Optimal Investment Allocation in a Jump Diffusion Risk Model with Investment: A Numerical Analysis of Several Examples

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Abstract

This article pertains to the optimal asset allocation of surplus from an insurance company model. The insurance company is represented by a compound Poisson risk process which is perturbed by diffusion and has investments. The investments are in both risky and risk-free types of assets similar to stocks/real estate and bonds. The insurance company can borrow at a constant interest rate in the event of a negative surplus. Numerical analysis appears to show that an optimal asset allocation range can be estimated for certain parameters and compared with insurance data. Using a conservative method to minimize the probability of ruin, a reasonable optimal asset allocation range for a typical insurance company is about 4.5 to 8.2 percent invested in risky stock/real estate assets. An inequality and the exact solution are obtained for the pure diffusion equation. In addition, the asymptotic form of the ruin probability is shown to be a power function.

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1 Introduction

The original classical risk model was first introduced by Lundberg near the beginning of the 20th century. Lundberg considered the risk aspect as a Poisson process for the number of claims against an insurance company and as a continuous process of probability between the insurance company and the policyholders. Today this is considered, "collective risk theory." Lundberg (1919, 1926) presented the ruin probability function for this continuous probability process and calculated results for the ruin probability function.

One of the research areas that have been developed throughout the years from studies on the Cramer-Lundberg and other insurance models is the optimization problem of ruin or the minimization of the probability of ruin. Recently, Browne (1995) considered the optimal ruin solution to the classical insurance risk model with investments in a risky asset that followed geometric Brownian motion and with a stochastic cash flow. He obtained policies that maximize the utility from terminal wealth when using an exponential utility function. Hipp and Plum (2000) considered the optimal investment for insurers to minimize the probability of ruin with risks modeled as a compound Poison Process, the Cramer-Lundberg model. They found an optimal strategy when the claim size followed an exponential distribution and the investment was in a market index which was modeled by geometric Brownian motion. Deelstra, Grasselli and Koehl (2000) studied the stock-bond optimal investment in a continuous-time framework where the interest rates followed CIR dynamics and found a closed-form solution by assuming the completeness of the market. They used the Cox-Huang methodology because it resulted in a simpler PDE than the non-linear Hamilton-Jacobi-Bellman PDE.

Even more recently, others have studied the minimization of ruin for insurance company models. Liu and Yang (2004) generalized the model and problem in Hipp and Plum (2000) by including a risk-free bond type asset. They studied optimal ruin strategies when the claim size followed exponential, Gamma and Pareto distributions. However, their model did not represent a jump diffusion process. Yang and Zhang (2005) studied optimal investment policies of an insurance company model by a jump diffusion risk process with bond and stock type investments and correlated Weiner processes. They obtained a closed form expression for the optimal policy when the utility function was exponential. Also, they provided numerical results that minimized the ruin probability for various claim size distributions when the amount invested in the stock type asset was a function of time. Schmidli (2005) considered the Cramer-Lundberg model with investments in a risky asset modeled by geometric Brownian motion and proportional reinsurance. He showed the numerical optimal strategies for light and heavy tailed claim size distributions.

In the NAAJ published discussion by Yang (2006), it was suggested that an alternative model to Cai and Xu (2006) be investigated. The closed form solutions to the alternative model were published by Gerber and Yang (2007) using only a risky-free investment and allowing the insurance company to borrow in the event of negative surplus. In this article, we investigated the solution to the full alternative model with both risky and risk-free investments. We numerically solved the full model with no borrowing and studied various aspects of the model. Further, we used the Pure Diffusion Models both with and without borrowing to make a general statement on the effect of borrowing. This general borrowing statement was used to estimate the effect that borrowing might have on the full model and was incorporated into the analysis.

The presentation of this article is in the general procedure that was used to derive the results. The development of the Jump Diffusion Model is reviewed with the diffusion and Brownian motion of the risky investment correlated. The risk-free and risky investments follow the usual model of bonds and stocks, respectively. Once the general risk model is presented, the integro-differential equation is shown. The full third order ordinary differential equation of the model is developed. An inequality for the variables is proved using the Pure Diffusion Model. Numerical solutions to the Pure Diffusion Models for optimal asset allocation are presented and compared. The full third order ODE without borrowing is numerically solved. Then, the full third order ODE with borrowing is estimated using the difference between the Pure Diffusion Models. A rule-of-thumb range for the amount that an average insurance company could reasonably have invested in risky assets such as stocks and real estate is formulated. The rule-of-thumb range is compared to real data from 2002 Best's Insurance Reports Life/Health. Finally, the asymptotic form of the ruin probability for the full third order ODE is derived and shown to be a power function.

2 Risk Model with Correlated Variables

In this section we develop the surplus risk model used in this article. First, we let all processes and random variables be on the filtered probability space (Ω, F, F_t, P) with $t \geq 0$ and satisfying the usual conditions such as F is right continuous. Now, the insurance company's basis surplus is driven by four factors: the initial surplus at time 0, the aggregate amount of the premiums up to time t, the aggregate amount of claims paid out up to time t denoted by Y_t, and the claim frequency process denoted by N_t. Y_t is modeled by the random sum of claim amounts $X_1 + X_2 + ... + X_{N(t)}$ where the X's are i.i.d. random variables. The insurance company's basis surplus model is

$$U_t = u + ct - Y_t, \quad t \geq 0$$

where u is the initial surplus, c is the claim rate which may equal $\lambda E(X)(1+\theta)$ and θ is the relative security loading. Y_t is a compound Poisson process with parameter λ and a cumulative distribution function for individual claims defined as $G(x) = Pr(X \leq x)$. Y_t, U_t and N_t are continuous-time stochastic variables. The initial surplus and the amount of aggregate premiums are deterministic variables.

The compound Poisson surplus process perturbed by diffusion is an extension of the basic surplus model. Dufresne and Gerber (1991) were instrumental to the introduction of the jump diffusion model which accounts for some market uncertainties in the application of the surplus risk model. The jump diffusion risk model is found by adding an independent Wiener process to the surplus risk model

$$U_t = u + ct - Y_t + \sigma_1 B_t^{(1)} \ , \quad t \geq 0$$

where σ_1 is a constant representing the diffusion volatility. $B_t^{(1)}$ is a standard Weiner process independent of the compound Poisson process Y_t with $B_t^{(1)} \sim N(0,t), \sigma B_t^{(1)} \sim N(0, \sigma^2 t)$. Here $N(\mu, \sigma^2)$ is defined as a normal distribution with mean μ and variance σ^2.

We assume that the risky and risk-free investments follow the usual stock and bond models with the following stochastic differential equations

$$dS_t = \mu S_t dt + \sigma_2 S_t dB_t^{(2)}$$
$$dR_t = rR_t dt$$

where R_t denotes the price of the bond at time t and S_t denotes the price of the stock at time t. $B_t^{(2)}$ is a standard Brownian motion. The stock price follows the usual geometric Brownian motion model and the bond price follows the deterministic process with an exponential growth model. In addition, we assume that $B_t^{(1)}$ is correlated to the Brownian motion of the stock price by $dB_t^{(1)} dB_t^{(2)} = \rho dt$.

The surplus model of the insurance company is denoted by X_t and represents a stationary Markov process. The combination of risky and risk-free investments is allocated proportionally by α in a stock or risky investment and 1 - α in a bond or risk-free investment with the assumption of continuous rebalancing to maintain the relative portions. Although perfect continuous rebalancing is impossible, the $B_t^{(1)}$ term adds some robustness to the model which should make the intended results appear indistinguishable from results that have slight aberrations of imperfect continuous rebalancing of the investments. A similar development can be found in Yang (2006).

