WEIGHTED PRICING FUNCTIONALS [1]

Edward Furman

Department of Mathematics and Statistics, York University, Toronto, Ontario M3J 1P3, Canada. E-mail: efurman@mathstat.yorku.ca

Ričardas Zitikis

Department of Statistical and Actuarial Sciences, University of Western Ontario, London, Ontario N6A 5B7, Canada. E-mail: zitikis@stats.uwo.ca

Abstract. We explore the concept of weighted distributions and their role in various phenomena occurring in insurance and finance. In particular, we relate weighted distributions to actuarial and economic premium calculation principles, and also to the capital asset pricing model (CAPM). Imitating the latter, we propose a weighted insurance pricing model (WIPM). Although general in formulation, we show that the WIPM can successfully be evaluated in a variety of situations, which we illustrate with a number of examples.

Keywords and phrases: Weighted distributions, weighted premiums, pricing functionals, actuarial premium calculation principles, economic premium calculation principles, capital asset pricing model, CAPM, weighted insurance pricing model, WIPM.

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Contents

1. Introduction

Pricing insurance risks is a significant and challenging problem. Inappropriately determined premiums, whether too high or too low, may result in insolvency of insurance policies, failure of business lines, and even bankruptcy of entire insurance enterprizes. Naturally, therefore, the problem has given rise to an active research area and, consequently, to numerous debates as to what pricing functionals, widely known as premium calculation principles (pcp's), should or should not be used in one situation or another (see, e.g., Gerber, 1979; Goovaerts et al., 1984; Kaas et al., 1994; Wang, 1996; Young, 2004; Denuit et al., 2005; Pflug and Römisch, 2007). Generally, actuarial pcp's are functionals (see Bühlmann, 1980)

$$\pi : \mathcal{X} \to [0, \infty]$$

from the set $\mathcal{X}$ of all non-negative random variables X (representing, e.g., risks or losses) to the interval of all non-negative extended real numbers. Another way of looking at the actuarial pricing functionals is to treat them as functionals from the set $\mathcal{F}$ of the cumulative distribution functions (cdf's) F_X of $X \in \mathcal{X}$. These two points of view highlight the fact that the actuarial price $\pi[X]$ depends on $X \in \mathcal{X}$ only via the cdf F_X, a property which is known in the literature as objectiveness (see, e.g., Denuit et al., 2005) or conditional state independence (see, e.g., Bülmann, 1980, 1984).

The objectiveness property, however, precludes the decision maker from taking into account factors such as insurer's financial position and attitude, general condition of economy, dependence on other risks: these may, and indeed do, influence the price of risks (see, e.g., Bülmann, 1980, 1984; Deprez and Gerber, 1985). Hence, in the present paper we give a particular attention to pricing functionals

$$\Pi : \mathcal{X} \times \mathcal{X} \to [0, \infty],$$

where the first coordinate X in the pricing functional value $\Pi[X, Y]$ is the risk under consideration and the second coordinate Y is insurer's overall risk or, generally, any random variable that influences the price of X. In addition to the properties inherited from $\pi : \mathcal{X} \to [0, \infty]$, and depending on the economic context under consideration, the pricing functional Π may be required, or desired, to satisfy additional properties, some of which we discuss later in this paper. We note that since $\Pi : \mathcal{X} \times \mathcal{X} \to [0, \infty]$ is a generalization of $\pi : \mathcal{X} \to [0, \infty]$, it is natural to require that, for all $X \in \mathcal{X}$,

$$\Pi[X, X] = \pi[X], \tag{1.1}$$

which we call the compatibility property of the actuarial and economic pricing functionals.

Constructing actuarial and economic pricing functional has been an interesting and fruitful area, and in the following sections we shall provide a number of functionals accompanied with several references. Recently, Furman and Zitikis (2008a) have shown that a large number of actuarial pricing functionals available in the literature can be unified into one actuarial weighted pricing functional, which we denote by

$$\pi_w : \mathcal{X} \to [0, \infty]$$

and define by the formula $\pi_w[X] = \mathbf{E}[Xw(X)]/\mathbf{E}[w(X)]$. Furthermore, Furman and Zitikis (2007, 2008e) argue that the functional π_w allows for a convenient departure from conditional state independence and thus facilitates introducing the economic weighted pricing functional

$$\Pi_w : \mathcal{X} \times \mathcal{X} \to [0, \infty]$$

defined by the formula $\Pi_w[X, Y] = \mathbf{E}[Xw(Y)]/\mathbf{E}[w(Y)]$. Clearly, the economic pricing functional Π_w is compatible with the actuarial functional π_w in the sense of equation (1.1).

The rest of the paper is organized as follows. In Section 2 we further elaborate on the notion and properties of the actuarial pricing functional π_w and also relate them to weighted and distorted distributions, as well as to distorted premiums. In Section 3 we explore a technique for computing actuarial weighted pricing functionals π_w in the context of log-exponential family (LEF) of distributions. In Section 4 we discuss the economic weighted pricing functional Π_w. In Section 5 we explore axiomatic properties of general pricing functionals Π and specialize them to the weighted pricing functional Π_w. Then we turn our attention into developing techniques for computing $\Pi_w[X, Y]$, given a joint distribution of the pair (X, Y). In Section 6 we connect the functional Π_w to general Stein-type decompositions of covariances. As a consequence, in Section 7 we arrive at a weighted insurance pricing model (WIPM), which we view as an insurance counterpart of the well-known capital asset pricing model (CAPM) in finance. Then, based on the notion of weighted distributions, in Section 8 we discuss ways for computing the generalized economic weighted pricing functional

$$\Pi_{v,w} : \mathcal{X} \times \mathcal{X} \to [0, \infty],$$

which is defined by the formula $\Pi_{v,w}[X, Y] = \mathbf{E}[v(X)w(Y)]/\mathbf{E}[w(Y)]$. Section 9 concludes the paper with a summary of main contributions.

2. Weighted distributions and actuarial weighted pricing functionals

As we have already hinted above, not every functional $\pi : \mathcal{X} \to [0, \infty]$ is admissible for pricing risks. Indeed, pricing functionals are always subjected to constraints, which depend on the situation at hand and/or decision maker's aims. For example, common sense suggests and mathematics confirms that in order to avoid insolvency, the pricing functional value $\pi[X]$ should not be smaller than the net premium $\mathbf{E}[X]$ for every $X \in \mathcal{X}$. This relationship between $\pi[X]$ and $\mathbf{E}[X]$ is known in the literature as the (non-negative) loading property.

In general, constructing loaded pricing functionals is not difficult. An obvious route for achieving this goal is by adding to the net premium $\mathbf{E}[X]$ a constant or, say, a fraction of the mean $\mathbf{E}[X]$, the variance $\mathbf{Var}[X]$, or the standard deviation $\mathbf{Var}^{1/2}[X]$ (see, e.g., Chapter 5 in Gerber, 1979). A considerable number of other differently constructed actuarial pricing functionals can be found in, e.g., Goovaerts et al. (1984), Kaas et al. (1994), Wang (1996), Young (2004), Denuit et al. (2005), Pflug and Römisch (2007). Naturally, we discuss only some pricing functionals without attempting to give a detailed account of the research area or literature.

