously worked in continuous time.

1. *Time Reversal*

In his paper, Dr. Frees deals with discounted values of random payments, when the discount rates are themselves random variables. To find the distribution of such "randomly discounted present values" is by no means an easy question, as the paper itself shows. However, in many cases it is possible to greatly simplify the calculations, by using a technique called "time reversal." This technique has a long history in probability theory. I have applied it to random present values in [5] and [6]. My goal here is to describe, in simple terms, how time reversal can be used to calculate the moments of random present values. No attempt is made to treat the most general case. The simplest case, that of i.i.d. rates of interest and level payments, is dealt with in part (a). Two extensions are then given, one to variable payments, part (b), and the other to MA rates of interest, part (c).

(a) First, consider accumulated values. Suppose one unit is invested at the end of each year, at i.i.d. rates of interest $I_k = U_k - 1$, producing a total amount S_k just after the k-th payment is made. Then

$$S_{k+1} = U_{k+1}\, S_k + 1,\; S_0 = 0. \tag{A}$$

S_k only depends on U_j, $j \le k$, and is thus independent of U_{k+1}. The mean values of $\{S_k\}$ thus satisfy

$$ES_{k+1} = u_1 ES_k + 1,\; u_1 = EU_{k+1};$$

that is, they grow at constant rate $i_1 = u_1 - 1$. Hence

$$\begin{aligned} ES_k &= s_{\overline{k}|i_1} \\ &= c_1 u_1^k + c_2 \end{aligned}$$

where c_1 and c_2 are constants. The first expression $s_{\overline{k}|}$ is traditional in actuarial science, but the second one is more useful in the present context (of course $c_1 = 1/(u_1 - 1)$ and $c_2 = -1/(u_1 - 1)$). To obtain second moments, square equation (A) and take expectations on both sides:

$$\begin{aligned} ES_{k+1}^2 &= u_2 ES_k^2 + 2u_1 ES_k + 1 \\ &= u_2 ES_k^2 + 2u_1(c_1 u_1^k + c_2) + 1, \end{aligned}$$

where $u_2 = EU_{k+1}^2$. Thus ES_{k+1}^2 is the accumulated value, at constant rate $i_2 = u_2 - 1$, of annual payments consisting of two parts, one constant (namely, $2u_1 c_2 + 1$) and the other growing at rate $i_1 = u_1 - 1$ (namely, $2u_1 c_1 u_1^k$). There is no standard actuarial notation for this accumulated value, but what can be said with certainty is that for some constants c_1', c_2' and c_3',

$$ES_k^2 = c_1'\, u_2^k + c_2'\, u_1^k + c_3'$$

(this is a standard result in the theory of difference equations; see, for example, [10, Section 3.4]). In this fashion all moments of S_k can be calculated, either recursively or by explicitly finding the proper constants c_{mj} in

$$ES_k^m = c_{m0} + c_{m1}\, u_1^k + c_{m2}\, u_2^k + \dots + c_{mm}\, u_m^k.$$

The latter approach is used in [6] and is also illustrated in Part 2 of this discussion. Going one step further, the distribution of S_k can be calculated numerically from Equation (A); this is essentially what Dr. Frees does in his Example 4.2.

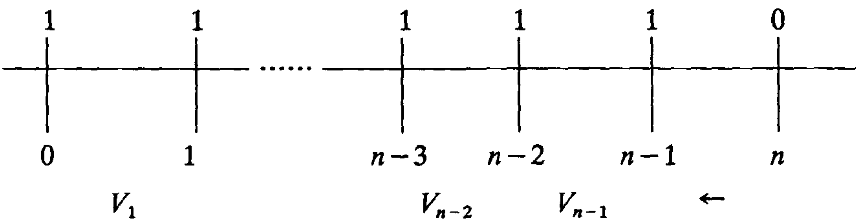
When the rate of interest is constant, there is a simple relationship between accumulating and discounting: discounting at rate i is the same as accumulating at rate $i' = -i/(1+i)$. As will be shown presently, to some extent this duality is preserved when rates of interest are random.