3 Ruin Probability

Ruin is defined as when the surplus goes below a specified negative level or zero, given an initial surplus. Ruin may occur due to the oscillation from the Brownian motion in $B_t^{(1)}$ or the risky investment. Ruin may occur due to a claim. After ruin, it is assumed that the insurance company no longer exists in the form of the proposed model. The ruin probability is defined as

$$\psi = Pr(T < \infty | X_0 = u)$$

where T is the stopping time. The stopping time is defined as follows: $T = inf(t : X_t < 0)$ and, if $X_t \geq 0 \, \forall \, t \geq 0$, the stopping time is infinity. The stopping time is defined with slight changes for a specified negative surplus level.

4 Differential Equation and Inequality

The Jump Diffusion Risk Model for an insurance company is the following

$$dX_t = cdt + \alpha\mu X_t dt + (1-\alpha)rX_t dt + \sigma_1\, dB_t^{(1)} + \alpha\sigma_2\, X_t dB_t^{(2)} - dY_t \tag{1}$$

with the initial surplus given by

$$X_0 = u$$

where the Brownian motions are correlated and it represents a fundamentally "incomplete market." The desired integro-differential equation of the insurance company model can be found by introducing an optimization problem that assumes the insurance company pays dividends at a continuous rate. Next, take the function that is to be optimized and use the Bellman's dynamic programming principle to obtain the Hamilton-Jacobi-Bellman equation with its usual second order for stochastic problems. Finally, optimize the dividend payout by choosing the payout policy that minimizes the ruin probability which is no dividends and use the verification theorem. The resulting integro-differential equation is

$$\begin{gathered}\frac{1}{2}(\sigma_1^2 + 2\rho\alpha\sigma_1\sigma_2 u + \alpha^2\sigma_2^2 u^2)\psi^{''} + \\ ([\alpha\mu + (1-\alpha)r]u + c)\psi^{'} + \lambda E[\psi(u-x) - \psi(u)] = 0\end{gathered} \tag{2}$$

A similar integro-differential equation can be found in Gerber and Yang (2007).

Using the cdf of the claims as an exponential function $G(x) = 1 - e^{-\beta x}$ with $x > 0$ and $\beta > 0$, the integro-differential equation is

$$\begin{gathered}\frac{1}{2}(\sigma_1^2 + 2\rho\alpha\sigma_1\sigma_2 u + \alpha^2\sigma_2^2 u^2)\psi^{''} + ([\alpha\mu + (1-\alpha)r]u + c)\psi^{'} + \\ \lambda e^{-\beta(u+c/\tau)} - \lambda\psi(u) + \lambda\beta\int_0^{u+c/\tau}\psi(u-x)e^{-\beta x}dx = 0\end{gathered}$$

After some manipulation and calculations, we find the desired full ordinary differential equation is

$$\begin{gathered}\frac{1}{2}[\sigma_1^2 + 2\rho\alpha\sigma_1\sigma_2 u + \alpha^2\sigma_2^2 u^2]\psi^{'''}(u) + \\ [\frac{1}{2}\beta\alpha^2\sigma_2^2 u^2 + (\alpha\mu + (1-\alpha)r + \alpha^2\sigma_2^2 + \rho\alpha\sigma_1\sigma_2\beta)u + c + \rho\alpha\sigma_1\sigma_2 + \frac{1}{2}\beta\sigma_1^2]\psi^{''}(u) + \\ [(\alpha\mu + (1-\alpha)r)\beta u + c\beta + \alpha\mu + (1-\alpha)r - \lambda]\psi^{'}(u) = 0\end{gathered} \tag{3}$$

with the boundary conditions

$$\psi(0) = 1,\;\; \psi(0_+)^{''} = 0,\;\; \psi(\infty) = 0.$$

In order to obtain an inequality for the variables of this differential equation that means $\psi(u) < 1$ with $u \geq 0$, we consider the Pure Diffusion Model with $\lambda = 0$.

Theorem 1:

Given

1. *The process model is the Pure Diffusion Model of the Jump Diffusion Risk Model;*
2. *The Jump Diffusion Risk Model is a stationary Markov process;*
3. *The solution has the boundary conditions $\psi(0) = 1$ and $\psi(\infty) = 0$;*
4. *The claims follow an exponential distribution;*

Then, an inequality relating the variables is

$$[\alpha\mu + (1-\alpha)r]/[\alpha^2\sigma_2^2] \;\geq\; 1 \tag{4}$$

which guarantees $\psi(u) < 1$.

Proof

The differential equation for the Pure Diffusion Model is

$$\frac{1}{2}(\sigma_1^2 + 2\rho\alpha\sigma_1\sigma_2 u + \alpha^2\sigma_2^2 u^2)\psi^{''} + ([\alpha\mu + (1-\alpha)r]u + c)\psi^{'} = 0. \tag{5}$$

with $\lambda = 0$. The boundary conditions are

$$\psi(0) = 1$$

$$\psi(\infty) = 0. \tag{6}$$

The solution to this differential equation is

$$\psi_{\lambda=0}(u) = \frac{A}{B}, \quad u \geq 0,$$

$$A = \int_u^\infty h(x)\; exp^{-J(x)} dx, \quad B = \int_0^\infty h(x)\; exp^{-J(x)} dx,$$

where

$$h(x) = (\sigma_1^2 + 2\rho\alpha\sigma_1\sigma_2 x + \alpha^2\sigma_2^2 x^2)^Q,$$

$$Q = [\alpha\mu + (1-\alpha)r]/[\alpha^2\sigma_2^2]$$

$$J(x) = (2\zeta)tan^{-1}([\alpha\sigma_2 x + \rho\sigma_1]/[\sigma_1\sqrt{1-\rho^2}]),$$

$$\zeta = [c\alpha\sigma_2 - \rho(\alpha\mu + (1-\alpha)r)\sigma_1]/[\alpha^2\sigma_1\sigma_2^2\sqrt{1-\rho^2}].$$

Substituting $v = tan^{-1}([\alpha\sigma_2 x + \rho\sigma_1]/[\sigma_1\sqrt{1-\rho^2}])$ into the integrals, we obtain

$$\psi_{\lambda=0}(u) = \frac{C}{D}, \quad u \geq 0,$$

$$C = \int_{tan^{-1}([\alpha\sigma_2 u+\rho\sigma_1]/[\sigma_1\sqrt{1-\rho^2}])}^{\pi/2} cos(v)^{2Q-2}\ exp^{-2\zeta v} dv,$$

$$D = \int_{tan^{-1}(\rho/\sqrt{1-\rho^2})}^{\pi/2} cos(v)^{2Q-2}\ exp^{-2\zeta v} dv$$

where the common constants have been canceled out of C and D. The integrands of C and D $\rightarrow \infty$ at $v = \pi/2$ if $2Q - 2 < 0$. Therefore for this model, we must have

$$[\alpha\mu + (1-\alpha)r]/[\alpha^2\sigma_2^2]\ \geq\ 1$$

which guarantees $\psi(u) < 1$. ■

Remark 1.1

The integral in equation (15) can be solved exactly for (2Q - 2) = (0, 1, 2,...) or (2Q - 2) $\in W$. For 2Q - 2 = 1, we find

$$C = -2\zeta/(\zeta^2+1)exp^{-2\zeta v}\ cos(v) + 1/(\zeta^2+1)exp^{-2\zeta v}\ sin(v)\ |_{tan^{-1}([\alpha\sigma_2 u+\rho\sigma_1]/[\sigma_1\sqrt{1-\rho^2}])}^{\pi/2}$$

for $u \geq 0$. The exact solution is very useful in finding the accuracy of any numerical solutions to the Pure Diffusion Models.

5 Numerical Analysis of Optimal Ruin Probability

We used Matlab to perform the simulations and the parameter values in examples 4.1 and 5.1 in Gerber and Yang (2007). The boundary value Matlab solver was a fourth order accurate finite difference subroutine. The simulations for the Pure Diffusion Model were checked with the exact solution at various values of alpha and found to have at least four significant digits of accuracy. The simulations for the Pure Diffusion Model with borrowing were checked with the exact solution at $\psi(0)$ for various values of alpha and found to have at least four significant digits of accuracy.