We proceed with one of the most general and widely considered methods for constructing loaded actuarial pricing functionals. It starts with the tail representation $\int_0^\infty \bar{F}_X(x)dx$ of the net premium $\mathbf{E}[X]$. By lifting up the de-cumulative distribution function (ddf) $\bar{F}_X = 1 - F_X$ with a function g such that $g(t) \geq t$ for all $t \in [0, 1]$, we obtain the loaded pricing functional $\pi_g : \mathcal{X} \to [0, \infty]$ defined by the formula (see Denneberg, 1994; Wang, 1995, 1996; Wang et al., 1997; Wang, 1998)

$$\pi_g[X] = \int_0^\infty g(\bar{F}_X(x))dx. \tag{2.1}$$

The distortion function $g : [0, 1] \to [0, 1]$ is usually right-continuous and such that $g(0) = 0$ and $g(1) = 1$, in addition to the already noted bound $g(t) \geq t$, which is automatically satisfied if the function g is assumed, or chosen, to be concave. Under these conditions, we define the 'distorted' cdf $F_{g,X}(x) = 1 - g(\bar{F}_X(x))$, which in turn implies the representation

$$\pi_g[X] = \int_0^\infty \bar{F}_{g,X}(x)dx. \tag{2.2}$$

Hence, with X_g denoting a random variable with the cdf $F_{g,X}$, the right-most integral of equation (2.2) can be written as the expectation $\mathbf{E}[X_g]$, thus implying that $\pi_g[X] = \mathbf{E}[X_g]$. The loading property of the distortion pricing functional can therefore be rewritten as the bound $\mathbf{E}[X_g] \geq \mathbf{E}[X]$ for all $X \in \mathcal{X}$.

Wang et al. (1997) have observed that if the general pricing functional π satisfies certain axioms, then the functional becomes the distorted pricing functional π_g for a distortion function g. For additional information on the axioms as well as for their critique, we refer to, for example, Young (2004), Denuit et al. (2005), and references therein.

There are loaded actuarial pricing functionals that do not admit representation (2.1) for any distortion function g, and we refer to Denuit et al. (2005) for examples. Some pricing functionals such as Esscher's and modified variance do not obviously imply reformulations in the form of π_g. However, note that the two aforementioned actuarial pricing functionals (i.e., Esscher's and modified variance) can be easily expressed as

$$\pi_w[X] = \frac{\mathbf{E}[Xw(X)]}{\mathbf{E}[w(X)]} \tag{2.3}$$

for some weight functions w. Indeed, with $w(x) = e^{tx}$ and $w(x) = x$, we have the Esscher and modified variance pricing functionals, respectively. In general, equation (2.3) defines the functional

$$\pi_w : \mathcal{X} \to [0, \infty],$$

which Furman and Zitikis (2008a) call the weighted pricing functional, assuming that the weight function w is non-negative and non-decreasing to ensure the loading property.

For example, the conditional tail expectation (CTE) $\mathbf{E}[X|X \geq F_X^{-1}(p)]$ defines a weighted pricing functional π_w with $w(x) = \mathbf{1}\{x \geq F_X^{-1}(p)\}$, where $\mathbf{1}$ is the indicator function and $F_X^{-1}(p)$ is the p-th quantile of the cdf F_X. Interestingly, the distortion pricing functional π_g also falls into the class of weighted pricing functionals π_w, provided that the distortion function g is differentiable and F_X is continuous. Indeed, we easily check the equation

$$\pi_g[X] = \mathbf{E}[Xg'(\bar{F}_X(X))], \tag{2.4}$$

which has been extensively utilized by Tsanakas and Barnett (2003), Tsanakas and Desli (2003), Tsanakas (2004), Tsanakas and Christofides (2006), Tsanakas (2008). The right-hand side of equation (2.4) is a weighted pricing functional. Indeed, choose $w(x) = g'(\overline{F}_X(x))$ as the weight function and note that $\mathbf{E}[w(X)] = 1$ due to $\mathbf{E}[w(X)] = \int_0^1 g'(t)dt$ and the boundary conditions $g(0) = 0$ and $g(1) = 1$. For a set of examples of the actuarial weighted pricing functional $\pi_w : \mathcal{X} \to [0, \infty]$, see Table 2.1.

Actuarial pricing functionals	$w(x)$	$\pi_w[X]$
Net	$const$	$\mathbf{E}[X]$
Modified variance	x	$\mathbf{E}[X] + \mathbf{Var}[X]/\mathbf{E}[X]$
Size-biased	x^t	$\mathbf{E}[X^{1+t}]/\mathbf{E}[X^t]$
Esscher	e^{tx}	$\mathbf{E}[Xe^{tX}]/\mathbf{E}[e^{tX}]$
Aumann-Shapley	$e^{tF_X(x)}$	$\mathbf{E}[Xe^{tF_X(X)}]/\mathbf{E}[e^{tF_X(X)}]$
Kamps	$1 - e^{-tx}$	$\mathbf{E}[X(1 - e^{-tX})]/\mathbf{E}[(1 - e^{-tX})]$
Excess-of-loss	$\mathbf{1}\{x \geq t\}$	$\mathbf{E}[X \mid X \geq t]$
Distorted	$g'(\overline{F}_X(x))$	$\mathbf{E}[Xg'(\overline{F}_X(X))]$
Proportional hazard	$p\overline{F}_X^{p-1}(x)$	$p\,\mathbf{E}[X\overline{F}_X^{p-1}(X)]$
Conditional tail expectation	$\mathbf{1}\{x \geq x_p\}$	$\mathbf{E}[X \mid X \geq x_p]$
Modified tail variance	$x\mathbf{1}\{x \geq x_p\}$	$\mathbf{E}[X \mid X \geq x_p] + \mathbf{Var}[X \mid X \geq x_p]/\mathbf{E}[X \mid X \geq x_p]$

TABLE 2.1. Examples of the actuarial weighted pricing functional $\pi_w : \mathcal{X} \to [0, \infty]$ for various weight functions w with the notation $x_p = F_X^{-1}(p)$. Both $t \in [0, \infty)$ and $p \in (0, 1]$ are fixed parameters.

We have already noted above that $\pi_g[X]$ can be written as the expectation $\mathbf{E}[X_g]$ of the 'distorted' random variable X_g. Likewise, the weighted pricing functional value $\pi_w[X]$ can be expressed as the expectation $\mathbf{E}[X_w]$ of a 'weighted' random variable X_w, whose cdf is

$$F_{w,X}(x) = \frac{\mathbf{E}[\mathbf{1}\{X \leq x\}w(X)]}{\mathbf{E}[w(X)]}$$

(see, e.g., Patil and Rao, 1978; Patil et al., 1988; Rao, 1997; Patil, 2002). The Fubini theorem implies that $\pi_w[X]$ is equal to the integral $\int_0^\infty \overline{F}_{w,X}(x)dx$, and so we have the equation

$$\pi_w[X] = \mathbf{E}[X_w]. \tag{2.5}$$

Consequently, the loading property can be written as the bound $\mathbf{E}[X_w] \geq \mathbf{E}[X]$ for all $X \in \mathcal{X}$. The weighted pricing functional π_w is loaded whenever the non-negative weight function w is non-decreasing (see Lemmas 1 and 3 in Lehmann, 1966). Note also that in view of the equation

$$\pi_w[X] = \mathbf{E}[X] + \frac{\mathbf{Cov}[X, w(X)]}{\mathbf{E}[w(X)]}, \tag{2.6}$$

the loading property $\pi_w[X] \geq \mathbf{E}[X]$ is equivalent to the non-negativity of the covariance $\mathbf{Cov}[X, w(X)]$. In later sections equation (2.6) will play other important roles. For recent results concerning the non-negativity of covariances, we refer to Zucca (2008).

