The Optimal Strategy and Capital Threshold of Multi-period Proportional Reinsurance

Jianfa Cong

(Department of Statistics and Actuarial Science, University of Waterloo)

Zhongfei Li

(Department of Finance, Lingnan (University) College Sun Yat-sen University)

Ken Seng Tan

(Department of Statistics and Actuarial Science, University of Waterloo)

Abstract This paper investigates the optimal multi-period proportional reinsurance strategy that minimizes the ruin probability of an insurer. Some conditions under which it is optimal to have no reinsurance are derived. Based on these results, a new concept, capital threshold of proportional reinsurance, is introduced. In the case of two periods, a lower bound of the capital threshold of proportional reinsurance is provided. Some economic implications and properties of this new concept are also discussed.

Key words Discrete-time risk process, multi-period proportional reinsurance, optimal strategy, capital threshold

1 Introduction

Reinsurance is the transfer of risk from a direct insurer to a second insurance company, the reinsurer. Many insurance companies employ reinsurance strategies to ensure their income remain relatively stable and protect themselves from the potential large losses. Therefore, the problem of optimal reinsurance strategies has been an area of active research in the last few decades.

Several earlier literatures involving optimal reinsurance are Dayananda (1970), Bühlmann (1970), Gerber (1980), Waters (1983), Dickson and Waters(1996), Hipp and Vogt (2003), Irgens and Paulsen (2004), Kaishev (2005), Krvavych and Sherris (2006), Taksar and Hunderup (2007), and Cai et al. (2008), which address several different kinds of reinsurance strategies under many different criteria. In particular, there are many papers discussing optimal proportional reinsurance problems. For example, Schmidli (2001) studies the proportional reinsurance strategy which minimizes the insurer's ruin probability in a continuous-time framework. He considers two types of risk models: a classical Cramér-Lundberg model and a diffusion approximation to the surplus process. Via dealing with the corresponding HJB equations, he derives the optimal reinsurance strategy in the diffusion case. In the Cramér-Lundberg case, though he does not solve the problem explicitly, he characterizes the optimal reinsurance strategy and the corresponding survival probability. He also points out that when the surplus of the insurance company is sufficiently small, the optimal choice of the insurance company is to retain all the risk. For other discussions on optimal proportional reinsurance strategies, we refer readers to Højgaard and Taksar (1998a), Højgaard and Taksar (1998b), Schmidli (2002), Bäulerle (2005), Bai and Guo (2008),and Cao and Wan (2009).

However, in the literature, optimal reinsurance problems are most commonly explored in the continuous-time framework. An early exception is Schäl (2004). He presents a formulation of the problem in which the arrival of claims and the asset price changes occur at discrete points of time. He studies the optimal proportional reinsurance and investment strategy which either maximizes expected exponential utility or minimizes ruin probability. But he does not give the optimal strategy explicitly. Under the objective of minimizing ruin probability, he gives a condition under which it is optimal to have no reinsurance when the time horizon is infinite. After Schäl (2004), there are several papers addressing optimal strategies and ruin probability in the discrete-time framework, such as Irgens and Paulsen (2005), Chan and Zhang (2006), Wei and Hu (2008), and Diasparra and Romera (2010). Recently, introducing proportional reinsurance in a discrete-time risk process, Li and Cong (2008) study the optimal proportional reinsurance strategy which minimizes the insurer's ruin probability in finite time horizon. They give some necessary conditions for the optimal multi-period proportional reinsurance strategy and prove that the dynamic programming approach can be used to solve minimal ruin probability.

Based on the necessary conditions of Li and Cong (2008), in this paper we further study the optimal proportional reinsurance strategy to the ruin probability problem in the discrete-time framework. Due to that obtaining explicit optimal solutions is very difficult, we focus on the conditions under which it is optimal to have no reinsurance when the time horizon is finite. Then we introduce a new concept: capital threshold of proportional reinsurance, and discuss its economic implications and properties.

2 Ruin model and preliminaries

We start with the classical discrete-time risk process

$$U(n) = u_0 + cn - \sum_{i=1}^{n} X_i, \tag{1}$$

where $U(n)$ is the surplus (size of the fund of reserves) of an insurance company at time n, $u_0 \geqslant 0$ is the insurer's initial surplus at time 0, $c > 0$ is the constant rate of premium income per period, X_i is the claim size occurring in period i (i.e., the time interval from time $i-1$ to i), and $\{X_i\}_{i=1}^{\infty}$ are assumed to be independent and identically distributed (i.i.d.) random variables with distribution function $F(x)$ and density function $f(x)$. We further assume that

$$f(x) = \alpha e^{-\alpha x}, \quad F(x) = 1 - e^{-\alpha x}, \quad x \geq 0, \tag{2}$$

where $\alpha > 0$ is a constant. In the discrete-time model (1), the ruin is supposed to occur only at the end of a period.

We now modify the risk process (1) via introducing proportional reinsurance. At the beginning of each period, the insurer can determine the retention level for the coming period. Denote by θ_i the retention level in period i and by $R(X_i, \theta_i)$ the part of claim X_i paid by reinsurer in period i. Hence, $I(X_i, \theta_i) = X_i - R(X_i, \theta_i)$ is the part paid by the insurer in period i. In particular, according to the definition of proportional reinsurance, we have

$$R(X_i, \theta_i) = (1 - \theta_i)X_i \ , \quad I(X_i, \theta_i) = \theta_i X_i \ , \quad 0 \leqslant \theta_i \leqslant 1. \tag{3}$$

At the beginning of each period i, under the retention level θ_i, the insurer pays a premium rate to the reinsurer, which has to be deducted from the premium c. This leads to the insurer's net premium income rate per period, $c(\theta_i)$. In this paper, we assume that premiums are calculated according to the expected value (or mean value) principle:

$$c(\theta_i) = c - (1 + \mu)E[R(X_i, \theta_i)] = (1 + \lambda)E[X_i] - (1 + \mu)(1 - \theta_i)E[X_i], \tag{4}$$

where $\lambda > 0$ and $\mu > 0$ are respectively the insurer's and reinsurer's safety loadings. According to the no-arbitrage principle, it is reasonable for us to assume that the reinsurance loading is higher than the insurance loading, i.e., $\mu > \lambda$. Noticing that $E[X_i] = \frac{1}{\alpha}$, equation (4) can be written as

$$c(\theta_i) = -\frac{\mu - \lambda}{\alpha} + \frac{1 + \mu}{\alpha}\theta_i. \tag{5}$$

Consider a fixed and finite time horizon of T periods where T is a integer. Let $\theta := \{\theta_i\}_{i=1}^{T}$ and call it an reinsurance strategy in the time horizon (strategies which are identical with probability 1 are considered to be the same strategy). Similarly, let $\theta_{n,T} := \{\theta_i\}_{i=n+1}^{T}$ and call it a substrategy in the time interval $[n, T]$ (when $n = T$, there is no need for offering strategies). A strategy θ is said to be admissible if $0 \leqslant \theta_i \leqslant 1$ for $i = 1, 2, \ldots, T$. We denote the set of all admissible strategies by Θ. For a $\theta \in \Theta$, we use $U_\theta(n)$ to denote the insurer's net surplus at time n under the strategy θ . When we do not concern with the strategy, we can use u_n to denote the insurer's net surplus at time n. Then, according to (3), we have the following recursive equation

$$U_\theta(n+1) = U_\theta(n) + c(\theta_{n+1}) - I(X_{n+1}, \theta_{n+1}) = U_\theta(n) + c(\theta_{n+1}) - \theta_{n+1}X_{n+1} \tag{6}$$

for $n = 0, 1, \ldots, T-1$, with $U_\theta(0) = u_0$.