Suppose V_1, V_2, ... are i.i.d. annual discount factors. The present value of an n-year annuity-certain is

$$Y_n = 1 + V_1 + V_1 V_2 + \dots + V_1 \dots V_{n-1}. \qquad \text{(B)}$$

(In Dr. Frees' notation, $V_k = e^{-\Delta_k}$). The progression of discounted values Y_1, Y_2, ... is different from that of S_1, S_2, ... Y_n is obtained by adding $V_1 \dots V_{n-1}$ to Y_{n-1}, whereas S_n results from multiplying S_{n-1} by U_n, and then adding 1. Accumulating is done by moving forward in time, whereas discounting is usually thought of as bringing back one unit from time n to time 0. Because of this difference between accumulating and discounting, the moments of Y_n are *a priori* a lot more difficult to calculate than those of S_n; the random variables $V_1 \dots V_{n-1}$ and Y_{n-1} are certainly not independent, and even the calculation of second moments gets messy; see Equation (2.7).

To remedy this situation, consider the following argument: imagine yourself at time n with an initial amount of 0. Move back to time $n-1$, adding the unit invested at time $n-1$. The result is 1. Now move backwards to time $n-2$, multiplying the previous amount by V_{n-1}, and then add 1. The result is $1+V_{n-1}$. Next move backwards one more period, multiplying by V_{n-2} and again adding 1. The result is $1+V_{n-2}+V_{n-2}V_{n-1}$. Moving backwards $n-3$ more periods yields Equation (B).



The progression of discounted values, starting from time n and then moving backwards, is seen to mimic the progression of accumulated values. The moments and distributions of $\{Y_n\}$ can therefore be obtained in the same way as those of $\{S_n\}$. More specifically, define a new process $\{B_n\}$ by

$$B_{k+1} = V_{k+1} B_k + 1, B_0 = 0.$$

("B" stands for "backwards.") By iterating this equation, we find

$$B_n = 1 + V_n + V_n V_{n-1} + \dots + V_n \dots V_2.$$

B_n is not equal to Y_n, because the discount factors are in reverse order. Nonetheless, the fact that the discount factors are i.i.d. assures us that B_n and Y_n have the same distribution. This implies $EY_n^m = EB_n^m$ for any m. Proceeding as with $\{S_n\}$, we find

$$EY_{k+1} = \phi_1 EY_k + 1,$$

$$EY_{k+1}^2 = \phi_2 EY_k^2 + 2\phi_1 EY_k + 1,$$

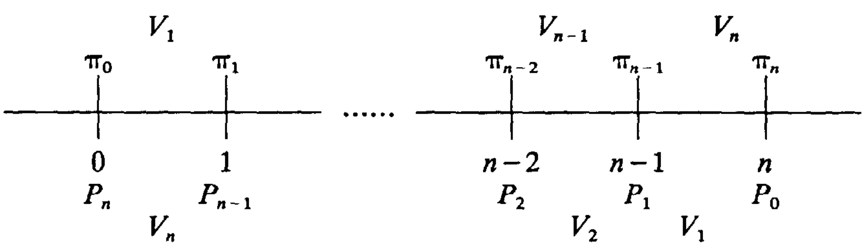
where $\phi_m = EV_k^m$. All moments of Y_k can either be found recursively, or else by determining the constants d_{mj} in

$$EY_k^m = d_{m0} + d_{m1}\,\phi_1^k + \dots + d_{mm}\,\phi_m^k.$$

(b) Now suppose payment π_k is made at time k, and let Y_n stand for the discounted value of $\pi_0, \dots, \pi_n$.

First, let the payments be deterministic. Define $P_0 = \pi_n$, $P_1 = \pi_{n-1}$, ..., $P_n = \pi_0$ and

$$B_{k+1} = V_{k+1} B_k + P_{k+1}, B_0 = P_0. \qquad \text{(C)}$$



The moments of $\{B_k\}$ an be found recursively; this yields all the moments of

$$Y_n = \pi_0 + \pi_1 V_1 + \dots + \pi_n V_1 \dots V_n,$$

since

$$B_n = P_n + P_{n-1} V_n + \dots + P_0 V_n \dots V_1$$

clearly has the same distribution as Y_n.