We conclude this section with notes related to equation (2.4) that has been frequently used for deriving statistical inferential results for $\pi_g[X]$. Namely, assuming that F_X is continuous, the right-hand side of equation (2.4) can be written as the expectation $\mathbf{E}[F_X^{-1}(U)g'(1-U)]$, where U is the uniform on $[0,1]$ random variable. The expectation, which is equal to $\int_0^1 F_X^{-1}(t)\psi(t)dt$ with the 'score' function $\psi(t) = g'(1-t)$, is the asymptotic mean of an L-statistic (see, e.g., Serfling, 1980). Hence, the well developed asymptotic theory of these statistics (see, e.g., Serfling, 1980) can now be utilized to derive desired statistical inferential results concerning $\pi_g[X]$. For further details on the topic, we refer to Jones and Zitikis (2003, 2005, 2007), Jones et al. (2006), Brazauskas et al. (2007), Brazauskas et al. (2009). Schechtman and Zitikis (2006) elaborate on the difference between the expectations $\mathbf{E}[F_X^{-1}(U)g'(1-U)]$ and $\mathbf{E}[Xg'(\bar{F}_X(X))]$ for discontinuous cdf's F_X. Brazauskas et al. (2008) develop statistical inferential results concerning the conditional tail expectation $\mathbf{E}[X|X \geq F_X^{-1}(p)]$.

3. Actuarial weighted functionals and the log-exponential family

Before generalizing $\pi_w : \mathcal{X} \to [0,\infty]$ into an economic pricing functional, which is the subject matter of the next section, we first provide several hints and techniques for calculating $\pi_w[X]$.

Given a sufficiently large data set, we can estimate $\pi_w[X]$ non-parametrically by, for example, replacing the cdf F in a formula for $\pi_w[X]$ by the empirical cdf F_n. But this non-parametric approach may not be always adequate. If so, then assuming a parametric distribution of X, one would then need to express $\pi_w[X]$ in terms of the distribution parameters, which we denote by $\theta_1, \ldots, \theta_k$, and then replace each parameter by an empirical estimator, such as the maximum likelihood estimator or some other one (see, e.g., Brazauskas et al., 2009).

Expressing $\pi_w[X]$ in terms of the parameters may, however, be challenging and for this reason we next discuss several efficient ways for solving the problem. We start with an earlier derived equation $\pi_w[X] = \mathbf{E}[X_w]$, which suggests that if we determine the distribution of the weighted random variable X_w, then calculating the expectation $\mathbf{E}[X_w]$ would be a standard exercise. To proceed, we further restrict ourselves to only those weight functions w that are of the form

$$w(x) = x^c$$

for some constant $c > 0$. Then there is a large class of parametric cdf's such that for any member F_X of the class the corresponding weighted cdf $F_{w,X}$ is also a member of the class, although with different parameters. Namely, let F_X be an absolutely continuous cdf with the density (pdf) f_X of the form

$$f_X(x; \theta_1, \theta_2, \ldots, \theta_k, a) = \frac{x^{h(\theta_1, \theta_2, \ldots, \theta_k)} a(x; \theta_2, \ldots, \theta_k)}{\int_0^\infty y^{h(\theta_1, \theta_2, \ldots, \theta_k)} a(y, \theta_2, \ldots, \theta_k) dy}, \tag{3.1}$$

where $x \mapsto a(x; \theta_2, \ldots, \theta_k)$ is a non-negative function that does not depend on θ_1, and $h(\theta_1, \theta_2, \ldots, \theta_k)$ is such that

$$h(\theta_1, \theta_2, \ldots, \theta_k) + c = h(\theta^*_{1,c}, \theta_2, \ldots, \theta_k) \tag{3.2}$$

for some $\theta^*_{1,c} = g(\theta_1, \theta_2, \ldots, \theta_k, c)$ and a function g. Under these assumptions, which are more detailed than in the paper by Patil and Ord (1978), we check that if a cdf F has pdf (3.1), then the size-biased cdf

$$F_{c,X}(x) = \frac{\mathbf{E}[\mathbf{1}\{X \leq x\} X^c]}{\mathbf{E}[X^c]}$$

also belongs to the same parametric family. The following examples, most of which follow Patil and Ord (1978), illustrate the above general class of distributions.

Example 3.1. The gamma random variable $X \sim Ga(\gamma, \alpha)$ has the pdf

$$\frac{\alpha^\gamma x^{\gamma-1}}{\Gamma(\gamma)} e^{-\alpha x} \mathbf{1}\{x > 0\},$$

which can be written as $f_X(x; \gamma, \alpha, a)$ with $h(\gamma, \alpha) = \gamma$ and the function $a(x) = x^{-1} e^{-\alpha x} \mathbf{1}\{x > 0\}$. We have the equation $h(\gamma, \alpha) + c = h(\gamma^*, \alpha)$ with $\gamma^* = \gamma + c$. Hence, $X_c \sim Ga(\gamma + c, \alpha)$.

Example 3.2. The Pareto random variable $X \sim Pa(\alpha, \beta)$ has the pdf

$$\frac{\alpha\beta^\alpha}{x^{\alpha+1}}\mathbf{1}\{x \geq \beta\},$$

which can be written as $f_X(x; \alpha, \beta, a)$ with $h(\alpha, \beta) = -\alpha$ and $a(x) = x^{-1}\mathbf{1}\{x \geq \beta\}$. We have the equation $h(\alpha, \beta) + c = h(\alpha^*, \beta)$ with $\alpha^* = \alpha - c$. Hence, $X_c \sim Pa(\alpha - c, \beta)$.

Example 3.3. The log-normal random variable $X \sim LogN(\mu, \sigma^2)$ has the pdf

$$\frac{1}{x\sqrt{2\pi}\sigma} \exp\Big\{ - \frac{1}{2\sigma^2}(\log x - \mu)^2\Big\}\mathbf{1}\{x > 0\},$$

which can be written as $f_X(x; \mu, \sigma^2, a)$ with $h(\mu, \sigma^2) = \mu/\sigma^2$ and the function $a(x) = x^{-1}\exp\{-(\log x)^2/(2\sigma^2)\}\mathbf{1}\{x > 0\}$. We have the equation $h(\mu, \sigma^2) + c = h(\mu + c\sigma^2, \sigma^2)$ with $\mu^* = \mu + c\sigma^2$. Hence, $X_c \sim LogN(\mu + c\sigma^2, \sigma^2)$.

Example 3.4. The inverse gamma random variable $X \sim IGa(\gamma, \alpha)$ has the pdf

$$\frac{\alpha^\gamma}{\Gamma(\gamma)x^{\gamma+1}}e^{-\alpha/x}\mathbf{1}\{x > 0\},$$

which can be written as $f_X(x; \gamma, \alpha, a)$ with $h(\gamma, \alpha) = -\gamma$ and $a(x) = x^{-1}e^{-\alpha/x}\mathbf{1}\{x > 0\}$. We have the equation $h(\gamma, \alpha) + c = h(\gamma^*, \alpha)$ with $\gamma^* = \gamma - c$. Hence, $X_c \sim IGa(\gamma - c, \beta)$.

The above general class of distributions can further be extended into the log-exponential family (LEF) of cdf's denoted by $F_X(x; \lambda, \nu)$ and defined by the equation

$$F_X(dx; \lambda, \nu) = \exp\{\lambda \log x - \kappa(\lambda)\}\nu(dx),$$

where λ is a parameter, ν is a measure, and $\kappa(\lambda) = \log \int_0^\infty x^\lambda \nu(dx)$ is the normalizing constant. Note that the pdf f_X of equation (3.1) is a member of LEF with the parameter $\lambda = h(\theta_1, \theta_2, \dots, \theta_k)$ and the measure $\nu(dx) = a(x)dx$.