For $n = 0, 1, \ldots, T$, let

$$\phi_n^\theta(u) := Pr(U_\theta(n) \geqslant 0, U_\theta(n+1) \geqslant 0, \ldots, U_\theta(T) \geqslant 0 | U_\theta(n) = u) \tag{7}$$

denote the survival probability in time interval $[n, T]$ under strategy θ, given the initial surplus $u \geqslant 0$ at time n. Clearly,

$$\phi_T^\theta(u) = \begin{cases} 0 & \text{if} \quad u < 0, \\ 1 & \text{if} \quad u \geqslant 0. \end{cases} \tag{8}$$

Applying the law of total probability to (7), we have the following recursive equation

$$\phi_{n-1}^{\theta}(u) = \int_0^{\frac{u+c(\theta_n)}{\theta_n}} \phi_n^{\theta}(u + c(\theta_n) - \theta_n x) f(x) dx, \qquad n = 1, 2, \ldots, T. \tag{9}$$

Let $\psi_n^{\theta}(u)$ denote the ruin probability in time interval $[n, T]$ under strategy θ, given the initial surplus $u \geqslant 0$. Then, for $n = 1, 2, \ldots, T$,

$$\begin{aligned} \psi_{n-1}^{\theta}(u) &= 1 - \phi_{n-1}^{\theta}(u) \\ &= 1 - F\left(\frac{u + c(\theta_n)}{\theta_n}\right) + \int_0^{\frac{u+c(\theta_n)}{\theta_n}} \psi_n^{\theta}(u + c(\theta_n) - \theta_n x) f(x) dx, \end{aligned} \tag{10}$$

$$\psi_T^{\theta}(u) = \begin{cases} 1 & \text{if} \quad u < 0, \\ 0 & \text{if} \quad u \geqslant 0. \end{cases} \tag{11}$$

It is clear that when given the initial surplus $u \geqslant 0$ at time n, $\phi_n^{\theta}(u)$ and $\psi_n^{\theta}(u)$ only depend on substrategy $\theta_{n,T}$, while do not rely on the strategy in time interval $[0, n]$. Therefore, we can denote

$$\phi_n^{\theta_{n,T}}(u) = \phi_n^{\theta}(u), \quad \psi_n^{\theta_{n,T}}(u) = \psi_n^{\theta}(u)$$

Accordingly, (10) can be written as

$$\psi_{n-1}^{\theta_{n-1,T}}(u) = 1 - F\left(\frac{u + c(\theta_n)}{\theta_n}\right) + \int_0^{\frac{u+c(\theta_n)}{\theta_n}} \psi_n^{\theta_{n,T}}(u + c(\theta_n) - \theta_n x) f(x) dx \tag{12}$$

for $n = 1, 2, \ldots, T$.

The substrategy $\theta_{n,T}$ that minimizes the ruin probability $\psi_n^{\theta_{n,T}}(u)$ in time interval $[n, T]$ is called the optimal reinsurance substrategy in time interval $[n, T]$. In particular, when $n = 0$, it is called the optimal reinsurance strategy.

Further, we define the symbol $\hat{\psi}_n(u)$ $(n = 0, 1, \ldots, T)$ by the following recursive expression

$$\hat{\psi}_{n-1}(u) = \min_{0 \leqslant \theta_n \leqslant 1} \left\{ 1 - F\left(\frac{u + c(\theta_n)}{\theta_n}\right) + \int_0^{\frac{u+c(\theta_n)}{\theta_n}} \hat{\psi}_n(u + c(\theta_n) - \theta_n x) f(x) dx \right\} \tag{13}$$

for $n = 1, 2, \ldots, T$, with

$$\hat{\psi}_T(u) = \begin{cases} 1 & \text{if} \quad u < 0, \\ 0 & \text{if} \quad u \geqslant 0. \end{cases} \tag{14}$$

In this paper, we will use some results in the reference [1]. In order to make it more convenient for the readers to refer, we list them as lemmas here and give their proofs in the appendix.

Lemma 1: For $n \in \{0.1, \ldots, T\}$, when given the initial surplus $u \geqslant 0$ at time n, then the minimal ruin probability in time interval $[n, T]$ is $\hat{\psi}_n(u)$.

Lemma 2: Suppose that $\theta^* = (\theta_1^*, \theta_2^*, \ldots, \theta_T^*)$ is an optimal reinsurance strategy. For each $n \in \{1, 2, \ldots, T\}$, if the insurer's initial surplus at time $n - 1$, u_{n-1}, satisfies that $0 \leqslant u_{n-1} < \frac{\mu - \lambda}{\alpha}$, then $\theta_n^* = 1$.

Lemma 3: For any reinsurance strategy θ, if when making policy at the beginning, the insurer can anticipate that: there is a $k \in \{1, 2, \ldots, T\}$ satisfying

$$c(\theta_k) < 0\ ,\ U_\theta(k-1) + c(\theta_k) < \frac{\mu - \lambda}{\alpha},$$

then strategy θ is not an optimal reinsurance strategy.

Lemma 1 indicates that the minimal ruin probability is

$$\hat{\psi}_n(u) = \min_{\theta \in \Theta} \psi_n^\theta(u)$$

and can be calculated according to the recursive equation (13), which is the same as the Bellman equation in dynamic programming. In other words, one can use the dynamic programming approach to solve the multi-period reinsurance problem minimizing the ruin probability. Moreover, in the light of the proof of Lemma 1, $\theta^* = (\theta_1^*(u_0), \theta_2^*(u_1), \ldots, \theta_T^*(u_{T-1}))$ is an optimal reinsurance strategy, where u_i is the the insurer's initial surplus at time i and $\theta_n^*(u)$ is the point at which the function

$$T_n(\theta_n) = 1 - F\left(\frac{u + c(\theta_n)}{\theta_n}\right) + \int_0^{\frac{u+c(\theta_n)}{\theta_n}} \hat{\psi}_n(u + c(\theta_n) - \theta_n x) f(x) dx$$

achieves its minimal value on the bounded closed set $[0, 1]$.

Lemma 2 provides a necessary condition for the optimal multi-period proportional reinsurance strategy. It actually gives an optimal reinsurance policy in a simple case. Lemma 3 also presents a necessary condition for the optimal reinsurance strategy, only in terms of the insurer's surplus.

3 Optimal dynamic proportional reinsurance strategy

Using the expressions of $f(x)$ and $F(x)$ in (2), we can simplify (13) as

$$\hat{\psi}_{n-1}(u) = \min_{0 \leqslant \theta_n \leqslant 1} \left\{ \exp\{-\alpha c(u, \theta_n)\} + \alpha \int_0^{c(u,\theta_n)} \hat{\psi}_n(\theta_n x) \exp\{\alpha x - \alpha c(u, \theta_n)\}\, dx \right\} \tag{15}$$

for $n = 1, 2, \ldots, T$, where $c(u, \theta_n) := \frac{u + c(\theta_n)}{\theta_n} = \frac{1}{\theta_n}\left(u - \frac{\mu - \lambda}{\alpha}\right) + \frac{1+\mu}{\alpha}$.