When the payments are random, but independent of the discount factors, the same procedure can be applied. The only additional ingredients required are the moments

$$EP_k, EP_j P_k, 0 \le j, k \le n.$$

Equation (C), raised to the powers 1 and 2, respectively, implies

$$EB_{k+1} = \phi_1 EB_k + EP_{k+1},$$

$$EB_{k+1}^2 = \phi_2 EB_k^2 + 2\phi_1 EP_{k+1} B_k + EP_{k+1}^2.$$

The only unknown quantity on the right-hand side of these equations is $EP_{k+1} \times B_k$. This can be found recursively from

$$EP_{k+1} B_0 = EP_{k+1} P_0$$

$$EP_{k+1} B_1 = \phi_1 EP_{k+1} B_0 + EP_{k+1} P_1$$

$$\vdots$$

$$EP_{k+1} B_k = \phi_1 EP_{k+1} B_{k-1} + EP_{k+1} P_k.$$

Third moments of $\{B_k\}$ can also be found recursively, if $\{EP_i P_j P_k, 0 \le i, j, k \le n\}$ are known, and so forth for higher moments. Finally, the moments of Y_n are again the same as those of B_n.

(c) Time reversal arguments also apply when geometric rates of interest form a moving average process. I illustrate this for an n-year annuity-certain when rates of interest are MA(1):

$$-\log V_k = \Delta_k = \mu + \epsilon_k + \tau\epsilon_{k-1}.$$

$\{\epsilon_k, k \ge 0\}$ is assumed i.i.d. The discounted value of the annuity is

$$Y_n = 1 + e^{-(\mu+\epsilon_1+\tau\epsilon_0)} + \dots + e^{-[(n-1)\mu + \sum_{j=1}^{n-1} \epsilon_j + \tau\epsilon_{j-1}]}.$$

After replacing $(\epsilon_0, \dots, \epsilon_{n-1})$ with $(\epsilon_n, \dots, \epsilon_1)$, it is seen that Y_n has the same distribution as

$$B_n = 1 + e^{-(\mu + \epsilon_{n-1} + \tau\epsilon_n)} + \ldots + e^{-[(n-1)\mu + \Sigma_{j=1}^{n-1} \epsilon_j + \tau\epsilon_{j+1}]}.$$

Furthermore

$$B_k = e^{-(\mu + \tau\epsilon_k + \epsilon_{k-1})} B_{k-1} + 1, B_0 = 0.$$

Observe that B_{k-1} is a function of $(\epsilon_1, \ldots, \epsilon_{k-1})$ only. By defining

$$C_k = e^{-\epsilon_k} B_k$$

we get the pair of equations

$$B_k = e^{-(\mu + \tau\epsilon_k)} C_{k-1} + 1$$

$$C_k = e^{-[\mu + (1+\tau)\epsilon_k]} C_{k-1} + e^{-\epsilon_k}.$$

The second equation allows the recursive calculation of the moments of $\{C_k\}$, from which the moments of $\{B_k\}$ can be calculated by using the first equation.

This ends my description of the technique of time reversal. For applications to continuous functions, the reader is referred to [5] and [6]. Other applications and extensions are also possible, for example, to MA processes of higher order. Some of these will be described in future articles.

2. *Specific Comments*

Section 2. Use of only the first two moments may not always give an adequate idea of the distribution of randomly discounted payments. I illustrate this point with a random present value possessing a closed-form distribution. Consider the random counterpart of

$$\bar{a}_{\overline{\infty}|} = \int_0^\infty e^{-\delta t}\, dt.$$

Let the log of the discount factors ($\log v_t$ in Dr. Frees' notation) form a continuous-time random walk, that is to say, a Brownian motion process $-W_t$ with mean $-\delta t$ and variance $\sigma^2 t$. This is the continuous-time equivalent of i.i.d. rates of interest. The discounted value of such a "continuous perpetuity" is

$$Y = \int_0^\infty e^{-W_t}\, dt.$$

If $\delta > 0$, it can be proved that

$$\frac{1}{Y} \sim \Gamma(2\delta/\sigma^2, \sigma^2/2)$$

(see Section 4 of [6]). Y has skewness coefficient

$$g = \frac{E(Y - EY)^3}{(\text{Var } Y)^{3/2}}$$

$$= \frac{4\sqrt{a-2}}{a-3}, \; a > 3,$$

where $a = 2\delta/\sigma^2$. A few values of g are shown in Table 1.