To illustrate the convenience of working with the LEF random variable X, we next calculate the excess-of-loss premium $\mathbf{E}[X|X \geq x]$. We have that

$$\begin{aligned}
\mathbf{E}[X|X \geq x] &= \frac{1}{\overline{F}_X(x; \lambda, \nu)} \int_{[x,\infty)} x e^{\lambda \log x - \kappa(\lambda)}\nu(dx) \\
&= \frac{e^{\kappa(\lambda+1) - \kappa(\lambda)}}{\overline{F}_X(x; \lambda, \nu)} \int_{[x,\infty)} e^{(\lambda+1)\log x - \kappa(\lambda+1)}\nu(dx) \\
&= \frac{\int_0^\infty x^{\lambda+1}\nu(dx)}{\int_0^\infty x^\lambda \nu(dx)} \; \frac{\overline{F}_X(x; \lambda + 1, \nu)}{\overline{F}_X(x; \lambda, \nu)}. \qquad (3.3)
\end{aligned}$$

For example, when $X \sim Ga(\gamma, \alpha)$, which is a member of the LEF, then the first ratio on the right-hand side of equation (3.3) is equal to γ/α. Since $F_X(x; \lambda + 1, \nu)$ is the gamma distribution $Ga(\gamma + 1, \alpha)$, we therefore have that

$$\mathbf{E}[X|X \geq x] = \frac{\gamma}{\alpha} \frac{\bar{G}(x; \gamma + 1, \alpha)}{\bar{G}(x; \gamma, \alpha)},$$

where $\bar{G}(x; \gamma, \alpha)$ denotes the gamma ddf. Setting $x = F_X^{-1}(p)$ in the above formula, we obtain the corresponding one for the actuarial CTE pricing functional derived by Landsman and Valdez (2005) within the framework of the exponential family of distributions (see Jørgensen, 1997).

Unlike the gamma distribution, the Pareto distribution $Pa(\alpha, \beta)$ is not a member of the just noted exponential family, but it nevertheless belongs to the LEF as we have noted in an earlier example. Hence, when $X \sim Pa(\alpha, \beta)$, we can proceed with equation (3.3). The first ratio on the right-hand side of the equation is equal to $\alpha\beta/(\alpha - 1)$, where we assume that $\alpha > 1$ for the expectation $\mathbf{E}[X]$ to be finite. Since $F_X(x; \lambda + 1, \nu)$ is the Pareto distribution $Pa(\alpha - 1, \beta)$, we have the equation

$$\mathbf{E}[X|X \geq x] = \frac{\alpha\beta}{\alpha - 1} \frac{\bar{P}(x; \alpha - 1, \beta)}{\bar{P}(x; \alpha, \beta)}, \tag{3.4}$$

where $\bar{P}(x; \alpha, \beta)$ denotes the Pareto ddf. The right-most ratio of equation (3.4) is equal to x/β. Interestingly, this linear form of the function $x \mapsto \mathbf{E}[X|X \geq x]$ is a characteristic property of the Pareto distribution (see, e.g., Arnold, 1983).

We conclude this section with a general note that in spite of the popularity of the exponential family, which has been used in several actuarial research areas including credibility theory, risk modeling and pricing (see, e.g., Landsman and Valdez, 2005, and references therein), this paper seems to be the first one to introduce the log-exponential family into the actuarial context.

4. Departing from conditional state independence

The actuarial weighted pricing functional $\pi_w : \mathcal{X} \to [0, \infty]$ can naturally be extended (see Furman and Zitikis, 2007) beyond conditional state independence using the functional

$$\Pi_w : \mathcal{X} \times \mathcal{X} \to [0, \infty]$$

defined by the formula

$$\Pi_w[X,Y] = \frac{\mathbf{E}[Xw(Y)]}{\mathbf{E}[w(Y)]}.$$

We call Π_w the economic weighted pricing functional. With appropriately chosen weight functions w, the functional Π_w reduces to a number of special economic pricing functionals, some of which are recorded in Table 4.1.

Economic pricing functionals	$w(y)$	$\Pi_w[X,Y]$
Net	$const$	$\mathbf{E}[X]$
Modified covariance	y	$\mathbf{E}[X] + \mathbf{Cov}[X,Y]/\mathbf{E}[Y]$
Size-biased	y^t	$\mathbf{E}[XY^t]/\mathbf{E}[Y^t]$
Esscher	e^{ty}	$\mathbf{E}[Xe^{tY}]/\mathbf{E}[e^{tY}]$
Aumann-Shapley	$e^{tF_Y(y)}$	$\mathbf{E}[Xe^{tF_Y(Y)}]/\mathbf{E}[e^{tF_Y(Y)}]$
Kamps	$1-e^{-ty}$	$\mathbf{E}[X(1-e^{-tY})]/\mathbf{E}[(1-e^{-tY})]$
Excess-of-loss	$\mathbf{1}\{y \geq t\}$	$\mathbf{E}[X \mid Y \geq t]$
Distorted	$g'(\overline{F}_Y(y))$	$\mathbf{E}[Xg'(\overline{F}_Y(Y))]$
Proportional hazard	$p\overline{F}_Y^{\,p-1}(y)$	$p\,\mathbf{E}[X\overline{F}_Y^{\,p-1}(Y)]$
Conditional tail expectation	$\mathbf{1}\{y \geq y_p\}$	$\mathbf{E}[X\|Y \geq y_p]$
Modified tail covariance	$y\mathbf{1}\{y \geq y_p\}$	$\mathbf{E}[X\|Y \geq y_p] + \mathbf{Cov}[X,Y\|Y \geq y_p]/\mathbf{E}[Y\|Y \geq y_p]$

Table 4.1. Examples of the economic weighted pricing functional $\Pi_w : \mathcal{X} \times \mathcal{X} \to [0,\infty]$ for various weight functions w with the notation $y_p = F_Y^{-1}(p)$. Both $t \in [0,\infty)$ and $p \in (0,1]$ are fixed parameters.

Note that some weight functions w in Table 4.1 are independent of any cdf and other ones are dependent on the cdf of Y. To make a distinction between the two cases may turn out to be crucial, especially when ordering risks, comparing economic weighted pricing functionals corresponding to different 'background' risks Y, developing statistical inference. Hence, we may sometimes need to indicate the dependence of the weight function w on the cdf F_Y by writing w_{F_Y} or w_Y instead of the simple w. In turn, we may need to use the more detailed notation

$$\Pi_{w_Y}[X,Y] = \frac{\mathbf{E}[Xw_Y(Y)]}{\mathbf{E}[w_Y(Y)]}$$

for the earlier introduced $\Pi_w[X,Y]$, if a confusion is possible. To get an insight into the aforementioned statistical inferential results and with them associated need for using w_{F_Y} instead of just w, we refer to Brazauskas et al. (2008) where the CTE pricing functional is estimated in the case $X = Y$, and also to Schechtman et al. (2008) where not necessarily equal X and Y are considered. We note in this regard that Schechtman et al. (2008) deal with $\mathbf{E}[X\mathbf{1}\{Y \le F_Y^{-1}(p)\}]$, which is the 'dual' version of $\mathbf{E}[X|\,Y \ge F_Y^{-1}(p)]$.

There are good reasons for considering even a more general economic pricing functional $\Pi_{v,w} : \mathcal{X} \times \mathcal{X} \to [0,\infty]$ defined by augmenting Π_w with a function $v : [0,\infty) \to [0,\infty)$ as follows: $\Pi_{v,w}[X,Y] = \mathbf{E}[v(X)w(Y)]/\mathbf{E}[w(Y)]$. Obviously, when $v(x) = x$, then $\Pi_{v,w}$ reduces to Π_w. Furthermore, when $v(x) = \mathbf{1}\{x \le y\}$, then $\Pi_{v,w}[Y,Y]$ becomes the weighted cdf $F_w(y)$. The function $v(x) = x^t$ emerges when considering conditional tail variance and higher order moments. The economic pricing functional $\Pi_{v,w}$ is of course compatible with the corresponding actuarial weighted pricing functional $\pi_{v,w} : \mathcal{X} \to [0,\infty]$ defined by

$$\pi_{v,w}[X] = \frac{\mathbf{E}[v(X)w(X)]}{\mathbf{E}[w(X)]}.$$

The latter formula can be traced back to Remark 1 in Heilmann (1989). Note that the functional $\pi_{v,w}$ is loaded when both functions v and w are non-decreasing (see Lemma 1(i & iii) and Lemma 3 in Lehmann, 1966). Same conditions on v and w, in addition to positive quadrant dependence of X and Y, imply the loading property for the general economic pricing functional $\Pi_{v,w}$. For positively quadrant and other types of dependence structures, we refer to, for example, Lehmann (1966), Mari and Kotz (2001), and references therein.