3.1 Optimal single-period reinsurance strategy

According to (15), we have $\hat{\psi}_{T-1}(u) = \min_{0 \leqslant \theta_T \leqslant 1} e^{-\alpha c(u, \theta_T)}$, which yields

$$\hat{\psi}_{T-1}(u) = \begin{cases} e^{-\alpha c(u,\theta_T)}\Big|_{\theta_T = 0} = 0 & \text{if} \quad u \geqslant \dfrac{\mu - \lambda}{\alpha}, \\ e^{-\alpha c(u,\theta_T)}\Big|_{\theta_T = 1} = \exp\{-\alpha(u + \dfrac{1+\lambda}{\alpha})\} & \text{if} \quad u < \dfrac{\mu - \lambda}{\alpha}\ . \end{cases} \tag{16}$$

When $T = 1$, the expressions above give the optimal single-period reinsurance strategy and the corresponding ruin probability.

3.2 Optimal two-period reinsurance strategy

In this subsection, we focus on the two-period reinsurance problem, which is the simplest multi-period case. We need to find the optimal two-period reinsurance strategy $\theta^* = (\theta_1^*, \theta_2^*)$ that gives the insurer's minimal ruin probability $\hat{\psi}_0(u)$. Clearly, using the results in Section 3.1, the insurer can determine θ_2^* according to $U_\theta(1)$. Therefore, we just need to work out θ_1^*. When $T = 2$, taking advantage of (15) and (16), we have

$$\hat{\psi}_0(u) = \min_{0 \leqslant \theta_1 \leqslant 1} \left\{ \exp\left\{-\alpha c(u, \theta_1)\right\} + \alpha \int_0^A \exp\left\{\alpha(1-\theta_1)x - (1 + \lambda + \alpha c(u, \theta_1))\right\} dx \right\}, \tag{17}$$

where $A = \min\left\{\frac{\mu-\lambda}{\alpha\theta_1}, c(u, \theta_1)\right\}$.

3.2.1 When the insurer's initial surplus u satisfies $u < \frac{\mu-\lambda}{\alpha}$

According to Lemma 2, when the insurer's initial surplus $u_0 = u < \frac{\mu-\lambda}{\alpha}$, we have $\theta_1^* = 1$. Substituting $\theta_1^* = 1$ into (17) gives the insurer's minimal ruin probability

$$\hat{\psi}_0(u) = \exp\left\{-\alpha\left(u + \frac{1+\lambda}{\alpha}\right)\right\} \cdot \left(1 + \alpha A_1 \cdot \exp\left\{-(1+\lambda)\right\}\right),$$

where $A_1 = \min\left\{\frac{\mu-\lambda}{\alpha}, u + \frac{1+\lambda}{\alpha}\right\}$.

$\theta_1^* = 1$ means that it is optimal to have no reinsurance at time 0. In this case, the insurer has so little initial surplus that he can not spend any on the reinsurance. It follows that if the insurer has too little initial surplus, reinsurance will not be able to diversify the risk for the insurer effectively.

3.2.2 When the insurer's initial surplus u satisfies $u \geqslant \frac{2(\mu-\lambda)}{\alpha}$

From (5), it is easy to derive that if the insurer wants to transfer all the risk in one period, the net amount of premium he needs to pay is $\frac{\mu-\lambda}{\alpha}$. Therefore, when the insurer's initial surplus $u \geqslant \frac{2(\mu-\lambda)}{\alpha}$, the insurer has enough reserve to transfer all risk in the two forthcoming periods. In this case, if the insurer wants to minimize the ruin probability, the best choice is to transfer all the risk, i.e. $\theta_1^* = \theta_2^* = 0, \hat{\psi}_0(u) = 0$. [①]

3.2.3 When the insurer's initial surplus u satisfies $\frac{\mu-\lambda}{\alpha} < u < \frac{2(\mu-\lambda)}{\alpha}$

Taking advantage of Lemma 3, we can simplify the expression of $\hat{\psi}_0(u)$. To this end, we denote, for brevity,

$$h = u - \frac{\mu-\lambda}{\alpha}, \quad y = \frac{2(\mu-\lambda)}{\alpha} - u,$$

$$\begin{aligned} H(u, \theta_1) = &\exp\left\{-\frac{\alpha h}{\theta_1} - (1+\mu)\right\} + \frac{1}{1-\theta_1}\exp\left\{\frac{\alpha y}{\theta_1} - 2(1+\mu)\right\} \\ &- \frac{1}{1-\theta_1}\exp\left\{-\left(2 + \lambda + \mu + \frac{\alpha h}{\theta_1}\right)\right\}. \end{aligned}$$

[①] In this case, the problem of optimal reinsurance is relatively simple and intuitive. Therefore, we do not offer the related strict mathematical proof.

Theorem 1: Let $\theta^* = \{\theta_1^*, \theta_2^*\}$ be an optimal reinsurance strategy, and let $\frac{\mu-\lambda}{\alpha} < u < \frac{2(\mu-\lambda)}{\alpha}$. Then

$$\hat{\psi}_0(u) = \begin{cases} \lim\limits_{\theta_1 \to 1} H(u, \theta_1), & \text{if } \theta_1^* = 1; \\ \min\limits_{\frac{y}{\alpha(1+\mu)} < \theta_1 < 1} H(u, \theta_1), & \text{if } \theta_1^* \neq 1. \end{cases}$$

Proof: If $c(\theta_1) \leqslant -\left(u - \frac{\mu-\lambda}{\alpha}\right)$, then $U_\theta(1) = u + c(\theta_1) - \theta_1 X_1 < \frac{\mu-\lambda}{\alpha}$. Therefore, according to Lemma 3, the optimal reinsurance strategy $\theta^* = \{\theta_1^*, \theta_2^*\}$ satisfies $c(\theta_1^*) > -\left(u - \frac{\mu-\lambda}{\alpha}\right)$, i.e., $\theta_1^* > \frac{2(\mu-\lambda)}{1+\mu} - \frac{\alpha u}{1+\mu} = \frac{y}{\alpha(1+\mu)}$. In this case, (17) can be rewritten as

$$\hat{\psi}_0(u) = \min_{\frac{y}{\alpha(1+\mu)} < \theta_1 \leqslant 1} \left\{ \exp\{-\alpha c(u, \theta_1)\} + \alpha \int_0^A \exp\{\alpha(1-\theta_1)x - (1 + \lambda + \alpha c(u, \theta_1))\}\, dx \right\} \tag{18}$$

where $A = \min\left\{\frac{\mu-\lambda}{\alpha\theta_1}, c(u, \theta_1)\right\}$. By the definition of $c(u, \theta_1)$, we have

$$c(u, \theta_1) - \frac{\mu-\lambda}{\alpha\theta_1} = \frac{1}{\theta_1}(u - \frac{\mu-\lambda}{\alpha}) + \frac{1+\mu}{\alpha} - \frac{\mu-\lambda}{\alpha\theta_1} = \frac{1}{\theta_1}(u - \frac{2(\mu-\lambda)}{\alpha}) + \frac{1+\mu}{\alpha}.$$