TABLE 1

SKEWNESS COEFFICIENT (g) OF CONTINUOUS PERPETUITY (Y) WHEN RETURNS ARE WHITE NOISE WITH MEAN δ AND VARIANCE σ^2

Mean (δ)	Standard Deviation (σ)	a	g
0.02	0.01	400	0.201
0.02	0.10	4	5.657
0.05	0.01	1,000	0.127
0.05	0.10	10	1.616
0.08	0.01	1,600	0.100
0.08	0.10	16	1.151

The skewness coefficient decreases to 0 at the same rate as $1/\sqrt{a} = \sigma/\sqrt{2\delta}$. Most likely this indicates that the normal approximation worsens as the ratio $\sigma^2/2\delta$ increases, that is, as the variance of returns increases relative to the mean. This is rather unexpected: one would think that the absolute size of the variance of returns would be the most important factor, but this is not so, at least in the case at hand. We tentatively conclude that, when average payments are approximately level over an extended period, the normal approximation may not be appropriate, especially if the variance of discount rates is large relative to the mean. (When applying this criterion, inflation and mortality should be taken into account.)

Closed-form expressions for all moments of Z_{K+1} and level payment annuities $a(K)$ are not too difficult to derive when returns are i.i.d; see Section

2 of [6]. As to reserves, here is how time reversal is applied to the calculation of the second moment of

$$ {}_kL = bv_{J+1} + c \sum_{s=0}^{J} v_s. $$

I am assuming a level death benefit b and level premiums $P = -c$. To calculate $E({}_kL^2|J=j)$, first assume $J=j$ (fixed) and then reverse the order of payments and discount rates.

Payments	c	c	......	c	c	b
Original Time Scale	0	1		$j-1$	j	$j+1$
New Time Scale	$j+1$	j		2	1	0

If B_t is as in Equation (C), with $P_0=b$ and $P_s=c$ for $1 \leq s \leq j+1$, then

$$ EB_{s+1} = \phi_1 EB_s + c, \; B_0 = b, $$

$$ => EB_s = \frac{c}{1-\phi_1} + \left(b - \frac{c}{1-\phi_1}\right) \phi_1^s $$

$$ = d_{10} + d_{11}\,\phi_1^s, \; 0 \leq s \leq j+1. $$

In the same way

$$ EB_{s+1}^2 = \phi_2 EB_s + 2c\phi_1\, EB_s + c^2 \tag{D} $$

$$ => EB_s^2 = d_{20} + d_{21}\,\phi_1^s + d_{22}\,\phi_2^s. $$

To obtain the constants d_{20} and d_{21}, substitute the expressions for EB_s, EB_s^2 and EB_{s+1}^2 on either side of Equation (D), and identify the coefficients of 1 and φ_1^s. This yields

$$ d_{20} = (2c\phi_1\, d_{10} + c^2)/(1 - \phi_2) $$

$$ = \frac{P^2\,(1 + \phi_1)}{(1 - \phi_1)(1 - \phi_2)} $$

$$d_{21} = 2c\phi_1 d_{11}/(\phi_1 - \phi_2)$$
$$= \frac{2P\phi_1}{\phi_2 - \phi_1}\left(b + \frac{P}{1 - \phi_1}\right).$$

To obtain d_{22}, use the initial condition $EB_0 = b$ to get

$$d_{22} = b^2 - d_{20} - d_{21}.$$

Thus

$$E({}_kL^2|J = j) = EB_{j+1}^2$$
$$= d_{20} + d_{21}\,\phi_1^{j+1} + d_{22}\,\phi_2^{j+1}$$
$$=> E_kL^2 = d_{20} + d_{21}\,A_{x+k} + d_{22}\,{}^2A_{x+k}$$

where A_{x+k} is valued at rate $i_1 = \varphi_1^{-1} - 1$ and ${}^2A_{x+k}$ at rate $i_2 = \varphi_2^{-1} - 1$. This formula for E_kL^2 is expressed in terms of insurance functions, which is the fashion adopted in *Actuarial Mathematics* and is equivalent to Equation (3.13) when $\theta = 0$. It is possible to determine the constants d_{ms} in

$$E\,{}_kL^m = d_{m0} + \sum_{s=1}^{m} d_{ms}\,{}^sA_{x+k},\; m \geq 1, \tag{E}$$

where ${}^s\mathrm{A}_{x+k}$ is valued at rate $i_s = \varphi_s^{-1} - 1$. The same arguments can be used for any pattern of benefits and premiums.

It is a simple matter to derive the counterpart of (2.5) when benefits are paid at the moment of death and returns are white noise; see [5, p. 196].