We conclude this section with several interpretations of $\Pi_{v,w}[X,Y]$ which are of course also applicable to $\Pi_w[X,Y]$ since the latter is $\Pi_{v,w}[X,Y]$ with $v(x) = x$. First, $\Pi_{v,w}[X,Y]$ can be viewed as the (only) solution of the following minimization problem:

$$\Pi_{v,w}[X,Y] = \arg\min_a \mathbf{E}[(v(X) - a)^2 w(Y)].$$

Second, $\Pi_{v,w}[X,Y]$ can be viewed as the mean of the regression function $r(y) = \mathbf{E}[v(X)|Y = y]$ with respect to the weighted distribution $F_{w,Y}$, that is,

$$\Pi_{v,w}[X,Y] = \mathbf{E}[r(Y_w)],$$

where Y_w is a random variable with the cdf $F_{w,Y}$. Note that when $v(x) = x - \mathbf{E}[X]$, then $r(y)$ is the centered regression function $r_{X|Y}(y) = \mathbf{E}[X - \mathbf{E}[X]|\,Y = y]$, which will play an important role in Section 6 below. Third, analogously to equation (2.6), we have

$$\Pi_{v,w}[X,Y] = \mathbf{E}[v(X)] + \frac{\mathbf{Cov}[v(X), w(Y)]}{\mathbf{E}[w(Y)]}, \tag{4.1}$$

where the ratio on the right-hand side can be thought of as the safety loading due to the risk X 'surrounded' by the background risk Y. When the two risks are uncorrelated, the loading is of course equal to zero.

5. Axiomatic properties of economic pricing functionals

Axiomatic properties of actuarial pricing functionals have been extensively studied in the literature (see, e.g., Gerber, 1979; Goovaerts et al., 1984; Kaas et al., 1994; Wang et al., 1997; Artzner et al., 1999; Young, 2004; Denuit et al., 2005; Pflug and Römisch, 2007). The literature discussing properties of economic pricing functionals is less voluminous, although it has been actively developing (see, e.g., Denault, 2001; Hesselager and Andersson, 2002; Dhaene et al., 2003; Goovaerts et al., 2003; Venter, 2004; Kalkbrener, 2005; Kim, 2007; Meucci, 2007; Pflug and Römisch, 2007).

In this section we formulate a number of properties that the economic pricing functional Π may be desired or required to satisfy, depending on circumstances or aims of the decision maker. Specifically, let $\{X_1, \ldots, X_K\}$ denote a pool of risks with the partial aggregate risk $S_\Delta = \sum_{k\in\Delta}^K X_k$, where $\Delta \subseteq \{1, \ldots, K\}$. The overall risk is $S = X_1 + \cdots + X_K$, which is of course S_Δ with $\Delta = \{1, \ldots, K\}$. We are interested in properties that the functional $(X_k, S) \mapsto \Pi[X_k, S]$ or, more generally, $(S_\Delta, S) \mapsto \Pi[S_\Delta, S]$ may be desired to satisfy. We start with the already noted non-negative loading and other simple properties, and then focus on more advanced ones by first formulating them for Π and then specializing to Π_w.

5.1. *Non-negative loading and no-unjustified loading.* The economic pricing functional Π is (non-negatively) loaded if the bound

$$\Pi[X_k, S] \geq \mathbf{E}[X_k] \tag{5.1}$$

holds for all pairs (X_k, S). In the context of the economic weighted pricing functional Π_w, we have bound (5.1) when X_k and $w(S)$ are positively correlated. This happens when, for example, X_k and S are positively quadrant dependent and the function w is non-decreasing (see Lemma 1(iii) and Lemma 3 in Lehmann, 1966).

Note that when $X_k = c_0$ (a constant), then bound (5.1) reduces to $\Pi[c_0, S] \geq c_0$. However, it is reasonable to require that the price of a constant risk is equal to the risk itself. This leads to the no-unjustified loading property:

$$\Pi[c_0, S] = c_0. \tag{5.2}$$

The economic weighted pricing functional Π_w obviously satisfies this property.

5.2. *Full-additivity and consistency.* It is often natural to require that the sum of the prices of individual risks is equal to the price of the aggregate risk:

$$\sum_{k=1}^{K} \Pi[X_k, S] = \Pi[S, S]. \tag{5.3}$$

Functionals Π satisfying equation (5.3) are called fully additive. Note that the right-hand side of equation (5.3) is equal to $\pi[S]$ due to the assumed compatibility of the economic and actuarial pricing functionals.

A more general condition than the full additivity is that of consistency, which means

$$\sum_{k\in\Delta} \Pi[X_k, S] = \Pi[S_\Delta, S] \tag{5.4}$$

for every subset $\Delta \subseteq \{1, \ldots, K\}$. The economic weighted pricing functional Π_w is obviously fully additive and consistent.

5.3. *No-undercut and consistent no-undercut.* The economic pricing functional Π satisfies the no-undercut property if

$$\sum_{k\in\Delta} \Pi[X_k, S] \leq \Pi[S_\Delta, S_\Delta] \tag{5.5}$$

for every subset $\Delta \subseteq \{1, \ldots, K\}$. When Π is fully additive, then the no-undercut property becomes equivalent to

$$\Pi[S_\Delta, S] \leq \Pi[S_\Delta, S_\Delta] \tag{5.6}$$

for every $\Delta \subseteq \{1, \ldots, K\}$. In general, we say that the (fully-additive or not) economic pricing functional Π satisfies the consistent no-undercut property if bound (5.6) holds for every $\Delta \subseteq \{1, \ldots, K\}$. Since the weighted pricing functional Π_w is fully additive, the no-undercut and consistent no-undercut properties coincide and can be written with appropriately specified weight functions as follows:

$$\Pi_{w_S}[S_\Delta, S] \leq \Pi_{w_{S_\Delta}}[S_\Delta, S_\Delta]. \tag{5.7}$$

Verification of bound (5.7) might be a challenging task. Furman and Zitikis (2008b) noted a sufficient condition, which states that if the function

$$r(x) = \frac{\mathbf{E}[w_{S_\Delta}(S_\Delta)|\ S_\Delta = x]}{\mathbf{E}[w_S(S)|\ S_\Delta = x]}$$

is non-decreasing, then bound (5.7) holds. This sufficient condition has proved particularly useful when w does not depend on any cdf. To illustrate the point, we specialize the condition to the case when the sums S_Δ and $S_{\complement\Delta}$ are independent. Under this assumption, and also assuming that w does not depend on any cdf, the function r reduces to

$$r(x) = \frac{w(x)}{\mathbf{E}[w(x+Z)]}, \tag{5.8}$$

where $Z = S - S_\Delta$. For example, let $w(x) = \mathbf{1}\{x \geq t\}$, which gives the economic excess-of-loss pricing functional. Then we have $r(x) = \mathbf{1}\{x \geq t\}/\mathbf{P}[Z \geq t - x]$, which is a non-decreasing function. Next, let $w(x) = x^t$, which yields the economic size-biased pricing functional. In this case we have $r(x) = 1/\mathbf{E}[(1 + x^{-1}Z)^t]$, which is a non-decreasing function. When $w(x) = 1 - \exp\{-tx\}$, then we have the economic Kamps pricing functional. The corresponding function r is increasing because its derivative $r'(x) = te^{-tx}(1 - a_t)/(1 - a_t e^{-tx})^2$ is positive, where $a_t = \mathbf{E}[e^{-tZ}]$. Finally, when $w(x) = \exp\{tx\}$, then we have the economic Esscher pricing functional with $r(x) \equiv c$, a constant.