When $\theta_1 > \frac{2(\mu-\lambda)}{1+\mu} - \frac{\alpha u}{1+\mu} = \frac{y}{\alpha(1+\mu)}$, we can easily obtain $c(u, \theta_1) > \frac{\mu-\lambda}{\alpha\theta_1}$, i.e., $A = \frac{\mu-\lambda}{\alpha\theta_1}$. Inserting this into (18) and using the definition of $c(u, \theta_1)$, it follows that when $\theta_1^* = 1$,

$$\hat{\psi}_0(u) = \exp\left\{-\alpha\left(u + \frac{1+\lambda}{\alpha}\right)\right\}(1 + (\mu - \lambda) \cdot \exp\{-(1+\lambda)\}) = \lim_{\theta_1 \to 1} H(u, \theta_1),$$

and when $\theta_1^* \neq 1$, $\hat{\psi}_0(u) = \min\limits_{\frac{y}{\alpha(1+\mu)} < \theta_1 < 1} H(u, \theta_1)$. □

When the insurer's initial surplus u satisfies $\frac{\mu-\lambda}{\alpha} < u < \frac{2(\mu-\lambda)}{\alpha}$, it is very difficult for us to derive the closed-form expressions of the optimal reinsurance strategy and its corresponding ruin probability. However, we can give the conditions under which it is optimal to have no reinsurance.

Theorem 2: When the insurer's initial surplus u satisfies $u < \frac{\mu-\lambda}{\alpha}\left(1 + e^{-1-\lambda}\right)$, i.e., $h < \frac{\mu-\lambda}{\alpha} e^{-1-\lambda}$, the strategy $\theta^* = (1, \theta_2^*)$, in which θ_2^* is determined according to $U_\theta(1)$ and the results in Section 3.1, is an optimal two-period proportional reinsurance strategy, that is, it is optimal to have no reinsurance at time 0.

In order to prove Theorem 2, we need to introduce the following two lemmas.

Lemma 4: When the insurer's initial surplus u satisfies $u < \frac{\mu-\lambda}{\alpha}\left(1 + e^{-1-\lambda}\right)$, i.e., $h < \frac{\mu-\lambda}{\alpha} e^{-1-\lambda}$, the strategy $\theta^* = (1, \theta_2^*)$, in which θ_2^* is determined according to $U_\theta(1)$ and the results in Section 3.1, is a local optimal two-period proportional reinsurance strategy, or equivalently, it is local optimal to have no reinsurance at time 0.

Proof: According to Theorem 1, we need only to prove that $\frac{\partial H(u,\theta_1)}{\partial \theta_1} < 0$ when θ_1 is in the left neighborhood of 1. It is easy to derive that

$$\begin{aligned} \frac{\partial H(u, \theta_1)}{\partial \theta_1} &= \left(\frac{\alpha h}{\theta_1^2} e^{1+\lambda} - \frac{1}{(1-\theta_1)^2} - \frac{\alpha h}{(1-\theta_1)\theta_1^2}\right) \exp\left\{-\left(2 + \mu + \lambda + \frac{\alpha h}{\theta_1}\right)\right\} \\ &\quad + \frac{1}{1-\theta_1}\left(\frac{1}{1-\theta_1} - \frac{\alpha y}{\theta_1^2}\right) \exp\left\{-\left(2 + 2\mu - \frac{\alpha y}{\theta_1}\right)\right\}. \end{aligned} \tag{19}$$

From (19), it is clear that $\frac{\partial H(u,\theta_1)}{\partial\theta_1}$ is a continuous function of θ_1 on $(0,1)$. Consequently, we need only to prove that $\lim\limits_{\theta_1\to 1^-}\dfrac{\partial H(u,\theta_1)}{\partial\theta_1}<0$. Notice that

$$\exp\left\{-\left(2+2\mu-\frac{\alpha y}{\theta_1}\right)\right\}=\exp\left\{-\left(2+\mu+\lambda+\frac{\alpha h}{\theta_1}\right)\right\}\exp\left\{\left(\frac{1}{\theta_1}-1\right)(\mu-\lambda)\right\}$$

and $h<\frac{\mu-\lambda}{\alpha}e^{-1-\lambda}$, it follows from (19) that

$$\begin{aligned}\frac{\partial H(u,\theta_1)}{\partial\theta_1}&=\left\{\frac{\alpha h e^{1+\lambda}}{\theta_1^2}-\frac{1}{(1-\theta_1)^2}-\frac{\alpha h}{(1-\theta_1)\theta_1^2}+\frac{1}{(1-\theta_1)^2}\exp\left[\left(\frac{1}{\theta_1}-1\right)(\mu-\lambda)\right]\right.\\&\qquad\left.-\frac{\alpha y}{(1-\theta_1)\theta_1^2}\exp\left[\left(\frac{1}{\theta_1}-1\right)(\mu-\lambda)\right]\right\}\cdot\exp\left\{-\left(2+\mu+\lambda+\frac{\alpha h}{\theta_1}\right)\right\}\\&<\left\{\frac{\mu-\lambda}{\theta_1^2}-\frac{1}{(1-\theta_1)^2}-\frac{\alpha h}{(1-\theta_1)\theta_1^2}+\frac{1}{(1-\theta_1)^2}\exp\left[\left(\frac{1}{\theta_1}-1\right)(\mu-\lambda)\right]\right.\\&\qquad\left.-\frac{\alpha y}{(1-\theta_1)\theta_1^2}\exp\left[\left(\frac{1}{\theta_1}-1\right)(\mu-\lambda)\right]\right\}\cdot\exp\left\{-\left(2+\mu+\lambda+\frac{\alpha h}{\theta_1}\right)\right\}.\end{aligned}$$

From $h<\frac{\mu-\lambda}{\alpha}e^{-1-\lambda}$ and $h+y=\frac{\mu-\lambda}{\alpha}$, we know that $y>\frac{\mu-\lambda}{\alpha}(1-e^{-1-\lambda})>\frac{\mu-\lambda}{2\alpha}$, i.e., $2\alpha y>\mu-\lambda$. Therefore, applying the L'Hospital's rule, we can obtain the following result

$$\begin{aligned}\lim_{\theta_1\to 1^-}&\left\{\frac{\mu-\lambda}{\theta_1^2}-\frac{1}{(1-\theta_1)^2}-\frac{\alpha h}{(1-\theta_1)\theta_1^2}+\frac{1}{(1-\theta_1)^2}\exp\left[\left(\frac{1}{\theta_1}-1\right)(\mu-\lambda)\right]\right.\\&\left.-\frac{\alpha y}{(1-\theta_1)\theta_1^2}\exp\left[\left(\frac{1}{\theta_1}-1\right)(\mu-\lambda)\right]\right\}=\frac{\mu-\lambda}{2}\left[(\mu-\lambda)-2\alpha y\right]<0.\end{aligned}$$

Obviously, $\exp\left\{-\left(2+\mu+\lambda+\frac{\alpha h}{\theta_1}\right)\right\}>0$ for all $\theta_1\in(0,1)$. Hence, $\lim\limits_{\theta_1\to 1^-}\dfrac{\partial H(u,\theta_1)}{\partial\theta_1}<0$. This completes the proof. □

Lemma 5: Assume that $h(x)$ is a continuously differential function on $(0,1)$ and satisfies
(i) $\lim\limits_{x\to 0^+}h(x)<0$, $\lim\limits_{x\to 1^-}h(x)<0$; and
(ii) for any $x_0\in(0,1)$, $h(x_0)=0$ implies $h'(x_0)>0$.
Then $h(x)<0$ for all $x\in(0,1)$.