5.1 Pure Diffusion Model with no Borrowing

Here we consider the Pure Diffusion Model with no claims, $\lambda = 0$. We used a premium rate c = 2, a risky return rate $\mu = 0.2$, a correlation $\rho = 0.1$, a risky volatility $\sigma_2 = 0.3$ and a diffusion $\sigma_1 = 10$. In addition various values of asset allocation α, interest rates r and surpluses u were used. The range of interest rates was from 0.01 to $\mu = 0.2$, the value of mean return from the risky asset. The differential equation and boundary conditions are equations (5) and (6). The first several numerical solutions to the ODE for an interest rate of 0.1 are graphically displayed in figure 1.

There are five different bands in figure 1. The optimal alpha band is the alpha value that gives the smallest ruin probability for integral surpluses. The upper and lower 0.25 percent bands are the alpha values that are 0.25 percent greater in ruin probability

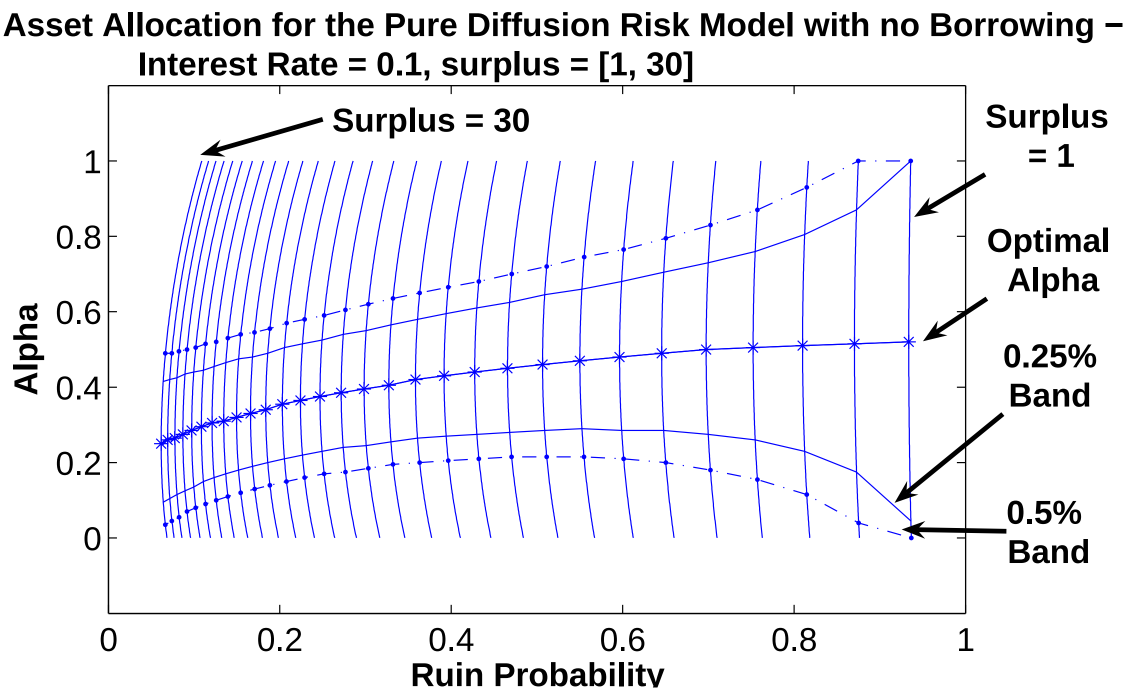

Figure 1: Alpha versus Ruin Probability - Pure Diffusion Model

relative to the optimal alpha. The upper and lower 0.5 percent bands are the alpha values that are 0.5 percent greater in ruin probability relative to the optimal alpha.

Some of the bands in figure 1 move in different directions. The optimal alpha starts at a little over 0.5 for a surplus of 1 and moves toward zero for a large surplus. The upper 0.25 and 0.5 percent bands start at one, move down, and back up to one for large values of surplus. The lower 0.25 and 0.5 percent bands start at or near zero, move up and back to zero for large surpluses. The 0.5 percent bands display some of the curvature of the integral surplus solutions. The 0.25 percent bands display how much one can change alpha and give up only 0.25 percent in ruin probability.

The 3-D surface in figure 2 shows how optimal alpha behaves with varying interest rate and surpluses. The 3-D surface in figure 2 has several interesting features. The surface has a plateau, colored red, near the interest rate of 0.01 to 0.04 and the surplus of 1 to 18. The surface goes to zero about the interest rate of 0.18 at low surpluses. At the surplus value of 60, the surface goes to zero about the interest rate of 0.15. Thus as the surpluses become larger, the surface tends toward zero at lower interest rates. The optimal alpha has a value of about 0.5 at a surplus of 60 with an interest rate of 0.01. Thus as the surpluses become larger, the surface tends to smaller values for low interest rates.

The 3-D surface in figure 3 shows how the maximum percentage alpha can change the ruin probability behaves with varying interest rate and surpluses. The 3-D surface in figure 3 has several interesting aspects. The general surface has a pseudo-saddle point at an interest rate of about 0.08 and a surplus of about 20. For the interest rate of 0.01, the largest maximum percentage alpha can change the ruin probability is about 17

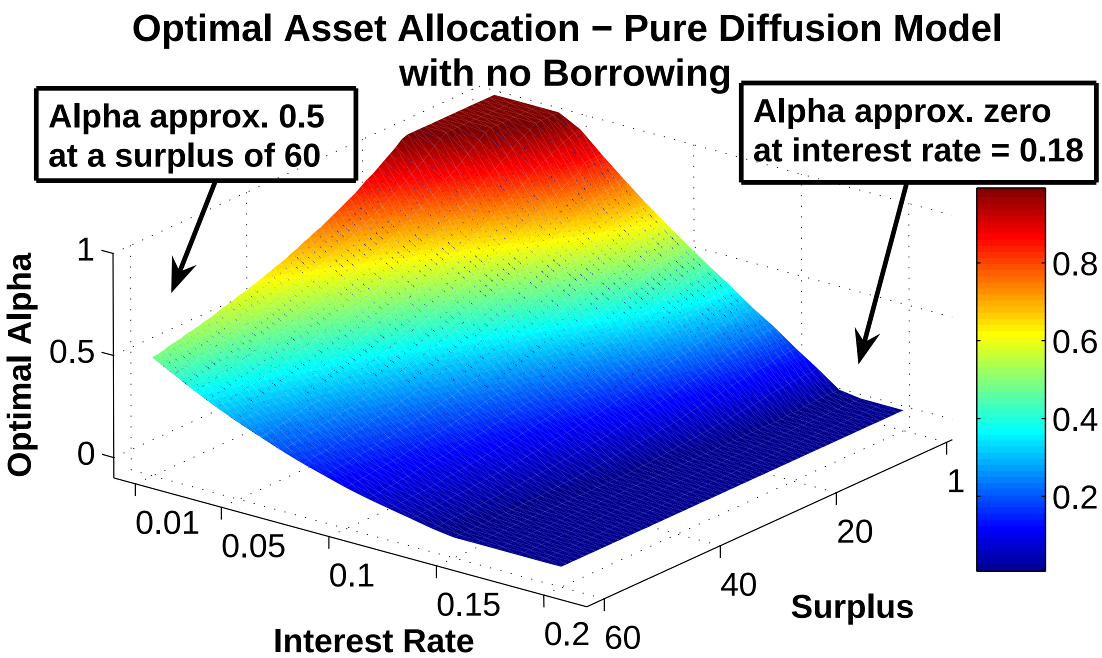

Figure 2: Optimal Alpha versus Interest Rate versus Surplus - Pure Diffusion Model

percent at the surplus of 17. For the interest rate of 0.2, the largest maximum percentage alpha can change the ruin probability is about 11 percent at the surplus of 17.