When the weight function w depends on F_Y, the aforementioned sufficient condition of Furman and Zitikis (2008b) is frequently either too strong to yield an appropriate result or too difficult to verify. For this reason, Furman and Zitikis (2008b) have used brute force calculations to show that, for example, the economic CTE pricing functional satisfies consistent no-undercut whereas the MTCov pricing functional violates it.

5.4. *Translativity.* The general economic pricing functional Π is (positively) sub-translative, translative, and super-translative if, with the sign $\bowtie$ standing for $\leq$, $=$ and $\geq$ respectively, we have that

$$\Pi[X_k + a, S + a] \ \bowtie\ a + \Pi[X_k, S] \tag{5.9}$$

for every constant $a \geq 0$. In the case of Π_w, statement (5.9) reduces to

$$\Pi_w[X_k, S + a] \ \bowtie\ \Pi_w[X_k, S]. \tag{5.10}$$

Consequently, for the economic Esscher pricing functional, we have statement (5.10) with the sign $\bowtie$ standing for '='. We also have the equality '=' for the economic CTE pricing functional since $S+a \geq F_{S+a}^{-1}(p)$ is equivalent to $S \geq F_S^{-1}(p)$. The equality '=' also holds for the economic distorted pricing functional since $\overline{F}_{S+a}(S+a) = \overline{F}_S(S)$. In more complex cases, relationships (5.10) can be checked by establishing the monotonicity of

$$r(x) = \frac{\mathbf{E}[w(S)|\, X_k = x]}{\mathbf{E}[w(S+a)|\, X_k = x]}. \tag{5.11}$$

Furman and Zitikis (2008b) have used this observation to analyze the Kamps pricing functional.

5.5. *Homogeneity.* The economic pricing functional Π is (positively) sub-homogeneous, homogeneous, and super-homogeneous if, with $\bowtie$ standing for $\leq$, $=$ and $\geq$ respectively, we have that

$$\Pi\Big[bX_k, \sum_{i\neq k} X_i + bX_k\Big] \;\bowtie\; b\,\Pi[X_k, S] \tag{5.12}$$

for every $b > 0$. For the weighted pricing functional Π_w, statement (5.12) reduces to

$$\Pi_w[X_k, S + (b-1)X_k] \;\bowtie\; \Pi_w[X_k, S]. \tag{5.13}$$

Relationship (5.13) can be checked using the above noted technique with an appropriately defined function r or using some ad hoc calculations.

5.6. *Additivity.* The economic pricing functional Π is (positively) sub-additive, additive and super-additive if, with $\bowtie$ standing for $\leq$, $=$ and $\geq$ respectively, we have

$$\Pi[X_k + Y, S + Y] \;\bowtie\; \Pi[X_k, S] + \Pi\Big[Y, \sum_{i\neq k} X_i + Y\Big] \tag{5.14}$$

for every $Y \in \mathcal{X}$. Note that property (5.14) can be viewed as a generalization of the earlier discussed translativity property: set $Y = a$.

6. Stein-type covariance decompositions

We have so far concentrated mainly on the axiomatic basis of the economic weighted pricing functional Π_w. From the applications point of view it is important to be able to calculate the functional given a bivariate distribution of (X, Y) and a weight function w. In general, this poses a challenging problem. For results and related discussions, mainly in the context of the economic CTE pricing functional, we refer to Panjer and Jing (2001), Panjer (2002), Landsman and Valdez (2003), Cai and Li (2005), Furman and Landsman

(2005, 2006b), Chiragiev and Landsman (2007), Vernic (2008), Dhaene et al. (2008), Furman and Zitikis (2008a, 2008b), and references therein.

In this section we discuss a general technique that helps with the task. For this, we first rewrite the economic weighted pricing functional in the form

$$\Pi_w[X,Y] = \mathbf{E}[X] + \frac{\mathbf{Cov}[X, w(Y)]}{\mathbf{E}[w(Y)]}, \tag{6.1}$$

which generalizes the earlier noted equation (2.6). Next we observe that in a number of situations the covariance $\mathbf{Cov}[X, w(Y)]$ can be split into the product of two components:

$$\mathbf{Cov}[X, w(Y)] = C(F_{X,Y})D(w, F_X, F_Y), \tag{6.2}$$

where $C(F_{X,Y})$ does not depend on weight function w, and $D(w, F_X, F_Y)$ does not depend on the joint cdf $F_{X,Y}$.

To illustrate the point, assume that (X, Y) follows a bivariate normal distribution and the weight function w is differentiable. Then Stein's lemma (see Stein, 1981) says that

$$\mathbf{Cov}[X, w(Y)] = \mathbf{Cov}[X, Y]\mathbf{E}[w'(Y)]. \tag{6.3}$$

Thus, decomposition (6.2) holds with $C(F_{X,Y}) = \mathbf{Cov}[X, Y]$ and $D(w, F_X, F_Y) = \mathbf{E}[w'(Y)]$. In fact, it is more in line with our following considerations to write $C(F_{X,Y})$ as the ratio $\mathbf{Cov}[X, Y]/\mathbf{Var}[Y]$ and $D(w, F_X, F_Y)$ as the product $\mathbf{Var}[Y]\mathbf{E}[w'(Y)]$. Equation (6.3), when applied on the right-hand side of equation (6.1), gives

$$\Pi_w[X,Y] = \mathbf{E}[X] + \mathbf{Cov}[X, Y]\frac{\mathbf{E}[w'(Y)]}{\mathbf{E}[w(Y)]}, \tag{6.4}$$

which accomplishes the desired separation of the weight function w from the dependence structure, which is condensed in the covariance $\mathbf{Cov}[X, Y]$. If, for example, the weight function is $w(y) = y$, which gives rise to the economic MCov pricing functional, then the ratio $\mathbf{E}[w'(Y)]/\mathbf{E}[w(Y)]$ is equal to $1/\mathbf{E}[Y]$, and thus $\Pi_w[X, Y] = \mathbf{E}[X]+\mathbf{Cov}[X, Y]/\mathbf{E}[Y]$. If the weight function is $w(y) = e^{ty}$, in which case we have the Esscher pricing functional, then the ratio $\mathbf{E}[w'(Y)]/\mathbf{E}[w(Y)]$ is equal to t, and thus $\Pi_w[X, Y] = \mathbf{E}[X] + t\cdot\mathbf{Cov}[X, Y]$.

Decomposition (6.3) holds only when the pair (X, Y) is bivariate normal and the weight function w differentiable. However, any of the two conditions may not be realistic, or may simply be violated, depending on the problem at hand. Landsman (2006), Landsman and Nešlehová (2008) have succeeded in relaxing the normality assumption by establishing an analog of Stein's lemma for bivariate elliptical distributions, which are symmetric. However, most insurance risks are positively skewed and non-negatively supported. Moreover, imposing the differentiability on w may simply be impossible, as is the case with the weight functions $w(y) = \mathbf{1}\{y \geq t\}$, $w(y) = \mathbf{1}\{y \geq F_Y^{-1}(p)\}$, and $w(y) = y\mathbf{1}\{y \geq F_Y^{-1}(p)\}$ corresponding to the excess-of-loss, conditional tail expectation, and modified tail covariance pricing functionals, respectively (see Table 4.1).