Section 3. In this section a small mistake has unfortunately gone unnoticed, and it affects several of the equations. The first term of

$$a(k) = \sum_{s=0}^{k} v_s\,a_s$$

is $1 \cdot a_0$. Since $Ev_0a_0 = a_0$, Equation (3.4) should be

$$Ea(K) = a_0 + C_1E\sum_{s=1}^{K} e^{-\delta_{1s}}\,a_s.$$

This implies

$$P_N = E[e^{-\delta_{1(K+1)}}\,b_{K+1}]\;/\;(a_0/C_1 + E\sum_{s=1}^{K} e^{-\delta_{1s}}\,a_S).$$

Hence it is only approximately true that "net premiums are calculated as in Section 2, except we use δ_1 in lieu of δ." The same comment applies to Equations (3.7) and (3.12):

$$_kV_x = C_1E(e^{-\delta_1(J+1)}\, b_{k+J+1}) - P[a_k + C_1E \sum_{s=1}^{J} e^{-\delta_1 s}\, a_{k+s}]$$

$$_kV_x = C_1A_{x+k} - P_x(1 + C_1 a_{x+k}).$$

The second equation relates to a whole life policy.

Similarly, Equation (3.6) does not hold when $s=0$. Thus, the terms $v_0 a_k$ have to be moved out of the two sums on the right-hand side of Equation (3.8) and dealt with separately. Equations (3.10), (3.11) and (3.13) are therefore incorrect. The effect on the numbers in Example 4.1 is most probably negligible, since C_1, C_2 and C_3 are very close to 1.

Concerning Example 3.1, readers might be interested in the numerical example contained in [7]. It shows accumulated values under the assumption that arithmetic rates of return are MA(1) with the same mean but varying covariances. Both examples indicate that our intuition needs to be reeducated when dealing with dependent rates of interest.

Closed-form expressions can be found for the moments of $_kL$ in the case of a whole life policy with level premiums. By using time reversal as in Section 1(c), a formula similar to (E) can be obtained. The formula for the second moment has only three terms (compare with (3.13)).

Section 4. The development leading to Proposition 4 and Example 4.1 should be compared with Section 4 of [12]. Proposition 5 is a direct application of the usual ergodic theorem for stationary processes, see, for example, [11, p. 87]. Convergence holds with probability one (not only in distribution) and EL^2 need not be finite. It should be emphasized that the limit Y has done away with all mortality fluctuations. Only random interest and expected cash flows are left. Example 4.2 is in effect an application of time reversal. Any discounted value can be treated in the same way, whether rates of interest are i.i.d. or MA(1). Some examples of perpetual bonds with closed-form distributions are given in [6].

For a block of business, I would suggest the following alternative to Proposition 4:

(i) Project cash flows: determine mean and covariance functions.
(ii) Use time reversal to calculate moments of discounted values.

The first step involves mortality and withdrawals only, whereas the second introduces random interest. I see a number of possible advantages to this approach:

(i) Multiple scenarios for cash flows and interest rates probably require less work;
(ii) Programming recursive equations may be less time-consuming than using explicit formulas;
(iii) Claims under different policies do not have to be independent;
(iv) Moments higher than the second can be calculated, if desired.

Section 6. Proposition 7 results from the central limit theorem for conditionally independent random variables. Suppose that, given θ, $X_1, X_2, \ldots$ are independent and have common distribution F_θ. Let θ have distribution G and

$$a^2 = \int \operatorname{Var} F_\theta \, dG(\theta).$$

Then ([9, p. 287, no. 21])

$$\frac{1}{a\sqrt{n}} \sum_{i=1}^{n} (X_i - E_\theta X_i) \underset{\text{distr.}}{\rightarrow} U$$

with

$$F_U(x) = \int \Phi(ax/\sqrt{\operatorname{Var} F_\theta}) dG(\theta).$$

In the case at hand θ is the vector of random rates of interest. The limit distribution is a weighted average of normal distributions. It may be obtained via simulations, or else by integrating with respect to the joint distribution of interest rates. Of course the latter method is usually a serious exercise in numerical analysis, requiring, for example, integrating in dimension 20, for a 20-year projection.

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ELIAS S. W. SHIU:

Dr. Frees is to be thanked for another contribution to the theory of stochastic life contingencies. The following are some thoughts on the paper.

