ASTAM Formula Sheet

$P(z)$ denotes the probability generating function. $M(z)$ denotes the moment generating function.

Q_α denotes the α-quantile of a distribution, also known as the α-Value at Risk

$\mathrm{ES}_\alpha[X]$ denotes the α-Expected Shortfall of X. This is also known as the $\alpha-$TailVaR, or the α-CTE.

The distribution function of the standard normal distribution is denoted $\Phi(x)$. The probability density function of the standard normal distribution is denoted $\phi(x)$.

The q-quantile of the standard normal distribution is denoted z_q, that is $\Phi\big(z_q\big) = q$.

For counting distributions, p_k denotes the probability function.

Continuous distributions

Pareto(α, θ) Distribution

$$f(x) = \frac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}, \qquad F(x) = 1 - \left(\frac{\theta}{\theta+x}\right)^\alpha,$$

$$\mathrm{E}[X] = \frac{\theta}{\alpha-1}, \quad \alpha > 1, \qquad \mathrm{Var}[X] = \left(\frac{\theta}{\alpha-1}\right)^2 \frac{\alpha}{\alpha-2}, \quad \alpha > 2,$$

$$\mathrm{E}\left[X \wedge x\right] = \frac{\theta}{\alpha-1}\left(1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha-1}\right), \qquad \alpha > 1,$$

$$\mathrm{E}\left[X^k\right] = \frac{k!\,\theta^k}{(\alpha-1)(\alpha-2)...(\alpha-k)}, \quad k = 1, 2, \cdots, \quad \alpha > k,$$

$$\mathrm{E}[X - Q|X > Q] = \frac{\theta+Q}{\alpha-1}, \quad \alpha > 1.$$

Lognormal(μ, σ) Distribution

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\log x - \mu}{\sigma}\right)^2\right\}, \qquad F(x) = \Phi\left(\frac{\log x - \mu}{\sigma}\right),$$

$$\mathrm{E}[X] = e^{\mu+\sigma^2/2}, \qquad \mathrm{Var}[X] = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right),$$

$$\mathrm{E}\left[X^k\right] = e^{k\mu+k^2\sigma^2/2},$$

$$\mathrm{E}\left[(X \wedge x)^k\right] = \exp\left(k\mu + \frac{1}{2}k^2\sigma^2\right)\Phi\left(\frac{\log x - \mu - k\sigma^2}{\sigma}\right) + x^k\left(1 - \Phi\left(\frac{\log x - \mu}{\sigma}\right)\right),$$

$$\mathrm{ES}_\alpha[X] = \frac{e^{\mu+\sigma^2/2}}{1-\alpha}\,\Phi\big(z_{1-\alpha} + \sigma\big).$$

Exponential(θ) Distribution

$$f(x) = \frac{e^{-x/\theta}}{\theta}, \qquad F(x) = 1 - e^{-x/\theta},$$

$$\mathrm{E}[X] = \theta, \qquad \mathrm{Var}[X] = \theta^2,$$

$$\mathrm{E}\left[X^k\right] = k!\,\theta^k,\ k = 1, 2, \cdots,$$

$$\mathrm{E}\left[X \wedge x\right] = \theta\left(1 - e^{-x/\theta}\right),$$

$$\mathrm{E}[X - Q | X > Q] = \theta,$$

$$M_X(t) = (1 - t\theta)^{-1}, \quad t < \frac{1}{\theta}.$$

Gamma(α, θ) Distribution

$$f(x) = \frac{x^{\alpha-1}\,e^{-x/\theta}}{\theta^\alpha\,\Gamma(\alpha)},$$

$$\mathrm{E}[X] = \alpha\,\theta, \qquad \mathrm{Var}[X] = \alpha\,\theta^2,$$

$$\mathrm{E}\left[X^k\right] = \alpha(\alpha+1)...(\alpha+k-1)\theta^k, \qquad k = 1, 2, \cdots,$$

$$M_X(t) = (1 - t\theta)^{-\alpha}, \quad t < \frac{1}{\theta}.$$

Chi-squared(ν) Distribution

Gamma distribution with $\alpha = \nu/2$ and $\theta = 2$.
$\nu \in \mathbb{N}^+$ is the degrees of freedom parameter.