Proof: If there is no any $x\in(0,1)$ satisfying $h(x)=0$, from the assumption conditions $\lim\limits_{x\to 0^+}h(x)<0$, $\lim_{x\to 1^-}h(x)<0$ and the Intermediate Value Theorem, we know that $h(x)<0$ for all $x\in(0,1)$. Therefore, in order to prove Lemma 5, we just need to prove that there is no any $x\in(0,1)$ satisfying $h(x)=0$. We show this by apagoge.

First, we prove that $h(x)$ does not have two or more different zero points in $(0,1)$. If this was not the case, obviously one can find a $\delta>0$ and $x_1\in(0,1)$ such that $h(x_1)=0$ and $h(x)\neq 0$ for all $x\in(x_1-\delta,x_1)\bigcup(x_1,x_1+\delta)$. We use x_2 to denote the zero point of $h(x)$ in $(0, 1)$ which is nearest to x_1. Without loss of generality, we assume that $x_2 > x_1$, i.e., $x_2 \in (x_1, 1), h(x_2) = 0$ and $h(x) \neq 0$, for all $x \in (x_1, x_2)$.

From $h(x_2) = 0$ and the conditions given in this lemma, we know that $h'(x_2) > 0$ and $h'(x)$ is a continuous function on $(0, 1)$. Therefore, there exists a $\delta_2 > 0$ such that $h'(x) > 0$ for all $x \in (x_2 - \delta_2, x_2 + \delta_2)$. It follows that $h(x) < 0$ for all $x \in (x_2 - \delta_2, x_2)$ and $h(x) > 0$ for all $x \in (x_2, x_2 + \delta_2)$. Similarly, there exists a $\delta_1 > 0$ such that $h(x) < 0$ for all $x \in (x_1 - \delta_1, x_1)$ and $h(x) > 0$ for all $x \in (x_1, x_1 + \delta_1)$. Notice that $h(x) \neq 0$ for all $x \in (x_1, x_2)$ and $h(x) < 0$ for all $x \in (x_2 - \delta_2, x_2)$, we know that $h(x) < 0$ for all $x \in (x_1, x_2)$, which contradicts with that $h(x) > 0$ for all $x \in (x_1, x_1 + \delta_1)$. Therefore, $h(x)$ does not have two or more different zero points in $(0, 1)$.

Now we suppose that there exists $x_0 \in (0, 1)$ satisfying $h(x_0) = 0$. From the discussion above, x_0 is the unique zero point of $h(x)$. According to the conditions given in this lemma, we know that $h'(x_0) > 0$ and $h'(x)$ is a continuous function on $(0, 1)$. Therefore, from the Intermediate Value Theorem, we know that $h(x) > 0$ for all $x \in (x_0, 1)$, which contradicts with the condition that $\lim\limits_{x \to 1^-} h(x) < 0$. Therefore, $h(x)$ does not have zero points in $(0, 1)$. This completes the proof. □

The following corollary can be easily derived from Lemma 5.

Corollary 1: Assume that $h(x)$ is a continuously differential function on $(0, 1)$ and satisfies
(i) $\lim\limits_{x \to 0^+} h(x) > 0$, $\lim\limits_{x \to 1^-} h(x) > 0$; and
(ii) for any $x_0 \in (0, 1)$, $h(x_0) = 0$ implies $h'(x_0) < 0$.
Then $h(x) > 0$ for all $x \in (0, 1)$.

Proof: Denote that $\hat{h}(x) = -h(x)$. It is clear that $\hat{h}(x)$ is a continuously differential function on $(0, 1)$ and satisfies

(i) $\lim\limits_{x \to 0^+} \hat{h}(x) < 0$, $\lim\limits_{x \to 1^-} \hat{h}(x) < 0$; and

(ii) for any $x_0 \in (0, 1)$, $\hat{h}(x_0) = 0$ implies $\hat{h}'(x_0) > 0$.

According to Lemma 5, $\hat{h}(x) < 0$ for all $x \in (0, 1)$. Hence, $h(x) > 0$ for all $x \in (0, 1)$. □

Now we can use Lemma 4, Lemma 5 and Corollary 1 to prove Theorem 2.

The proof of Theorem 2:

According to Theorem 1, in order to prove Theorem 2, we just need to prove that $\frac{\partial H(u,\theta_1)}{\partial \theta_1} < 0$ for all $\theta_1 \in (0, 1)$.

From equation (19), we can easily derive that $\lim\limits_{\theta_1 \to 0^+} \dfrac{\partial H(u, \theta_1)}{\partial \theta_1} < 0$ and that $\frac{\partial H(u,\theta_1)}{\partial \theta_1}$ is a continuously differential function of θ_1 on $(0, 1)$. From the proof of Lemma 4, we know that $\lim\limits_{\theta_1 \to 1^-} \dfrac{\partial H(u, \theta_1)}{\partial \theta_1} < 0$. Therefore, according to Lemma 5, in order to prove that $\frac{\partial H(u,\theta_1)}{\partial \theta_1} < 0$ for all $\theta_1 \in (0, 1)$, we just need to prove that $\frac{\partial^2 H(u,\theta_1)}{\partial \theta_1^2} > 0$ for all $\theta_1 \in \Omega := \left\{ \theta_1 \in (0, 1) | \frac{\partial H(u,\theta_1)}{\partial \theta_1} = 0 \right\}$.

Notice that $h + y = \frac{\mu-\lambda}{\alpha}$, we can rewrite (19) as

$$\begin{aligned}\frac{\partial H(u,\theta_1)}{\partial\theta_1} =& \exp\left[-\left(2+\mu+\lambda+\frac{\alpha h}{\theta_1}\right)\right]\cdot\left\{\frac{\alpha h}{\theta_1^2}\exp(1+\lambda)\right.\\ &\left.+\frac{\exp\left[\left(\frac{1}{\theta_1}-1\right)(\mu-\lambda)\right]-1}{(1-\theta_1)^2}-\frac{\alpha h+\alpha y\exp\left[\left(\frac{1}{\theta_1}-1\right)(\mu-\lambda)\right]}{(1-\theta_1)\theta_1^2}\right\},\end{aligned} \tag{20}$$

which implies that $\frac{\partial H(u,\theta_1)}{\partial\theta_1} = 0$ is equivalent to

$$\frac{\alpha h}{\theta_1^2}\exp(1+\lambda)+\frac{\exp\left[\left(\frac{1}{\theta_1}-1\right)(\mu-\lambda)\right]-1}{(1-\theta_1)^2}-\frac{\alpha h+\alpha y\exp\left[\left(\frac{1}{\theta_1}-1\right)(\mu-\lambda)\right]}{(1-\theta_1)\theta_1^2}=0. \tag{21}$$