In the context of life contingencies, it seems to me that Δ_k should mean the rate of *total* return in the k-th period of the investment portfolio that funds the insurance policy. The total return of an investment portfolio in a period is determined by the interest and dividend income received during the period and the market values of the portfolio at the beginning and end of the period. In other words, \$exp(Δ_k) should be the amount that one gets at time k if one invests \$1 at time $k-1$. In practice, it may be difficult for an insurance company to come up with such numbers. Although it is relatively easy to determine interest and dividend income, capital maturities and initial investment values, there are many assets whose market values are difficult to assess because they are not traded publicly. It may be useful to point out that, in Example 3.2 of the paper, some of the annual returns of bonds were negative because the bond portfolio incurred capital losses and not because interest rates became negative.

It is stated in Section 5 of the paper that the one-period spot rates,

$$h_S(s, s + 1) = \Delta_{s+1}, \qquad s = 0, 1, 2, \ldots ,$$

are used for discounting asset and liability flows. I have three remarks. First, note that the one-period spot rate $h_s(s, s+1)$ is known at time s. In particular, $h_0(0, 1) = \Delta_1$ is fixed at time 0; it is not a random variable. Second, because spot rates should not be negative, one may object to modeling them as normal random variables. Third, the expression

$$E\left[\exp\left(-\sum_{i=1}^{k}\Delta_i\right)\right]$$

gives the "actuarial" present value at time 0 of 1 to be paid at time k. Is it the same as the spot *price*, at time 0, of a k-period (noncallable and default-free) zero coupon bond

$$\exp[-kh_0(0, k)]?$$

In general, for $t>s\geq 0$, is

$$E\left[\exp\left(-\sum_{i=s+1}^{t}\Delta_i\right)\middle|\Delta_1, \Delta_2, \ldots, \Delta_s\right] = \exp[-(t-s)h_s(s,t)]? \quad \text{(D.1)}$$

Under the no-arbitrage hypothesis, the paper [6] shows how one may construct a term structure evolution model, in which (D.1) is satisfied for all t and s, $t>s\geq 0$. However, a probability measure under which (D.1) holds is not likely to be the "actual" probability measure (because we do not live in a risk-neutral world). Also, in such models, even though the expected values are market values, the meaning of variances, etc. is not clear; see also [4].

It is stated in Section 5 that Fisher and Weil posited that the movement of the term structure of interest rates is governed by

$$h_j(j, t) = h_{j-1}(j, t) + \delta_j, \; t \geq j, \; j = 0, 1, 2, \ldots,$$

where δ_j does not depend on t. I would like to add that such term structure movements necessarily admit arbitrages; see [6, p. 236].

I now try to rephrase Proposition 6 in the language of *functional analysis*. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $L^2(\Omega, \mathcal{F}, P)$ be the Hilbert space with the inner product

$$\langle X, Y\rangle = E(XY).$$

Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. The functions in $L^2(\Omega, \mathcal{F}, P)$, which are $\mathcal{G}$ measurable, form a closed subspace $\mathbf{N}$, that is, $\mathbf{N} = L^2(\Omega, \mathcal{G}, P)$. Let

$$\Pi : L^2(\Omega, \mathcal{F}, P) \to L^2(\Omega, \mathcal{F}, P)$$

be the orthogonal projection onto **N**. Then, for $X \in L^2(\Omega, \mathcal{F}, P)$,

$$E(X|\mathcal{G}) = \Pi X$$

almost surely. Consequently, we have the Pythagorean Theorem:

$$\|X - E(X|\mathcal{G})\|^2 + \|E(X|\mathcal{G}) - Y\|^2 = \|X - Y\|^2 \qquad \text{(D.2)}$$

for all $Y \in$ **N**. With $X = S_L$ and $\mathcal{G} = \mathcal{I}$, (D.2) becomes

$$\|S_L - E(S_L|\mathcal{I})\|^2 + \|E(S_L|\mathcal{I}) - g(\Delta_1, \Delta_2, \Delta_3, \ldots)\|^2 = \|S_L - g(\Delta_1, \Delta_2, \Delta_3, \ldots)\|^2$$

for all Borel measurable functions g.