The contour plot in figure 4 shows how the maximum percentage alpha can change the ruin probability behaves with varying interest rate and surpluses. The contour plot illustrates some aspects that figure 3 does not easily show. There appears to be a small well from an interest rate of about 0.1 and a surplus of about 4 to an interest rate of about 0.04 and a surplus of about 52. The well appears to have an anomaly at an interest rate of about 0.08 and a surplus of about 21. If we take a slice of figure 3 at the interest rate of 0.08, the anomaly shows up as a dip in the curve where the derivative is undefined.

5.2 Pure Diffusion Model with Borrowing

The Pure Diffusion Model with borrowing has the system of ODE's as follows

$$\frac{1}{2}(\sigma_1^2 + 2\rho\alpha\sigma_1\sigma_2 u + \alpha^2\sigma_2^2 u^2)\psi'' + ([\alpha\mu + (1-\alpha)r]u + c)\psi' = 0, \tag{7}$$

for $u \geq 0$,

$$\frac{1}{2}\sigma_1^2\psi'' + (\tau u + c)\psi' = 0, \tag{8}$$

for $-c/\tau < u < 0$, with boundary conditions

$$\psi(-c/\tau) = 1, \;\; \psi(\infty) = 0.$$

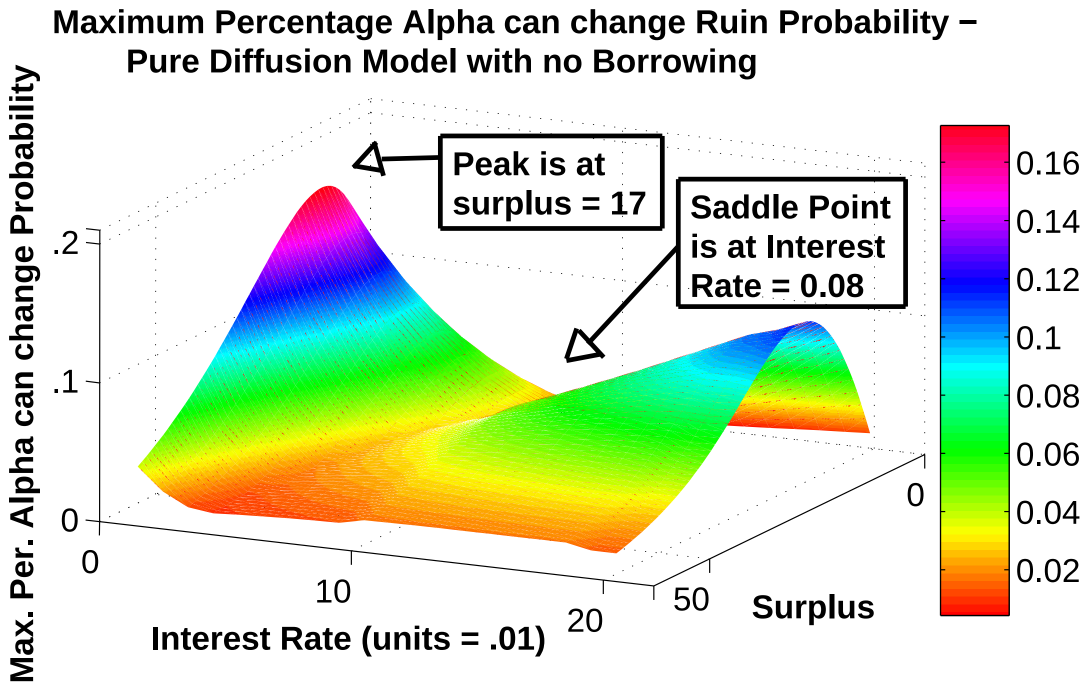

Figure 3: Maximum Percentage Alpha can change the Ruin Probability versus Interest Rate versus Surplus - Pure Diffusion Model

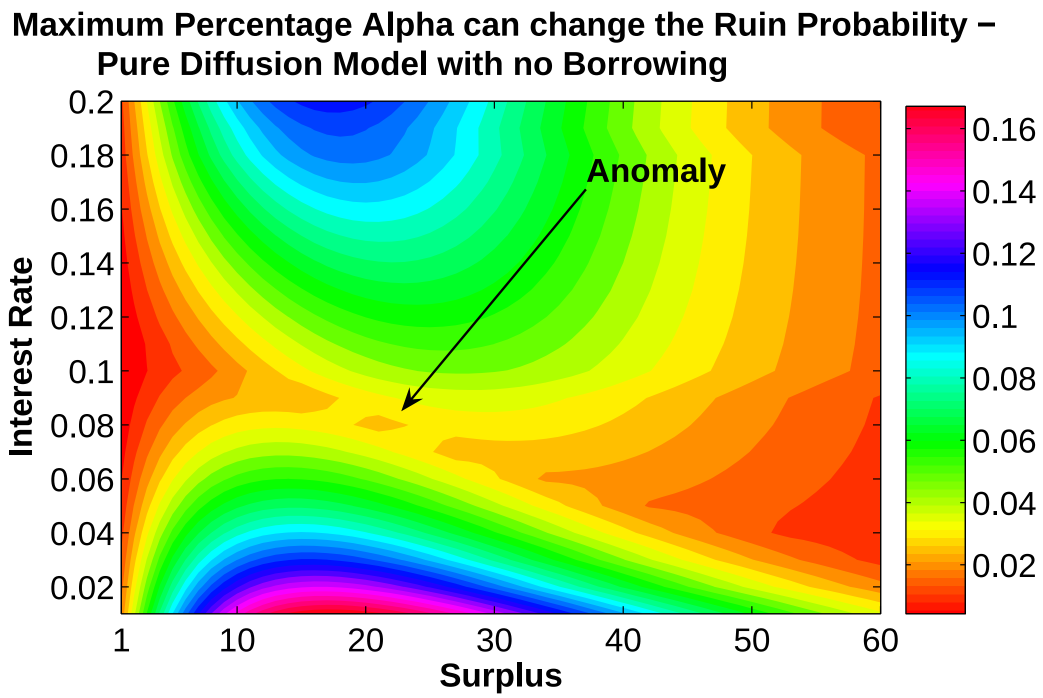

Figure 4: Contour Plot of Interest Rate versus Surplus for the Maximum Percentage Alpha can change the Ruin Probability - Pure Diffusion Model

For smoothness and continuity, there are the conditions

$$\psi(0_-) = \psi(0_+), \;\; \psi^{'}(0_-) = \psi^{'}(0_+)$$

at the interface between the two ODE's.

The parameter values were the same as with the Pure Diffusion Model with no borrowing except there was $\tau = 0.1$, the borrowing rate. The range of interest rates was from 0.01 to 0.2 as before. The range of interest rates means that we were assuming that the insurance company could get a return on their investments that were greater than the rates of borrowing, similar to a credit card company.

The 3-D surface in figure 5 shows how optimal alpha behaves with varying interest rate and surpluses for the Pure Diffusion Model with borrowing. The 3-D surface in figure 5 has similar features as the Pure Diffusion Model. A volume comparison of figures 5 and 2 shows that figure 5 has 4.8 percent more volume. Figure 5 and figure 2 are very similar.