$$\mathrm{E}[X] = \nu, \qquad \mathrm{Var}[X] = 2\nu$$

$$M_X(t) = (1-2t)^{-\nu/2}, \quad t < \frac{1}{2}.$$

Beta(a, b) Distribution

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, \qquad 0 < x < 1,$$

$$\mathrm{E}[X] = \frac{a}{a+b}, \qquad \mathrm{Var}[X] = \frac{a\,b}{(a+b)^2(a+b+1)}.$$

Normal(μ, σ^2) Distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}, \qquad F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right),$$

$$\mathrm{E}[X] = \mu, \qquad \mathrm{Var}[X] = \sigma^2,$$

$$\mathrm{ES}_\alpha[X] = \mu + \frac{\sigma}{1-\alpha}\phi\left(z_\alpha\right),$$

$$M_X(t) = e^{t\mu + t^2\sigma^2/2}.$$

Weibull(θ, τ) Distribution

$$f(x) = \frac{\tau x^{\tau-1}}{\theta^\tau} e^{-\left(\frac{x}{\theta}\right)^\tau}, \qquad F(x) = 1 - e^{-(x/\theta)^\tau},$$

$$\mathrm{E}[X] = \theta\,\Gamma\left(1+\frac{1}{\tau}\right), \qquad \mathrm{Var}[X] = \theta^2\left(\Gamma\left(1+\frac{2}{\tau}\right) - \left(\Gamma\left(1+\frac{1}{\tau}\right)\right)^2\right).$$

Counting Distributions

Poisson(λ) Distribution

$$p_k = \frac{\lambda^k\, e^{-\lambda}}{k!}, \qquad a = 0, \quad b = \lambda,$$

$$\mathrm{E}[N] = \lambda, \qquad \mathrm{Var}[N] = \lambda,$$

$$P_N(z) = \exp\{\lambda(z-1)\}, \qquad M_N(z) = \exp\{\lambda(e^z-1)\}.$$

Binomial(m,q) Distribution

$$p_k = \binom{m}{k} q^k(1-q)^{m-k}, \qquad a = -\frac{q}{1-q}, \quad b = \frac{(m+1)q}{1-q},$$

$$\mathrm{E}[N] = mq, \qquad \mathrm{Var}[N] = mq(1-q)$$

$$P_N(z) = \big(1+q(z-1)\big)^m, \qquad M_N(z) = \big(1+q(e^z-1)\big)^m.$$

Bernoulli(q) Distribution
Binomial Distribution with $m = 1$.

Negative Binomial(r,β) Distribution

$$p_0 = \left(\frac{1}{1+\beta}\right)^r, \qquad p_k = \frac{r(r+1)\cdots(r+k-1)}{k!}\left(\frac{\beta}{1+\beta}\right)^k\left(\frac{1}{1+\beta}\right)^r, \quad k = 1, 2, \ldots,$$

$$a = \frac{\beta}{1+\beta}, \quad b = \frac{(r-1)\beta}{1+\beta},$$

$$\mathrm{E}[N] = r\beta, \qquad \mathrm{Var}[N] = r\beta(1+\beta),$$

$$P_N(z) = \big(1 - \beta\,(z-1)\big)^{-r}, \qquad M_N(z) = \big(1 - \beta\,(e^z - 1)\big)^{-r}.$$

Geometric Distribution
Negative Binomial Distribution with $r = 1$;

Recursions for Compound Distributions

For $N \sim (a, b, 0)$: $$f_S(x) = \frac{\sum_{y=1}^{x \wedge m}\left(a + \frac{by}{x}\right) f_X(y)\, f_S(x-y)}{1 - a f_X(0)}$$

For $N \sim (a, b, 1)$: $$f_S(x) = \frac{\big(p_1 - (a+b)p_0\big) f_X(x) + \sum_{y=1}^{x \wedge m}\left(a + \frac{by}{x}\right) f_X(y)\, f_S(x-y)}{1 - a f_X(0)}$$

Empirical Bayes Credibility

Empirical Bayes parameter estimation for the Bühlmann model:

$$\bar{X}_i = \frac{\sum\limits_{j=1}^{n} X_{ij}}{n}, \qquad \bar{X} = \frac{\sum\limits_{i=1}^{r} \bar{X}_i}{r},$$

$$\hat{v} = \frac{1}{r(n-1)} \sum_{i=1}^{r} \sum_{j=1}^{n} \left(X_{ij} - \bar{X}_i \right)^2, \qquad \hat{a} = \frac{1}{r-1} \sum_{i=1}^{r} \left(\bar{X}_i - \bar{X} \right)^2 - \frac{\hat{v}}{n},$$