It turns out that neither bivariate ellipticity of (X, Y) nor differentiability of w are necessary for deriving Stein-type decompositions that suit our purpose. Indeed, as noted by Furman and Zitikis (2008c,d), the possibility of separating the joint distribution of (X, Y) from the weight function w is actually based not on a particular bivariate distribution but on a particular form of the centered regression function $r_{X|Y}(y) = \mathbf{E}[X - \mathbf{E}[X] \,|\, Y = y]$. Namely, assume that the function admits the decomposition

$$r_{X|Y}(y) = C(F_{X,Y})q(y, F_X, F_Y), \tag{6.5}$$

where $C(F_{X,Y})$ is a constant that does not depend on w, and $y \mapsto q(y, F_X, F_Y)$ is a function that does not depend on the joint cdf $F_{X,Y}$. Note in passing that $\mathbf{E}[q(Y, F_X, F_Y)] = 0$. Under assumption (6.5), it is straightforward to check that decomposition (6.2) holds with

$$D(w, F_X, F_Y) = \mathbf{E}[w(Y)q(Y, F_X, F_Y)].$$

Equation (6.5) is satisfied for a number of symmetric and non-symmetric bivariate distributions and also for non-differentiable weight functions. To illustrate, consider the case when (X, Y) is bivariate normal but the function w may not be differentiable. Then equation (6.5) holds with $C(F_{X,Y}) = \mathbf{Cov}[X, Y]/\mathbf{Var}[Y]$ and $q(y, F_X, F_Y) = y - \mathbf{E}[Y]$. Hence, we have that

$$\mathbf{Cov}[X, w(Y)] = \mathbf{Cov}[X, Y]\frac{\mathbf{Cov}[Y, w(Y)]}{\mathbf{Var}[Y]}. \tag{6.6}$$

Unlike Stein's equation (6.3), equation (6.6) does not require differentiability of w and can therefore be utilized to evaluate, for example, the economic CTE pricing functional. Indeed, using equations (6.1) and (6.6) with $w(y) = \mathbf{1}\{y \geq F_Y^{-1}(p)\}$, we obtain that

$$CTE_p[X, Y] = \mathbf{E}[X] + \mathbf{Cov}[X, Y]\frac{f_Y(F_Y^{-1}(p))}{1 - p}. \tag{6.7}$$

Equation (6.7) has been established using direct calculations by Panjer and Jing (2001), Panjer (2002).

For corresponding results in the elliptical case, we refer to Landsman and Valdez (2003). We only note here that when the pair (X,Y) follows the bivariate elliptical distribution $E_2(\boldsymbol{\mu},\mathbf{B},\psi)$ (see, e.g., Fang et al., 1987, for details), then equation (6.5) holds with

$$C(F_{X,Y})=\frac{\beta_{X,Y}}{\beta_Y^2},$$

where $\beta_{X,Y}$ is an off-diagonal entry and β_Y^2 is a diagonal entry of the positive definite matrix $\mathbf{B}$. In the normal case we have $\beta_{X,Y}=\mathbf{Cov}[X,Y]$ and $\beta_Y^2=\mathbf{Var}[Y]$.

To demonstrate that the above general idea works with non-symmetric distributions, let (X,Y) follow the bivariate Pareto distribution of the second kind $Pa_2(II)(\boldsymbol{\mu},\boldsymbol{\theta},a)$ (see Arnold, 1983). Then equation (6.5) holds with $C(F_{X,Y})=\mathbf{Cov}[X,Y]/\mathbf{Var}[Y]$, which can be rewritten as

$$C(F_{X,Y})=\frac{\theta_1}{a\theta_2},$$

where the parameters have the same meaning as on p. 603 of Kotz et al. (2000).

For another example, assume that (X,Y) follows the bivariate gamma distribution $Ga_2(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})$ of Mathai and Moscopoulos (1991). We easily check that equation (6.5) holds with $C(F_{X,Y})=\mathbf{Cov}[X,Y]/\mathbf{Var}[Y]$, which can be rewritten as

$$C(F_{X,Y})=\frac{\alpha_0\beta_1}{(\alpha_0+\alpha_2)\beta_2},$$

where the parameters have the same meaning as in Corollary 1 on p. 143 of Mathai and Moscopoulos (1991). We conclude this example with a note that the bivariate gamma distribution can be utilized for modeling risks when individual underlying risks are independent but their observable outcomes are contaminated by a background risk, thus making the observations dependent. For theory and applications concerning the (multivariate) gamma distribution, we refer to Mathai and Moscopoulos (1991).

7. The weighted insurance pricing model

The covariance decomposition discussed in the previous section is well suited for developing a pricing methodology for insurance risks analogously to the celebrated capital asset pricing model (CAPM) in finance (see Sharpe, 1964; Lintner, 1965; Mossin, 1966). Intuitively, the CAPM relates the expected return $\mathbf{E}[R_i]$ on asset i to the expected return $\mathbf{E}[R_m]$ on the entire market portfolio m using the equation

$$\mathbf{E}[R_i]=r_f+\beta_{i,m}(\mathbf{E}[R_m]-r_f),\tag{7.1}$$

where r_f is the riskless rate of return and $\beta_{i,m}$ is a proportionality coefficient known as 'beta' and assumed to be of the form

$$\beta_{i,m} = \frac{\mathbf{Cov}[R_i, R_m]}{\mathbf{Cov}[R_m, R_m]} \tag{7.2}$$

(see Sharpe, 1964; Lintner, 1965; Mossin, 1966; Owen and Rabonovitch, 1983). Interestingly, while the model initially incorporates investor's utility function, the above pricing equation does not seem to rely on the function. This is indeed true under certain assumptions on the form of investor's utility function and/or the class of distributions of (R_i, R_m). In particular, it has been shown that the CAPM equation holds independently of the investor's utility when the pair (R_i, R_m) follows symmetric distributions such as the bivariate normal or, more generally, bivariate elliptical (see, e.g., Fama, 1970; Owen and Rabinovitch, 1983; Hamada and Valdez, 2008).

Recently, Furman and Zitikis (2008d) have demonstrated that the CAPM is also valid for a class of bivariate distributions spanning well beyond the symmetric ones. Making use of this finding, we now initiate an insurance pricing model, which imitates the CAPM in its implications and also contributes to the computational tractability of the economic weighted pricing functional Π_w.

To elucidate our main line of reasoning, we start assuming bivariate normality, then rewrite $\pi_w[Y]$ in the form $\mathbf{E}[Y] + \mathbf{Cov}[Y, w(Y)]/\mathbf{E}[w(Y)]$ and, finally, using equations (6.1) and (6.6), we obtain that

$$\begin{aligned} \Pi_w[X, Y] - \mathbf{E}[X] &= \frac{\mathbf{Cov}[X, Y]}{\mathbf{Var}[Y]} \frac{\mathbf{Cov}[Y, w(Y)]}{\mathbf{E}[w(Y)]} \\ &= \frac{\mathbf{Cov}[X, Y]}{\mathbf{Var}[Y]} \big(\pi_w[Y] - \mathbf{E}[Y]\big). \end{aligned} \tag{7.3}$$

Equation (7.3) implies that in order to calculate $\Pi_w[X, Y]$, the only serious problem is to know how to calculate $\pi_w[X]$, but we have already discussed the topic in Section 3. The remaining quantities, that are the mean, the variance, and the covariance on the right-hand side of equation (7.3), are parameters of the bivariate normal distribution and therefore are either known or estimated from data. As a corollary to equation (7.3), we have, for example, the following convenient formula for calculating the economic CTE pricing functional:

$$CTE_p[X, Y] = \mathbf{E}[X] + \frac{\mathbf{Cov}[X, Y]}{\mathbf{Var}[Y]} \big(CTE_p[Y] - \mathbf{E}[Y]\big).$$

Equation (7.3) implies that the ratio of $\Pi_w[X,Y]-\mathbf{E}[X]$ and $\pi_w[Y]-\mathbf{E}[Y]$ does not depend on the weight function w, which is in line with the CAPM idea and also elucidates the main idea of the weighted insurance pricing model. Generally, if the pair (X,Y) possesses a linear regression function

$$r_{X|Y}(y) = C(F_{X,Y})\big(y-\mathbf{E}[Y]\big), \tag{7.4}$$

then we have that

$$\Pi_w[X,Y] = \mathbf{E}[X] + C(F_{X,Y})\big(\pi_w[Y]-\mathbf{E}[Y]\big). \tag{7.5}$$

We call equation (7.5) the weighted insurance pricing model (WIPM). It literally means that the economic premium due to the risk X in the pool of risks exceeds the net premium $\mathbf{E}[X]$ by a term which is proportional to the safety loading of the pool's overall risk Y.