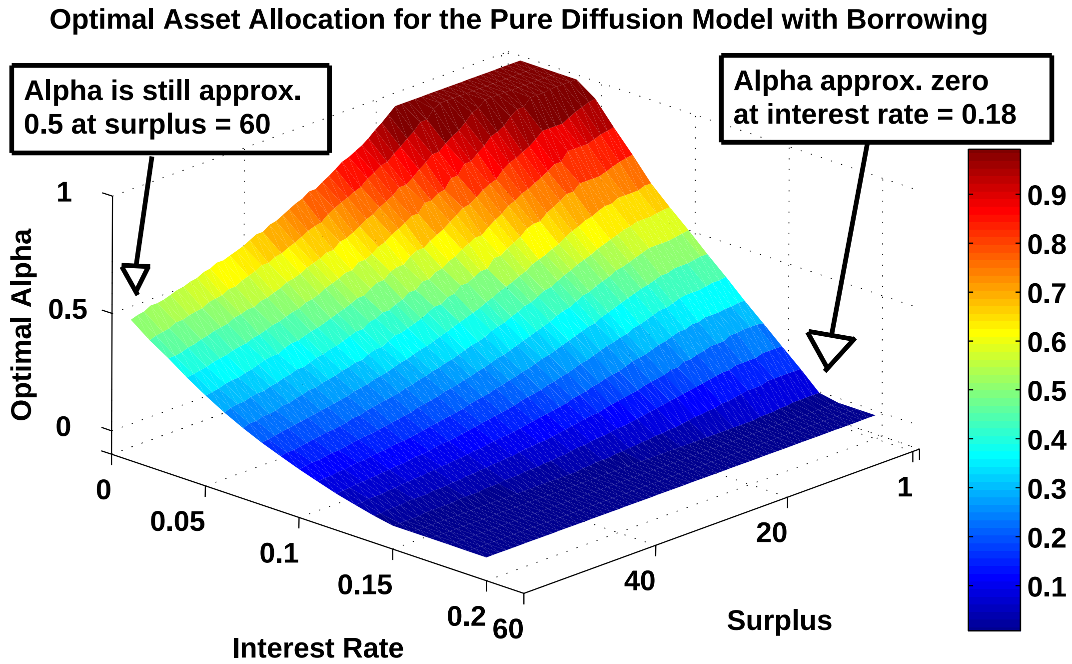

Figure 5: Optimal Alpha versus Interest Rate versus Surplus - Pure Diffusion Model with Borrowing

The 3-D surface in figure 6 shows how the maximum percentage alpha can change the ruin probability behaves with varying interest rate and surpluses. Figures 6 and 3 have the same general features. The general surface has a pseudo-saddle point at an interest rate of about 0.09 and a surplus of about 14. For the interest rate of 0.01, the largest maximum percentage alpha can change the ruin probability is about 11.5 percent at the surplus of 10. For the interest rate of 0.2, the largest maximum percentage alpha can change the ruin probability is about 5 percent at the surplus of 12.

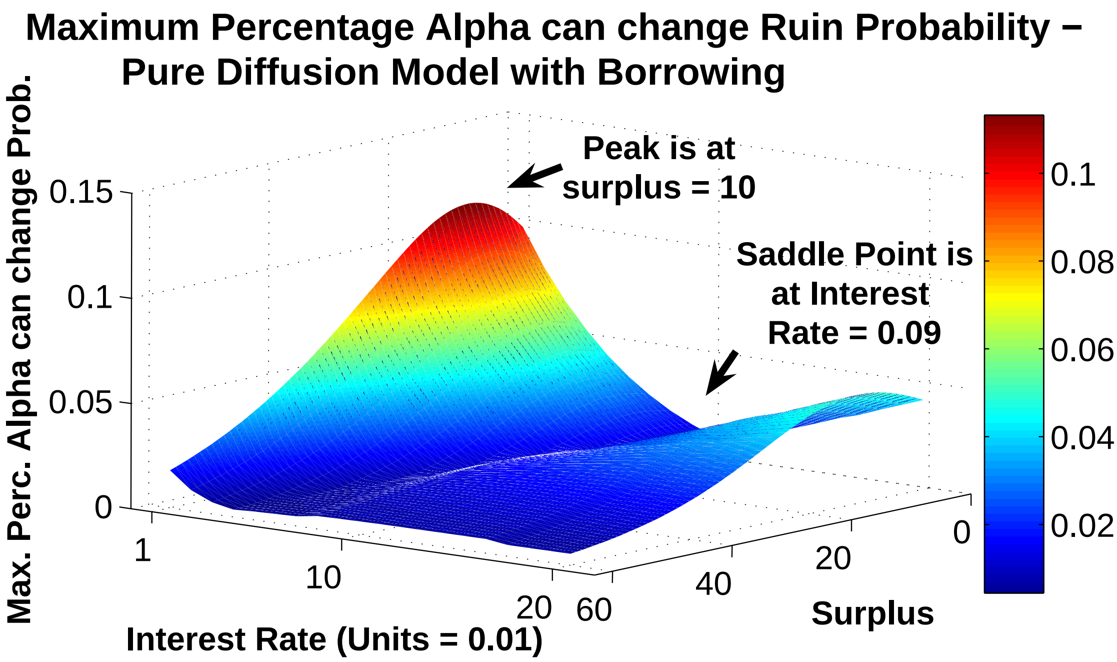

Figure 6: Maximum Percentage Alpha can change the Ruin Probability versus Interest Rate versus Surplus - Pure Diffusion Model with Borrowing

5.3 Jump Diffusion Risk Model

We used a premium rate c = 2.1, a risky return rate $\mu = 0.2$, a correlation $\rho = 0.1$, a risky volatility $\sigma_2 = 0.3$, a borrowing rate $\tau = 0.2$, an exponential claims parameter $\beta = 1$, a compound Poisson claims process given by $\lambda = 2$ and a diffusion $\sigma_1 = 2$. In addition various values of asset allocation α, interest rates r and surpluses u were used. Again, we assumed $\mu \geq r$ because there should be an incentive to invest in a risky asset versus a risk-free asset. The differential equation is equations (3) with the associated boundary conditions at zero and infinity. The first several numerical solutions to the ODE for an interest rate of 0.1 are graphically displayed in figure 7.

Figure 7 shows the first 15 solutions to the ODE. There are five bands that behave similarly to the Pure Diffusion Model. The most notable difference between figures 7 and 1 is that figure 7 is more sensitive to surplus loss and gain for small surpluses as shown by the larger gaps between the small surplus solutions.

The 3-D surface in figure 8 shows how optimal alpha behaves with varying interest rate and surpluses. The 3-D surface in figure 8 has similar features as in figures 2 and 5 but there are some differences. The surface has a smaller red colored plateau that is near the interest rate of 0.01 to 0.06 and the surplus of 1 to 7. The surface goes to zero about interest rate of 0.18 at low surpluses. At the surplus value of 40, the surface goes to zero about the interest rate of 0.12. Thus as the surpluses become larger, the surface goes to zero at lower interest rates. The optimal alpha has a value of about 0.25 at a surplus of 40 with an interest rate of 0.01. Thus as the surpluses become larger, the surface tends