Therefore, when $\theta_1 \in \Omega$, after differentiating (20) and using (21), we have

$$\frac{\partial^2 H(u,\theta_1)}{\partial\theta_1^2} = \exp\left[-\left(2+\mu+\lambda+\frac{\alpha h}{\theta_1}\right)\right]w(\theta_1),$$

where

$$\begin{aligned}w(\theta_1) =& \frac{2\alpha h[1-\exp(1+\lambda)]+2\alpha y\exp\left[\left(\frac{1}{\theta_1}-1\right)(\mu-\lambda)\right]}{(1-\theta_1)\theta_1^3}\\ &+\frac{\alpha h+[\alpha y-(\mu-\lambda)]\exp\left[\left(\frac{1}{\theta_1}-1\right)(\mu-\lambda)\right]}{(1-\theta_1)^2\theta_1^2}\\ &+\frac{\alpha y(\mu-\lambda)\exp\left[\left(\frac{1}{\theta_1}-1\right)(\mu-\lambda)\right]}{(1-\theta_1)\theta_1^4}.\end{aligned}$$

It is clear that $\exp\left[-\left(2+\mu+\lambda+\frac{\alpha h}{\theta_1}\right)\right] > 0$ for all $\theta_1 \in \Omega$. Therefore, in order to prove that $\frac{\partial^2 H(u,\theta_1)}{\partial\theta_1^2} > 0$ for all $\theta_1 \in \Omega$, we just need to prove that $w(\theta_1) > 0$ for all $\theta_1 \in \Omega$.

Applying $h < \frac{\mu-\lambda}{\alpha}e^{-1-\lambda}$ (i.e. $\alpha h\exp(1+\lambda) < \mu-\lambda$) and $\alpha h+\alpha y = \mu-\lambda$ to the expression of $w(\theta_1)$, we have

$$\begin{aligned}w(\theta_1) >& \frac{2\alpha h-2(\mu-\lambda)+2\alpha y\exp\left[\left(\frac{1}{\theta_1}-1\right)(\mu-\lambda)\right]}{(1-\theta_1)\theta_1^3}\\ &+\frac{\alpha h+[\alpha y-(\mu-\lambda)]\exp\left[\left(\frac{1}{\theta_1}-1\right)(\mu-\lambda)\right]}{(1-\theta_1)^2\theta_1^2}+\frac{\alpha y(\mu-\lambda)\exp\left[\left(\frac{1}{\theta_1}-1\right)(\mu-\lambda)\right]}{(1-\theta_1)\theta_1^4}\\ =& \frac{\alpha h+2\alpha y\left\{\exp\left[\left(\frac{1}{\theta_1}-1\right)(\mu-\lambda)\right]-1\right\}-\alpha h\exp\left[\left(\frac{1}{\theta_1}-1\right)(\mu-\lambda)\right]}{(1-\theta_1)^2\theta_1^2}\\ &+\frac{\alpha y(\mu-\lambda)\exp\left[\left(\frac{1}{\theta_1}-1\right)(\mu-\lambda)\right]}{(1-\theta_1)\theta_1^4}\\ >& \frac{\alpha y(\mu-\lambda)\exp\left[\left(\frac{1}{\theta_1}-1\right)(\mu-\lambda)\right]}{(1-\theta_1)\theta_1^4}-\frac{\alpha h\left\{\exp\left[\left(\frac{1}{\theta_1}-1\right)(\mu-\lambda)\right]-1\right\}}{(1-\theta_1)^2\theta_1^2}.\end{aligned}$$

Denote $g(\theta_1) = \frac{\alpha y(\mu-\lambda)\exp\left[\left(\frac{1}{\theta_1}-1\right)(\mu-\lambda)\right]}{\theta_1^2} - \frac{\alpha h\left\{\exp\left[\left(\frac{1}{\theta_1}-1\right)(\mu-\lambda)\right]-1\right\}}{(1-\theta_1)}$. From the definition of the set Ω, we know that $\Omega \subseteq (0,1)$. Therefore, in order to prove that $w(\theta_1) > 0$ for all $\theta_1 \in \Omega$, we just need to prove that $g(\theta_1) > 0$ for all $\theta_1 \in (0,1)$. We do it by using Corollary 1.

Obviously, $\lim_{\theta_1 \to 0^+} g(\theta_1) > 0$ and $g(\theta_1)$ is a continuously differential function on $(0,1)$. By the L'Hospital's rule, we can easily derive that $\lim_{\theta \to 1^-} g(\theta_1) = \alpha(y-h)(\mu-\lambda)$. From $h < \frac{\mu-\lambda}{\alpha} e^{-1-\lambda}$ and $h + y = \frac{\mu-\lambda}{\alpha}$, we know that $h < y$, i.e., $\lim_{\theta_1 \to 1^-} g(\theta_1) > 0$. Therefore, according to Corollary 1, in order to prove that $g(\theta_1) > 0$ for all $\theta_1 \in (0,1)$, we just need to prove that for any $\tilde{\theta}_1 \in (0,1)$, $g(\tilde{\theta}_1) = 0$ implies $g'(\tilde{\theta}_1) < 0$.

Suppose that $\tilde{\theta}_1 \in (0,1)$ satisfies $g(\tilde{\theta}_1) = 0$. This together with $h < y$ leads to

$$\begin{aligned}\frac{\alpha h(\mu-\lambda)\exp\left[\left(\frac{1}{\tilde{\theta}_1}-1\right)(\mu-\lambda)\right]}{(1-\tilde{\theta}_1)\tilde{\theta}_1^2} &= \frac{h}{y} \cdot \frac{\alpha h\left\{\exp\left[\left(\frac{1}{\tilde{\theta}_1}-1\right)(\mu-\lambda)\right]-1\right\}}{(1-\tilde{\theta}_1)^2} \\ &< \frac{\alpha h\left\{\exp\left[\left(\frac{1}{\tilde{\theta}_1}-1\right)(\mu-\lambda)\right]-1\right\}}{(1-\tilde{\theta}_1)^2}.\end{aligned}$$

Differentiating $g(\theta)$, letting $\theta = \tilde{\theta}_1$ and using the inequality above, we have

$$\begin{aligned}g'(\tilde{\theta}_1) =& \frac{\alpha h(\mu-\lambda)\exp\left[\left(\frac{1}{\tilde{\theta}_1}-1\right)(\mu-\lambda)\right]}{(1-\tilde{\theta}_1)\tilde{\theta}_1^2} - \frac{\alpha h\left\{\exp\left[\left(\frac{1}{\tilde{\theta}_1}-1\right)(\mu-\lambda)\right]-1\right\}}{(1-\tilde{\theta}_1)^2} \\ &- \frac{[2\tilde{\theta}_1 + (\mu-\lambda)] \cdot \alpha y(\mu-\lambda)\exp\left[\left(\frac{1}{\tilde{\theta}_1}-1\right)(\mu-\lambda)\right]}{\tilde{\theta}_1^4} \\ &< - \frac{[2\tilde{\theta}_1 + (\mu-\lambda)] \cdot \alpha y(\mu-\lambda)\exp\left[\left(\frac{1}{\tilde{\theta}_1}-1\right)(\mu-\lambda)\right]}{\tilde{\theta}_1^4} \\ &< 0.\end{aligned}$$

This completes the proof. □

Remark:

1. Theorem 2 offers a sufficient condition under which it is optimal to have no reinsurance.

2. Theorem 2 coincides with the results in Schmidli (2001), which studies the proportional reinsurance strategy minimizing the insurer's ruin probability in a continuous-time framework. Schmidli (2001) points out that when the surplus of the insurance company is sufficiently small, it is optimal to have no reinsurance. However, Schmidli (2001) does not explicitly indicate how small the surplus should be if the optimal choice of the insurance company is to have no reinsurance.