$$\hat{\mu} = \bar{X}, \qquad \hat{Z}_i = \frac{n}{n + \hat{v}/\hat{a}}.$$

Empirical Bayes parameter estimation for the Bühlmann-Straub model:

$$m_i = \sum_{j=1}^{n_i} m_{ij}, \qquad m = \sum_{i=1}^{r} m_i, \qquad \bar{X}_i = \frac{\sum\limits_{j=1}^{n_i} m_{ij} X_{ij}}{m_i}, \quad \bar{X} = \frac{\sum\limits_{i=1}^{r} m_i \bar{X}_i}{m},$$

$$\hat{v} = \frac{\sum\limits_{i=1}^{r} \sum\limits_{j=1}^{n_i} m_{ij} \left(X_{ij} - \bar{X}_i \right)^2}{\sum\limits_{i=1}^{r} (n_i - 1)}, \qquad \hat{a} = \frac{\sum\limits_{i=1}^{r} m_i \left(\bar{X}_i - \bar{X} \right)^2 - \hat{v}(r-1)}{m - \frac{1}{m} \sum\limits_{i=1}^{r} m_i^2}$$

$$\hat{Z}_i = \frac{m_i}{m_i + \hat{v}/\hat{a}}, \quad \hat{\mu} = \frac{\sum\limits_{i=1}^{r} \hat{Z}_i \bar{X}_i}{\sum\limits_{i=1}^{r} \hat{Z}_i}.$$

Extreme Value Theory

The Gumbel Distribution

$$F(x) = \exp\left\{-\exp\left(-\frac{x-\mu}{\theta}\right)\right\}, \quad \theta > 0.$$

The Fréchet Distribution

$$F(x) = \exp\left\{-\left(\frac{x-\mu}{\theta}\right)^{-\alpha}\right\}, \quad x > \mu;\ \alpha > 0;\ \theta > 0.$$

The Weibull EV Distribution

$$F(x) = \exp\left\{-\left(\frac{\mu-x}{\theta}\right)^{\tau}\right\}, \quad x < \mu;\ \tau > 0;\ \theta > 0.$$

The Generalized Extreme Value Distribution

The distribution function is $H(x)$ where

$$H_\xi(x) = \begin{cases} \exp\left(-(1+\xi x)^{-\frac{1}{\xi}}\right) & \xi \neq 0,\ \xi x > -1, \\ \exp\left(-e^{-x}\right) & \xi = 0. \end{cases}$$

The GEV can be adjusted for scale and location, to give $H_{\xi,\mu,\theta}$ where

$$H_{\xi,\mu,\theta}(x) = \begin{cases} \exp\left(-(1+\xi(x-\mu)/\theta)^{-\frac{1}{\xi}}\right) & \xi \neq 0,\ (1+\xi\,(x-\mu)/\theta) > 0, \\ \exp\left(-e^{-(x-\mu)/\theta}\right) & \xi = 0, \end{cases}$$

The Generalized Pareto Distribution (GPD)

$$G(x) = \begin{cases} 1-\left(1+\xi x/\beta\right)^{-\frac{1}{\xi}} & \xi \neq 0 \\ 1-e^{-\frac{x}{\beta}} & \xi = 0 \end{cases}$$

$$\mathrm{E}[X] = \frac{\beta}{1-\xi} \quad \text{for } 0 < \xi < 1$$

If $X - d | X > d \sim \mathrm{GPD}(\xi, \beta)$ then

$$Q_\alpha = d + \frac{\beta}{\xi}\left(\left(\frac{S_X(d)}{1-\alpha}\right)^{\xi} - 1\right)$$

$$\mathrm{ES}_\alpha = \frac{1}{1-\xi}\left(Q_\alpha + \beta - \xi\, d\right)$$

The Hill Estimator

$$\hat{\alpha}_j^H = \left(\sum_{k=j}^{n} \frac{\log(x_{(k)})}{n-j+1} - \log(x_{(j)}) \right)^{-1}$$

(Note: the version of the Hill estimator in QERM is incorrect.)