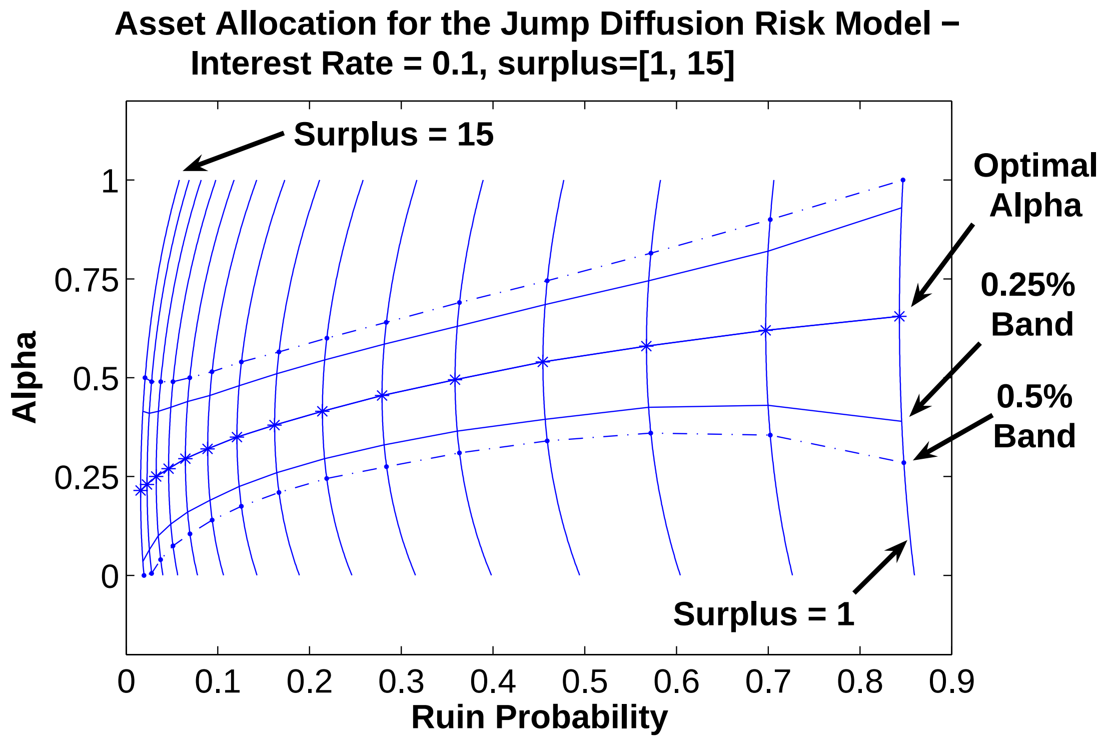

Figure 7: Alpha versus Ruin Probability - Jump Diffusion Risk Model

to smaller optimal alpha values faster than in figures 2 and 5 for low interest rates.

Browne (1995) showed that in one region the optimal investment policy toward the risky asset was proportional to $(\mu - r)$ when the stopping time was achieved and in another region all assets were invested in risky assets. This general conclusion appears to be in agreement with the red plateau and the slope of figure 8 given fixed low surpluses. However, it should be noted that Browne's choice of the utility function is subjective. Also intuitively as the surplus becomes larger, it would seem reasonable that the stationary process in figure 8 would become more invested in bonds than stocks because the company would have enough assets to cover any losses and would not want any risk. The general features of figure 8 appear to be generally justified.

The 3-D surface in figure 9 shows how the maximum percentage alpha can change the ruin probability behaves with varying interest rate and surpluses. Figures 3, 6 and 9 have the same general features. The general surface has a pseudo saddle point at an interest rate of about 0.09 and a surplus of about 8. For the interest rate of 0.01, the largest maximum percentage alpha can change the ruin probability is about 38 percent at the surplus of 10. For the interest rate of 0.2, the largest maximum percentage alpha can change the ruin probability is about 15 percent at the surplus of 6.

The 3-D surface in figure 10 shows how the 0.25 percent upper band for optimal alpha behaves with varying interest rates and surpluses. For a ruin probability of about 0.1 percent and an interest range of 0.1 to 0.17, the surplus range is about 23 to 20. For a ruin probability of about 5 percent and an interest range of 0.1 to 0.17, the surplus range

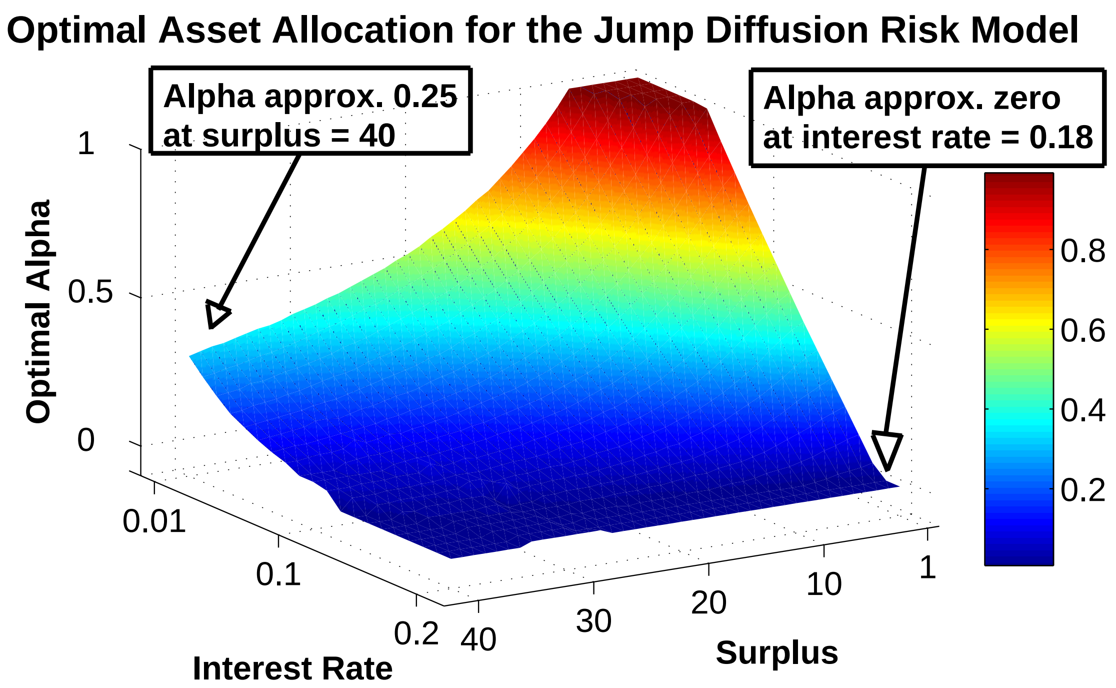

Figure 8: Optimal Alpha versus Interest Rate versus Surplus - Jump Diffusion Risk Model

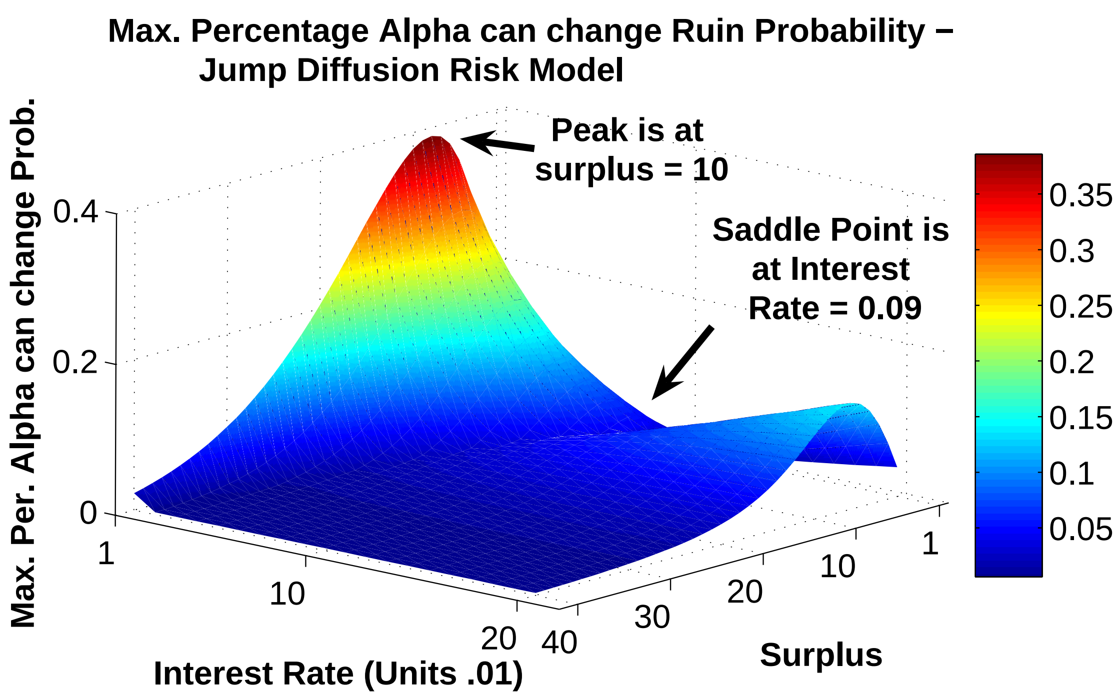

Figure 9: Maximum Percentage Alpha can change the Ruin Probability versus Interest Rate versus Surplus - Jump Diffusion Risk Model

is about 10. Both of these ruin probabilities are on the left side of the well and in the color range from yellow to light blue. Figure 10 is above figure 8.