To derive (5.3) of the paper, write

$$T = S - E(S),$$

$$T_A = S_A - E(S_A)$$

and

$$T_L = S_L - E(S_L).$$

Then

$$\begin{aligned} \mathrm{Var}(S) &= \|T\|^2 \\ &= \|T_A - T_L\|^2 \\ &= \|T_A\|^2 + \|T_L\|^2 - 2 \langle T_A, T_L \rangle . \end{aligned}$$

Now, if we assume that S_A is of the form

$$S_A = g(\Delta_1, \Delta_2, \Delta_3, \ldots),$$

then $T_A = \Pi T_T$. Hence

$$\langle T_A, T_L \rangle = \langle \Pi T_A, T_L \rangle = \langle T_A, \Pi^* T_L \rangle .$$

Since Π is a self-adjoint operator, that is

$$\Pi^* = \Pi,$$

we have

$$\langle T_A, T_L \rangle = \langle T_A, \Pi T_L \rangle = \langle T_A, E(T_L|\mathcal{I}) \rangle = \mathrm{Cov}(S_A, E(S_L|\mathcal{I})).$$

Dr. Frees points out that, under mild assumptions,

$$\mathrm{Var}(S_A - E(S_L|\mathcal{F}))$$

is zero if and only if $c_k = f_k$ for all k. Some suggestions on how one may match c_k with f_k can be found in Section VI of [5].

Finally, I wish to mention that two recent papers, by Beekman and Fuelling [1] and by Dufresne [3], contain results related to those in the present one. Black's paper [2] is also of interest.

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(AUTHOR'S REVIEW OF DISCUSSION)

EDWARD W. FREES:

The discussions by Dr. Dufresne and Dr. Shiu serve to expand and to focus certain aspects of the paper. They expand the paper by providing details of related areas that I had neither the inclination nor expertise to delve into here. Both Dr. Dufresne and Dr. Shiu demonstrated how we can sharpen our understanding of the models that we use by using deeper results in mathematics, probability, and statistical theory. On the other hand, one of my goals in writing this paper was to use only the tools that students learn today in their actuarial curriculum. For students of the Society of Actuaries, the only tool not encountered in Courses 100, 110, 120, 121, 140 and 150 is the Bounded Convergence Theorem used in the proof of Proposition 7. I found both discussions to be intellectually stimulating in that they forced me to focus on certain aspects of the paper that I had not given enough thought to. Although not planned, the two discussions complemented the paper nicely in that Dr. Dufresne's discussion essentially focuses on *Part I. Stochastic Life Contingencies*, while Dr. Shiu mainly addresses *Part II. Solvency Valuation*. I thank both discussants for their comments.

Dr. Dufresne offers a discussion on a technique for simplifying certain probability calculations. I am fortunate that he chose to exercise this technique on the ideas presented in this paper, because it led him to discover errors in some of the formulas for net premium and reserve calculations in the MA(1) model. The point that I missed when originally trying to sort out these ideas was how to jump-start a moving average series. I assumed the existence of a noise term at time 0, ϵ_0, so that the random term at time one, Δ_1, had an identical distribution to subsequent discount factors. The mathematically inconsistent thing that I did was to assume that the discount factor at time 0, Δ_0, was identically equal to zero. This, of course, produces the logical result of $v_0 = 1$. To be somewhat more consistent mathematically, one could assume that $\Delta_0 = \epsilon_0 - \theta\epsilon_{-1}$. With this definition, the expected value of Δ_0 is 0, but there is a distribution around it. Further, one could define

$$v_k = \exp\left(-\sum_{s=0}^{k} \Delta_s\right) \qquad \text{for } k = 0, 1, 2, \ldots .$$

By doing so, similar to Proposition 1, one can check that $E(v_k) = C_1^* \exp(-k\,\delta_1)$ where $C_1^* = M(\theta)\,M(-1)$ for $k = 0,1,2, \ldots$. Further, Proposition 2 holds by similarly redefining C_2 and C_3. This then provides a model so that the intuitively appealing formulas that appear in Section 3 are valid using new definitions of the C factors. I think the solution offered by Dr. Dufresne is the more practical one. With his solution, one only needs to remember in our spreadsheet formula that time 0 is a little different from the others because we assume that $v_0 = 1$. The interpretation, that we can accommodate an MA(1) structure by merely reinterpreting the force of interest, that I offered in the paper is still essentially valid. An alternative solution, again from a spreadsheet standpoint, is to note that in calculating variances we can take cash flows at time zero to be equal to zero. This is because variances simply are measured of dispersion of unknown quantities. Following the suggestion of Dr. Dufresne, I computed some of the percentage differences for annuities due in Example 4.2 and found the differences between the exact results and approximations to be of the order 10^{-5}. Still, an equality should represent sameness, and I thank Dr. Dufresne for pointing out this inconsistency in the paper.