Outstanding Claims Reserves

Functions of development factors

If all claims are settled by the end of DY J, then

for $j = 0, 1, ..., J-1,\ \lambda_j = \prod_{k=j}^{J-1} f_j;\ \ \lambda_J = 1$

for $j = 0, 1, ..., J, \qquad \beta_j = \dfrac{1}{\lambda_j}; \qquad \gamma_j = \beta_j - \beta_{j-1}$

Tests for correlated development factors

$$T_j = r_j \sqrt{\frac{\nu_j}{1 - r_j^2}}, \qquad T = \frac{\sum_{\nu_j \geq 3} T_j(\nu_j - 2)/\nu_j}{\sum_{\nu_j \geq 3} (\nu_j - 2)/\nu_j}$$

Under the null hypothesis, $T \approx N(0, v)$ where $v = \dfrac{1}{\sum_{\nu_j \geq 3} ((\nu_j - 2)/\nu_j)}$.

Test for calendar year effects

$Z_k = \min(S_k, L_k);\ Z = \sum_{k=1}^{I-1} Z_k$.

Under the null hypothesis, approximately:

$$\mathrm{E}[Z_k] = \frac{n_k}{2} - \binom{n_k - 1}{m_k} \frac{n_k}{2^{n_k}}$$

$$\mathrm{Var}[Z_k] = \frac{n_k(n_k - 1)}{4} - \binom{n_k - 1}{m_k} \frac{n_k(n_k - 1)}{2^{n_k}} + \mathrm{E}[Z_k] - \mathrm{E}[Z_k]^2$$

$$\mathrm{E}[Z] = \sum_{k=1}^{I-1} \mathrm{E}[Z_k]; \quad \mathrm{Var}[Z] = \sum_{k=1}^{I-1} \mathrm{Var}[Z_k]; \quad \frac{Z - \mathrm{E}[Z]}{\sqrt{\mathrm{Var}[Z]}} \sim N(0, 1)$$

The Bühlmann-Straub Model of Outstanding Claims

$$m_i = \hat{\beta}_{I-i}; \quad s_i^2 = \frac{1}{I-i} \sum_{j=0}^{I-i} \hat{\gamma}_j \left(\frac{X_{i,j}}{\hat{\gamma}_j} - \widehat{C}_{i,J} \right)^2$$

$$m = \sum_{i=0}^{I} m_i; \qquad \overline{C} = \frac{\sum_{i=0}^{I} C_{i,I-i}}{m}$$

$$\hat{v} = \frac{1}{I} \sum_{i=0}^{I-1} s_i^2; \qquad \hat{a} = \frac{\sum_{i=0}^{I} m_i \left(\widehat{C}_{i,J} - \overline{C} \right)^2 - I\hat{v}}{m - \frac{1}{m} \sum_{i=0}^{I} m_i^2}$$

$$Z_i = \frac{\hat{\beta}_{I-i}}{\hat{\beta}_{I-i} + \hat{v}/\hat{a}} \qquad \hat{\mu} = \frac{\sum_{i=0}^{I} Z_i \, \widehat{C}_{i,J}}{\sum_{i=0}^{I} Z_i}$$

Mack's Model

$$\hat{\sigma}_j^2 = \frac{1}{I-1-j} \sum_{i=0}^{I-1-j} C_{i,j} \left(f_{i,j} - \hat{f}_j \right)^2 \qquad \text{for } j \le I-2$$

$$\mathrm{Var}\big[C_{i,J} - C_{i,I-i} | C_{i,I-i}\big] \approx \widehat{C}_{i,J}^2 \sum_{j=I-i}^{J-1} \frac{\hat{\sigma}_j^2}{\hat{f}_j^2 \, \widehat{C}_{i,j}}.$$

$$\left(\widehat{C}_{i,J} - \mathrm{E}\left[C_{i,J} | \mathcal{D}_I \right] \right)^2 \approx \widehat{C}_{i,J}^2 \sum_{j=I-i}^{J-1} \frac{\hat{\sigma}_j^2}{\hat{f}_j^2 \, S_j}$$

$$\mathrm{MSEP}\left(\widehat{R} \,\middle|\, \mathcal{D}_I \right) \approx \sum_{i=1}^{I} \widehat{C}_{i,J}^2 \sum_{j=I-i}^{J-1} \frac{\hat{\sigma}_j^2}{\hat{f}_j^2} \left(\frac{1}{\widehat{C}_{i,j}} + \frac{1}{S_j} \right) + 2 \sum_{i=1}^{I-1} \widehat{C}_{i,J} \left(\sum_{j=I-i}^{J-1} \frac{\hat{\sigma}_j^2}{\hat{f}_j^2 \, S_j} \right) \left(\sum_{l=i+1}^{I} \widehat{C}_{l,J} \right)$$