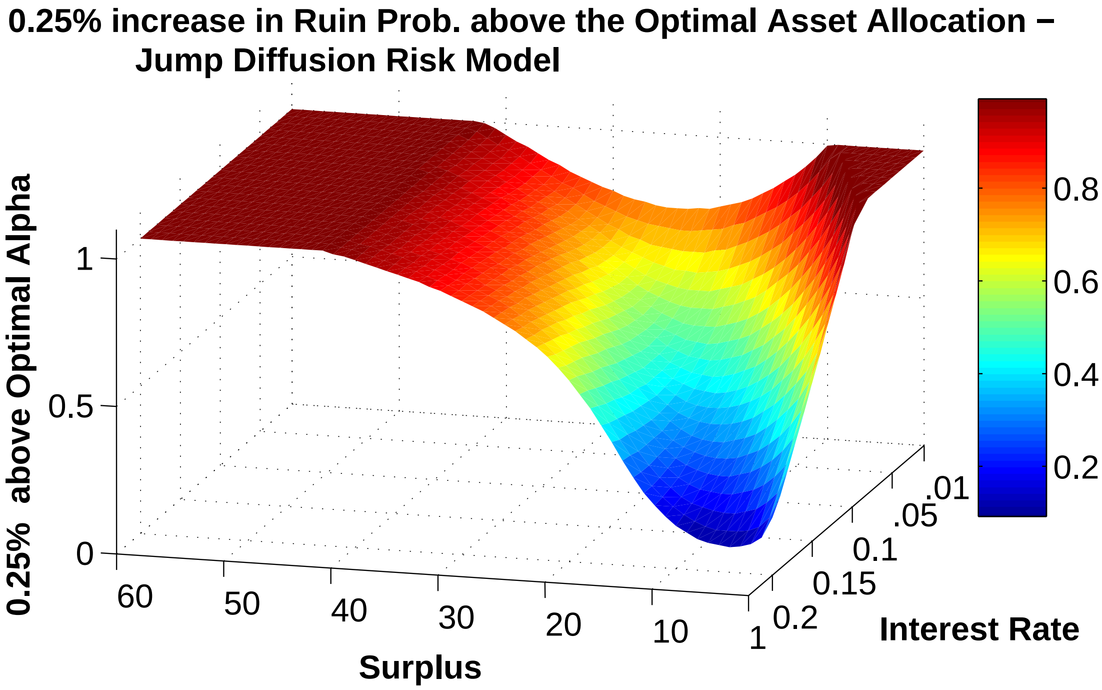

Figure 10: The 0.25 Percent Upper Band for Optimal Alpha versus Surplus versus Interest Rate

The 3-D surface in figure 11 shows how the 0.25 percent lower band for optimal alpha behaves with varying interest rates and surpluses. This surface goes to zero relatively quickly as the surplus is increased for all interest rates. Figure 11 is below figure 8.

5.4 Estimate of a Reasonable Range of Optimal Alpha

Firstly, we determined the limits on the probability of ruin. The article by Young (2004) suggested an upper limit of 5 percent probability of ruin might be considered. On the lower end, a 0.1 percent probability of ruin seemed like a reasonable limit to be considered. Therefore, we considered a ruin probability range of 5 to 0.1 percent probability of ruin.

To find the optimal alpha range, we wanted a method that was bias towards the 0.1 percent ruin probability and captured some of the dynamics of the optimal alpha in the 1, 2, 3, 4 and 5 percent ruin probabilities. We wanted a simple method that did not include making a probability distribution out of the alpha range of interest. So, we decided to take the averages of optimal alpha for ruin probabilities and interest rates in diminishingly smaller ranges.

We solved the full third order ODE without borrowing. However, we needed an estimate or the solution to the system of ODE's with borrowing to find a reasonable

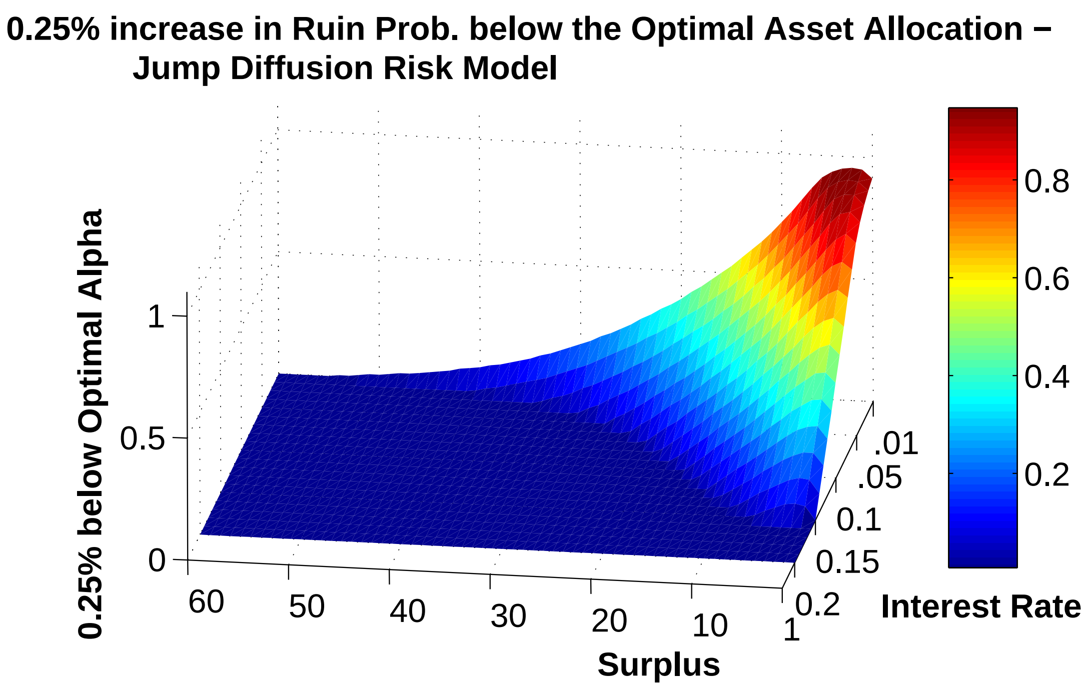

Figure 11: The 0.25 Percent Lower Band for Optimal Alpha versus Surplus versus Interest Rate

range of optimal alpha. To estimate the system of ODE's with borrowing, we used the difference between the Pure Diffusion Model with no borrowing and the Pure Diffusion Model with borrowing as indirectly suggested in Gerber and Yang (2007).

For the difference between the Pure Diffusion Models, we used the interest range of 0.10 to 0.15 because it was past the middle of the stock drift parameter and still had significant values for optimal alpha. We decided to use the interest ranges of 0.10 to 0.15, 0.11 to 0.15, ... , and 0.14 to 0.15 to find out how much borrowing changed the optimal alpha. With each interest range, we took the average optimal alpha for the ruin probability ranges of 0.1 to 1, 0.1 to 2, ... , and 0.1 to 5 percent and averaged the averages. We did the same for both Pure Diffusion Models. We found that borrowing increased the average optimal alpha relative to the interest ranges by 46, 43, 43, 41, and 45 percent, respectively. On average, borrowing increased the average optimal alpha by 44 percent.

We used the same averaging technique for the Jump Diffusion Risk Model. However, we used the interest ranges of 0.1 to 0.17 and included the ruin probability ranges of 0.1 to 0.5 and 0.1 to 0.2 percent because we wanted a more conservative analysis. The summary of the data are in Table 1.