In view of the fact that the validity of WIPM is based only on the linearity of the centered regression function $r_{X|Y}(y)$, the WIPM holds for a large number of bivariate distributions. For example, elliptical, Pareto and gamma distributions have linear regression functions. Pearson's bivariate distributions, of which the Pareto of the second kind is a member, are also characterized by linear regression functions (see Kotz et al., 2000). The already noted bivariate gamma distribution is a member of a large class of distributions constructed using the so-called trivariate reduction method (see Mardia, 1970). Necessary and sufficient conditions for these distributions to have linear regression functions have been discussed by Fix (1949).

The WIPM has another interesting application. Namely, suppose that we are interested in comparing the safety loadings of two risks, X^* and X^{**}. Using the WIPM equation, the ratio of the loadings $\Pi_w[X^*,Y]-\mathbf{E}[X^*]$ and $\Pi_w[X^{**},Y]-\mathbf{E}[X^{**}]$ is the ratio of the coefficients $C(F_{X^*,Y})$ and $C(F_{X^{**},Y})$. This ratio is free of the weight function w and can therefore be readily evaluated, given bivariate distributions of (X^*,Y) and (X^{**},Y).

8. Computing pricing functionals via weighted distributions

We have so far investigated the actuarial and economic weighted pricing functional π_w and Π_w, and demonstrated their tractability in a variety of situations. In this section we analyze the general economic pricing functional $\Pi_{v,w}$ that has been introduced at the end of Section 1.

The functional $\Pi_{v,w}$ is based on two weight functions, v and w, and it is therefore more difficult to separate the two from $F_{X,Y}$ than to separate just w from $F_{X,Y}$ as we have done in the case of Π_w. To make the task fruitful, throughout this section we therefore restrict ourselves to the case when $X_1, \dots, X_K$ are independent although not necessarily identically distributed risks, or losses, and discuss a technique for calculating $\Pi_{v,w}[X, Y]$ when $X = X_i$ and $Y = \sum_{k=1}^K c_k X_k$, where c_k are non-negative constants. The technique hinges on the observation that under the above assumptions the expectation $\mathbf{E}[v(X_i)w(Y)]$ can be written in the form

$$\mathbf{E}[v(X_i)w(Y)] = \mathbf{E}[v(X_i)]\mathbf{E}\left[w\left(c_i X_{i,v} + \sum_{k\neq i} c_k X_k\right)\right],$$

where the 'weighted' random variable $X_{i,v}$ is independent of the other ones and has the 'weighted' cdf

$$F_{X_{i,v}}(x) = \frac{\mathbf{E}[v(X_i)\mathbf{1}\{X_i \leq x\}]}{\mathbf{E}[v(X_i)]}. \tag{8.1}$$

Consequently, we have the formula

$$\Pi_{v,w}[X_i, Y] = \frac{\mathbf{E}[v(X_i)]}{\mathbf{E}[w(Y)]}\mathbf{E}\left[w\left(c_i X_{i,v} + \sum_{k\neq i} c_k X_k\right)\right], \tag{8.2}$$

and thus the computability of $\Pi_{v,w}[X_i, Y]$ mainly hinges on our successful determination of the cdf of the random variable $X_{i,v}$ which we have discussed in detail in Section 3.

To illustrate how equation (8.2) works in special cases, let $v(x) = x^\nu$ and $w(x) = \mathbf{1}\{x \geq t\}$ for some fixed $c > 0$ and $t > 0$. Then from the equation we have that

$$\mathbf{E}\left[X_i^\nu \mid \sum_{k=1}^K c_k X_k > t\right] = \mathbf{E}[X_i^\nu] \frac{\bar{F}_{c_i X_{i,\nu} + \sum_{k\neq i}^K c_k X_k}(t)}{\bar{F}_{\sum_{k=1}^K c_k X_k}(t)}, \tag{8.3}$$

where the size-biased (see Patil and Rao, 1978) random variable $X_{i,\nu}$ is independent of $X_1, \dots, X_K$ and has the 'weighted' cdf defined by equation (8.1) with $v(x) = x^\nu$. Equation (8.3) has been utilized by Furman and Zitikis (2008a,b), where we also find other references dealing with the equation.

Given distributions of the random variables $X_1, \dots, X_K$, we can calculate the νth moment of X_k as well as the ddf of $\sum_{k=1}^K c_k X_k$ using standard techniques. The ddf of $c_i X_{i,\nu} + \sum_{k\neq i}^K c_k X_k$ can also be calculated using standard techniques, provided that we know the distribution of $X_{i,\nu}$, which we have discussed in Section 3. Note in this regard that equation (8.3) and LEF related considerations in Section 3 provide laconic proofs of results by Furman and Landsman (2005) (see, also, Hürlimann, 2001).

Although equation (8.2) is valid for independent risks, it can be applied more generally, when the risks possess certain dependence structures, such as the one in the background economy model (see, e.g., Gollier and Pratt, 1996; Heaton and Lucas, 2000; Tsanakas, 2008). Specifically, assume that there is a background risk Z_0 and also independent but not necessarily identically distributed individual risks $Z_1, \dots, Z_K$. Assume that all the risks are independent. The background economy model implies that the individual risks $Z_1, \dots, Z_K$ have been contaminated by the background risk Z_0 and the following random variables $X_i = \rho_i Z_0 + Z_i$ have been observed, where ρ_i is the share of contamination on the individual risk Z_i by the background risk Z_0. Hence, even though we are dealing with the pool $\{X_1, \dots, X_K\}$ of dependent risks, under the above assumptions the overall portfolio risk $Y = \sum_{k=1}^K c_k X_k$ is nevertheless the linear combination $\sum_{k=0}^K d_k Z_k$ of independent risks $Z_0, Z_1, \dots, Z_K$, where $d_0 = \sum_{k=1}^K c_k \rho_k$ and $d_i = c_i$ for all $1 \le i \le K$. When $v(x) = x$, then we have

$$\Pi_w[X_i, Y] = \rho_i \Pi_w\left[Z_0, \sum_{k=0}^K d_k Z_k\right] + \Pi_w\left[Z_i, \sum_{k=0}^K d_k Z_k\right]. \tag{8.4}$$

Since $Z_0, Z_1, \dots, Z_K$ are independent, the two values of the pricing functional Π_w on the right-hand side of equation (8.4) can, for example, be handled using equation (8.2).

9. Summary

In this paper we have explored the role of weighted distributions and the encompassing nature of weighted pricing functionals in insurance and finance. Many well known actuarial and economic pricing functionals have been shown to be special cases of weighted pricing functionals. Moreover, via a straightforward and natural reformulation of weighted pricing functionals in terms of covariances, we have established a link between economic weighted pricing functionals and the celebrated capital asset pricing model, which has in turn inspired us to suggest a weighted insurance pricing model. Various techniques of computation of actuarial and economic weighted pricing functionals have been discussed in detail and illustrated on a number of specific pricing functionals and parametric families of distributions.

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