3. Theorem 2 her is different from Theorem 6 in Schäl (2004), though they both describe conditions under which it is optimal to have no reinsurance. The Theorem 6 in Schäl (2004) does not involve the surplus of the insurance company. The condition in Theorem 6 of Schäl (2004) is only about the safety loadings of the insurer and the

reinsurer. However, the condition in Theorem 2 here involve not only the safety loadings of the insurer and the reinsurer, but also the surplus of the insurance company and the expectation of the claim size. Theorem 6 in Schäl (2004) essentially gives a reasonable range of the price of reinsurance, while Theorem 2 here offers an optimal control of the insurer. The difference between Theorem 2 here and the Theorem 6 in Schäl (2004) stems from whether the time horizon is finite or not.

Next, we will use a numerical example to make some further discussions on Theorem 2.

3.3 Numerical example

We consider a numerical example in two-period case, i.e. $T = 2$. The model parameters are $\alpha = 0.8$, $\lambda = 0.15$, $\mu = 0.2$.②

From Theorem 1, we have

$$\hat{\psi}_0(u) = \min_{0 \leqslant \theta_1 \leqslant 1} \left\{ \exp\{-\alpha c(u, \theta_1)\} + \alpha \int_0^{\frac{\mu-\lambda}{\alpha\theta_1}} \exp\{\alpha(1-\theta_1)x - (1+\lambda+\alpha c(u,\theta_1))\}\, dx \right\}.$$

According to the expression above, we plot the following figures.③ Figure 1 shows the relation between the insurer's initial surplus and the optimal reinsurance strategy and Figure 2 shows the relation between the insurer's initial surplus and the minimal ruin probability.

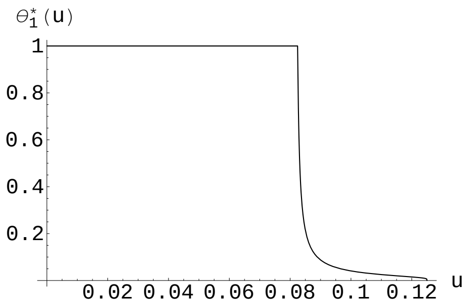

Figure 1: initial surplus and the optimal reinsurance strategy.

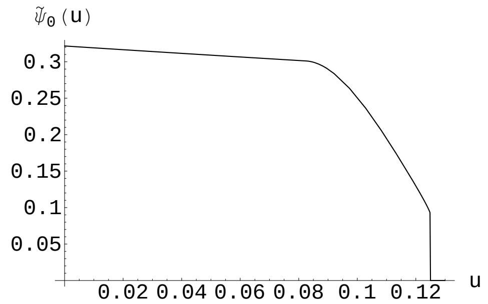

Figure 2: initial surplus and the ruin probability.

Figure 1 reflects Theorem 2 intuitively. From Figure 1, we can see that the optimal retention in the first period, θ_1^*, is decreasing with the insurer's initial surplus u.

From Figure 2, we find that the minimal ruin probability is decreasing with the insurer's initial surplus. Besides, Figure 2 also shows that when the insurer's initial surplus u satisfies $u < 0.082$, $|\frac{\partial\hat{\psi}_0(u)}{\partial u}|$ is relatively smaller, which means that the effect of increasing the initial surplus on decreasing the minimal ruin probability is relatively smaller; and when the insurer's initial surplus u satisfies $u > 0.082$, $|\frac{\partial\hat{\psi}_0(u)}{\partial u}|$ is relatively larger, which means that the effect of increasing the initial surplus on decreasing the minimal ruin probability is relatively larger.

②In this case, $\frac{\mu-\lambda}{\alpha} = 0.0625$, $\frac{\mu-\lambda}{\alpha}[1 + e^{-(1+\lambda)}] = 0.0823$.

③In this paper, we use mathematica 5.0 to deal with the numerical example

4 Capital threshold of multi-period proportional reinsurance

4.1 Definition

When considering a problem of optimal proportional reinsurance in the case of T periods, we denote

$$Z_T = \sup\left\{ Z \;\middle|\; \frac{\partial \psi_0^\theta(u)}{\partial \theta_1} < 0, \forall \theta \in \Theta, \forall u \in [0, Z) \right\}.$$

Definition 1: Z_T is said to be the capital threshold of proportional reinsurance in the case of T periods.

Obviously, for $T \geqslant 1$, when $u = \frac{(\mu-\lambda)T}{\alpha}$, we have $\frac{\partial \psi_0^\theta(u)}{\partial \theta_1} < 0$ for all $\theta \in \Theta$. It follows that $Z_T < \frac{(\mu-\lambda)T}{\alpha}$. Therefore, Z_T is bounded for any $T \geqslant 1$.

According to Definition 1, when we consider a problem of optimal proportional reinsurance in the case of T periods, if the insurer's initial surplus u satisfies $u < Z_T$, then $\psi_0^\theta(u)$ is a strict decreasing function of θ_1, which means that the optimal proportional reinsurance strategy θ^* satisfies $\theta_1^* = 1$. It follows that when the insurer's initial surplus u satisfies $u < Z_T$, it is optimal to have no reinsurance in the first period. In this case, whether the reinsurance available or not will not affect the insurer's decision-making. On the contrary, if the insurer's initial surplus u satisfies $u \geqslant Z_T$, then $\psi_0^\theta(u)$ is not a strict decreasing function of θ_1, which means that it may be optimal to have some reinsurance. To sum up, if the insurer's initial surplus u is less than Z_T, the insurer will not be able to afford the reinsurance in the first period. That is why we call Z_T the capital threshold of proportional reinsurance.

In fact, Theorem 2 offers a lower bound of the capital threshold of proportional reinsurance in the case of two periods.

4.2 Properties and application

From the solution of optimal single-period reinsurance strategy in Section 3.1, we know that $Z_1 = \frac{\mu-\lambda}{\alpha}$. From Theorem 2, we know that $Z_2 \geqslant \frac{\mu-\lambda}{\alpha}(1 + e^{-1-\lambda})$. In the case of $T \geqslant 3$, we are not able to make a good estimation of Z_T. However, we can derive the following results.

Theorem 3: $\{Z_T\}_{T=1}^{\infty}$ is an increasing series.

Proof: The conclusion is equivalent to $Z_T \leq Z_{T+1}$ ($T = 1, 2, \ldots$). To show this, it suffices to prove that if the insurer's initial surplus u satisfies $u < Z_T$, then $\psi_0^\theta(u)$ is a decreasing function of θ_1.