One contribution of the discussion of Dr. Shiu is to rephrase the statement of Proposition 6 using the language of functional analysis. By doing so, he has implicitly provided the foundation for switching from the discrete time model of the paper to a continuous time model. This is a natural extension in that many of the models used in financial economics are continuous. It will be interesting to see if the new measure of duration leads to anything of use in the future. Another important contribution of his discussion is to challenge my vague statements in Section 5 concerning the link between a so-called actuarial approach and a financial economics approach to valuing surplus. Before responding to these remarks, let me first summarize some of the difficulties that I encountered in arriving at a consistent model of reality.

In financial economics, no-arbitrage-type arguments have been available for almost 20 years to value widely traded securities using risk-neutral probability measures. For derivative securities that depend on exogenous processes that are not widely traded, such as interest rates, no-arbitrage-type arguments have appeared much more recently; see, for example, the Cox, Ingersoll and Ross [9] paper. There are at least four main sources of difficulty in valuing surplus using these types of arguments. First, the surplus process clearly depends on an interest rate process, which is exogenous in the sense that it is not subject to the usual preferences of investors. Second, the fact that the term structure changes at each valuation date further complicates the model. Third, the liability portion of the surplus process is not widely traded and hence it is not clear that no-arbitrage models offer a reasonable representation of reality. Fourth, actuaries perform valuations for many reasons. A solvency valuation seems the closest to a financial economics valuation because both rely on market forces to determine the price function.

Now, papers such as that by Pedersen, Shiu and Thorlacius ([6] in Shiu's list of references) address the first and second concerns, at least if one is willing to live in a lattice framework. The third concern is a deeper one and perhaps not well enough formulated to be able to model with mathematics. Insurance theoreticians have traditionally addressed the economic concerns of different players through their respective utility functions. One of the desirable features of the no-arbitrage arguments is that they seem to be robust to the shape of these various utility functions as long as players desire to make more riskless money. One of the goals of this paper was to incorporate as much of the no-arbitrage pricing as seems reasonable, while recognizing that this valuation technique is not directly applicable to other classes of assets/liabilities. This goal is tempered by the fourth concern that, as actuaries, we have the notion that a reserve should represent our best forecast of the future at a given time. This position would lend itself to using our knowledge of the entire term structure of interest rates when calculating reserves for solvency valuation. Conversely, this attitude could be viewed as myopic in that we know that the entire term structure will undoubtably change at subsequent valuations. For the purposes of this paper, I advocate the clumsy yet practical solution of using our knowledge of the term structure for one class of assets and disregarding this information, as unreliable, for another. Because of this, no attempt was made to move into the risk-neutral world that is so convenient for probability calculations.

With this background, my response to Dr. Shiu's third remark is that spot rates in this paper are meant to be realizations of an actual company investment policy. The probability measure governing that measure corresponds to that investment policy, not one that lives in a risk-neutral world. The harder problem that I have not addressed here is how to link the spot prices to actuarial present values. With respect to the first remark, I modeled $\Delta_1 = h_0(0,1)$ as a random variable for at least two reasons. First, I wanted to be consistent with the division of assets into two types: one that uses knowledge of the term structure and one that does not. Second, even though we use discrete models in practice, most of us believe the world is better approximated through the use of continuous models. If we think of our discrete time interval as a quarter of a year, we might interpret $h_0(0,1)$ as a 90-day Treasury bill spot rate. Of course, this is known at time 0, but by day 1 we must replace it by an 89-day Treasury bill, an unknown quantity at time 0. Now, if I did wish to assume that $h_0(0,1)$ is known, I also have to assume that $h_0(0,2)$, $h_0(0,3)$, and so on, are known. For consistency, I assumed the entire structure was known for one set of assets and unknown for the other. With my notation, $h_s(s,t)$ is assumed known at time s. While assumed known at time s, it is also considered unreliable for forecasting the asset/liability match of one class of assets.

Overall, I found the discussions thoughtful, well-written and enlightening. While I have not commented on all aspects of each discussion, this is generally because they are self-contained and I could add little by way of a response. I hope that this response serves to sharpen some of the issues underlying solvency valuation and suggest some important new areas in actuarial research.