To determine that lower limit of a reasonable range for the optimal alpha, we averaged the averages from Table 1. We averaged the averages from the ruin probability of 0.1 to 0.2 percent. This resulted in an optimal alpha range of [0.045 - 0.057]. We increased the optimal alpha range by 44 percent to estimate the optimal alpha in the Jump Diffusion Risk Model with borrowing and used the extremes. Thus, a reasonable range

	Interest Range						
Probability Range	.1 - .17	.11 - .17	.12 - .17	.13 - .17	.14 - .17	.15 - .17	.16 - .17
0.1 - 5	0.1065	0.0917	0.0786	0.0663	0.0525	0.0396	0.0279
0.1 - 4	0.1030	0.0892	0.0773	0.0635	0.0508	0.0393	0.0260
0.1 - 3	0.0994	0.0868	0.0734	0.0606	0.0493	0.0362	0.0239
0.1 - 2	0.0953	0.0830	0.0700	0.0574	0.0466	0.0339	0.0221
0.1 - 1	0.0874	0.0766	0.0653	0.0548	0.0434	0.0332	0.0211
0.1 - 0.5	0.0820	0.0698	0.0610	0.0491	0.0404	0.0290	0.0200
0.1 - 0.2	0.0709	0.0640	0.0572	0.0443	0.0375	0.0250	0.0183
Average	0.0921	0.0802	0.0690	0.0566	0.0458	0.0337	0.0228

Table 1: Optimal Alpha Averages - Jump Diffusion Risk Model

Company - Data summarized from 2002 Best's Insurance Reports	Percent of assets in Stocks and Real Estate	Assets in Billions
Amercian Life Ins. Co.	(High) 20.5	28.5
Berkshire Hathway Life Ins. Co. (NE)	(High) 11.3	1.6
J.C. Penny Life Ins. Co.	(Low) 2.9	1.8
John Hancock Life Ins. Co.	Rule-of-Thumb 4.6	64.3
Kansas City Life Ins. Co.	(High) 8.4	2.6
Metropolitan Life Ins. Co.	Rule-of-Thumb 6.9	184.7
New York Life Ins. Co.	Rule-of-Thumb 7.2	77.9
Physicians Life Ins. Co.	(High) 23.9	1.0
State Farm Life Ins. Co.	Rule-of-Thumb 4.6	30.4
Union Central Life Ins. Co.	(Low) 2.8	5.6
Total = 10 Insurance Companies	Average = 9.3	Total = 398.4

Table 2: Data Summary of 2002 Best's Insurance Reports - Life/Health

of optimal alpha for an insurance company is [0.045 - 0.082]. While minimizing the probability of ruin, a rule-of-thumb for an insurance company's distribution of risky assets is 4.5 to 8.2 percent of the assets may be invested in stocks and real estate.

We compared the risky asset rule-of-thumb range to real data of insurance companies from 2002 Best's Insurance Reports - Life/Health in Table 2. A comparison of ten well known insurance companies shows that four companies had more of their assets invested in risky assets than the rule-of-thumb range. Two companies had less of their assets invested in risky assets than the rule-of-thumb range. However, approximately 90 percent of the 389 billion in assets are within the rule-of-thumb range of 4.5 to 8.2 percent of the assets invested in stocks and real estate.

6 Asymptotic Form of the Ruin Probability

There is more than one way to find the asymptotic form of the solution to an ordinary differential equation. One way is to use asymptotic integrand relations and follow through with the integrations. Another way is to find the form of the asymptotic differential equation and given enough information about the solution, follow through with the solution to the asymptotic differential equation with the boundary conditions. Here we are working with the asymptotic form of the solution to differential equation (3). The asymptotic differential equation of (3) is

$$\frac{1}{2}(\alpha^2\sigma_2^2 u + 2\rho\alpha\sigma_1\sigma_2)\psi^{'''}(u)+$$
$$(\frac{1}{2}\beta\alpha^2\sigma_2^2 u + (\alpha\mu + (1-\alpha)r + \alpha^2\sigma_2^2 + \rho\alpha\sigma_1\sigma_2\beta))\psi^{''}(u)+$$
$$((\alpha\mu + (1-\alpha)r)\beta)\psi^{'}(u) = 0 \tag{9}$$

Theorem 2:

Given

1. $\psi(u) < 1$ *if* $2(\alpha\mu + (1-\alpha)r)/\alpha^2\sigma_2^2 > 1$ *for* $u \geq 0$*;*
2. *The process model is the Jump Diffusion Risk Model with risk-free and risky investments which follow deterministic exponential growth and geometric Brownian motion, respectively;*
3. *The Jump Diffusion Risk Model is a stationary Markov process;*
4. *The asymptotic solution has the boundary condition* $\psi(u) \to 0$ *as* $u \to \infty$ *and it is a real quantity;*

Then, the asymptotic ruin probability takes the form

$$\psi(u) \sim Cu^{1-2(\alpha\mu+(1-\alpha)r)/(\alpha^2\sigma_2^2)}, \tag{10}$$

of a power function where $0 < \alpha \leq 1$ *and* $C > 0$ *is a constant.*

Proof

First, substitute $\psi^{'} = \phi$ into equation (9) to obtain the second order differential equation

$$\frac{1}{2}(\alpha^2\sigma_2^2 u + 2\rho\alpha\sigma_1\sigma_2)\phi^{''}(u)+$$
$$(\frac{1}{2}\beta\alpha^2\sigma_2^2 u + (\alpha\mu + (1-\alpha)r + \alpha^2\sigma_2^2 + \rho\alpha\sigma_1\sigma_2\beta))\phi^{'}(u)+$$

$$((\alpha\mu + (1-\alpha)r)\beta)\phi(u) = 0. \tag{11}$$

Substitution $x = -\beta u - 2\beta\rho\sigma_1/(\alpha\sigma_2)$ into equation (11) to obtain

$$x\phi'' + [2(\alpha\mu + (1-\alpha)r)/(\alpha^2\sigma_2^2) + 2 - x]\phi' - [2(\alpha\mu + (1-\alpha)r)/(\alpha^2\sigma_2^2)]\phi = 0 \tag{12}$$

which is of the form of Kummer's confluent hypergeometric equation. The form of the confluent hypergeometric solutions after substitution of u for x are

$$\phi_1(u) \sim C_1[{}_1F_1(2(\alpha\mu + (1-\alpha)r)/(\alpha^2\sigma_2^2); 2(\alpha\mu + (1-\alpha)r)/(\alpha^2\sigma_2^2) + 2; -\beta u)] \tag{13}$$

and

$$\phi_2(u) \sim C_2 u^{-1-2(\alpha\mu+(1-\alpha)r)/(\alpha^2\sigma_2^2)}[{}_1F_1(-1; -2(\alpha\mu + (1-\alpha)r)/(\alpha^2\sigma_2^2); -\beta u)] \tag{14}$$

where $[{}_1F_1]$ is a confluent hypergeometric function of the first kind and the C's are constants. Next, take the solutions as real, use some of the transformations in Abramowitz (1972) and integrate $\int \phi_{1,2} du = \psi_{1,2}$. Given $1 - 2(\alpha\mu + (1-\alpha)r)/\alpha^2\sigma_2^2 < 0$ for $\psi(u) < 1$, the relationships as $u \to \infty$ are

$$C_4 u^{1-2(\alpha\mu+(1-\alpha)r)/(\alpha^2\sigma_2^2)} > C_3 u^{-2(\alpha\mu+(1-\alpha)r)/(\alpha^2\sigma_2^2)} > \psi_1(u) > 0 \tag{15}$$

Thus, the asymptotic form of the solution is the form of the slowest decaying term in the solutions at infinity

$$\psi \sim C u^{1-2(\alpha\mu+(1-\alpha)r)/(\alpha^2\sigma_2^2)} \tag{16}$$

where $C > 0$ is a constant and $0 < \alpha \leq 1$. ■

Remark 2

This asymptotic result is related to previous recent articles. With $\rho = 0$ and $\alpha = 1$, the result reduces to the asymptotic results in the model presented by Cai and Xu (2006)

$$\psi \sim D u^{1-2\mu/\sigma_2^2} \tag{17}$$

where $D > 0$ is a constant.

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