Let $u < Z_T$. By the definition of $\psi_0^\theta(u)$ and $\phi_0^\theta(u)$, we have $\psi_0^\theta(u) = 1 - \phi_0^\theta(u)$. Besides, from (7), we can easily derive that

$$\phi_0^\theta(u) = \Pr\{U_\theta(0) \geqslant 0, U_\theta(1) \geqslant 0, \ldots, U_\theta(T+1) \geqslant 0 | U_\theta(0) = u\} = \Pr(A \cap B),$$

where $A = \{U_\theta(0) \geqslant 0, U_\theta(1) \geqslant 0, \ldots, U_\theta(T) \geqslant 0 | U_\theta(0) = u\}$ and $B = \{U_\theta(1) \geqslant 0, U_\theta(2) \geqslant 0, \ldots, U_\theta(T+1) \geqslant 0 | U_\theta(0) = u\}$ are two events. From $u < Z_T$ and the definition of Z_T, we know that $\Pr(A)$ is increasing with θ_1. Notice that

$$E[U_\theta(1)] = E[U_\theta(0) + c(\theta_1) - \theta_1 X_1] = u + \lambda E[X_1] + (\theta_1 - 1)\mu E[X_1],$$

$E[U_\theta(1)]$ is increasing with θ_1. Obviously, $\Pr(B)$ is increasing as $E[U_\theta(1)]$ increases, which indicates that $\Pr(B)$ is increasing as θ_1 increases. Therefore, $\phi_0^\theta(u)$ is increasing with θ_1, or equivalently $\psi_0^\theta(u)$ is decreasing with θ_1. □

From Theorem 3, we can obtain the following corollary.

Corollary 2: Consider a problem of optimal proportional reinsurance in the case of T ($T \geqslant 2$) periods. If the insurer's initial surplus u satisfies $u < \frac{\mu-\lambda}{\alpha}(1 + e^{-1-\lambda})$, then the optimal reinsurance strategy θ^* satisfies $\theta_1^* = 1$, namely, it is optimal to have no reinsurance in the first period.

Proof: By Theorem 2, we know that $Z_2 \geqslant \frac{\mu-\lambda}{\alpha}(1+e^{-1-\lambda})$. So $u < Z_2$. From Theorem 3, we know that $Z_{T+1} > Z_T$ ($T = 1, 2, \ldots$). Therefore, we have $u < Z_T$. According to the definition of Z_T, the conclusion of the corollary 2 holds. □

Corollary 2 offers an optimal multi-period proportional reinsurance strategy in a special case. If only applying dynamic programming to solve the optimal multi-period reinsurance strategy without introducing the concept, the capital threshold of proportional reinsurance, we may not be able to obtain the similar result in Corollary 2.

Theorem 4: If $1 + \mu > (1 + \lambda)^2$, i.e., $\mu > 2\lambda + \lambda^2$, then the series $\{Z_T\}_{T=1}^{\infty}$ diverges, i.e., $\lim_{T\to\infty} Z_T = +\infty$.

Proof: According to Theorem 6 in Schäl (2004), when the time horizon is infinite, it is optimal to have no reinsurance if the condition $1 + \mu > (1 + \lambda)^2$ holds. Therefore, according to the definition of $\{Z_T\}_{T=1}^{\infty}$, it is clear that $\lim_{T\to\infty} Z_T = +\infty$ if the condition $1 + \mu > (1 + \lambda)^2$ holds. □

4.3 Economic implication

After computing a lot of numerical examples, we find the following phenomenon.

Assume that the insurer is facing a problem of finding the optimal proportional reinsurance strategy in the case of T periods. When the insurer's initial surplus u satisfies $u < (>) Z_T$, $|\frac{\partial \hat{\psi}_0(u)}{\partial u}|$ is relatively smaller (larger), which means that the effect of increasing the initial surplus on decreasing the minimal ruin probability is relatively smaller (larger).

Now we will offer some concise analysis on the immanent mechanism of the above phenomenon. From the definition of Z_T, we know that when the insurer's initial surplus u satisfies $u < Z_T$, it is optimal for the insurer to have no reinsurance in the first period. In this case, the minimal ruin probability $\hat{\psi}_0(u)$ is decreasing as the insurer's initial surplus u increases because the increase of the initial surplus enhances the insurer's ability of paying the policyholders' claims. When the insurer's initial surplus u satisfies $u \geqslant Z_T$, it may be optimal for the insurer to have some reinsurance. In this case, the increase of the initial surplus not only enhances the insurer's ability of paying the policyholders' claims, but also allows the insurer to have more reinsurance. That is why the effect of increasing the initial surplus on decreasing the minimal ruin probability is relatively larger when the insurer's initial surplus u satisfies $u > Z_T$.

This property of Z_T can give us some enlightenment. For example, when considering the case of T periods, in order to control the risk of insurers, the government can stipulate that the insurers must maintain their surplus above Z_T at the beginning of every period. In this case, reinsurance can be truly helpful for the insurers to reduce their risk.

5 Concluding remarks

In this paper we incorporate proportional reinsurance into the multi-period ruin model, and then apply the dynamic programming approach to deal with the problem of finding the optimal proportional reinsurance strategy under the assumption that the claim size follows the exponential distribution. We give some conditions under which it is optimal to have no reinsurance. Also we introduce a new concept, capital threshold of proportional reinsurance, and discuss its properties and economic implication.

To conclude this article, we point out several possible directions of further research. For example, (1) it is of interest to make some good estimation of Z_T $(T \geqslant 3)$. If some better estimation of $Z_T(T \geqslant 3)$ is obtained, a result that is stronger than Corollary 2 can be derived. Besides, it may be noteworthy to consider the global properties and asymptotic behavior of $\{Z_T\}_{T=1}^{\infty}$, such as its uniform upper bound and limit, if exist. (2) Our study in this article focuses on the proportional reinsurance. It deserves to study other kinds of reinsurance, such as stop-loss reinsurance. (3) It is interesting to relax the assumption that the claim size follows the exponential distribution. (4) Instead of considering the ruin model, it is worth to investigate other models, such as mean-risk model.

Appendix:

The proof of Lemma 1:

Assume that for each $n \in \{1, 2, \ldots, T\}$, the function

$$T_n(\theta_n) = 1 - F\left(\frac{u + c(\theta_n)}{\theta_n}\right) + \int_0^{\frac{u+c(\theta_n)}{\theta_n}} \hat{\psi}_n(u + c(\theta_n) - \theta_n x) f(x) dx$$

achieves its minimal value on the bounded closed set $[0, 1]$ at $\theta_n = \theta_n^*(u)$.

First, via induction, we prove that for $n \in \{0, 1, \ldots, T-1\}$ and $u_n \geqslant 0$, the sub-strategy $\theta_{n,T}^* = (\theta_{n+1}^*(u_n), \theta_{n+2}^*(u_{u+1}), \ldots, \theta_T^*(u_{T-1}))$ satisfies that $\psi_n^{\theta_{n,T}^*}(u_n) = \hat{\psi}_n(u_n)$, where u_n is the the insurer's initial surplus at time n. When $n = T$, (11) and (14) give

$$\psi_T^{\theta_{T,T}^*}(u_T) = \hat{\psi}_T(u_T) = \begin{cases} 1 & \text{if} \quad u_T < 0, \\ 0 & \text{if} \quad u_T \geqslant 0. \end{cases}$$

Therefore, when $n = T - 1$, we have

$$\hat{\psi}_{T-1}(u) = \min_{0 \leqslant \theta_T \leqslant 1} \left\{ 1 - F\left(\frac{u + c(\theta_T)}{\theta_T}\right) \right\} = \psi_{T-1}^{\theta_{T-1,T}^*}(u).$$

Assume that for $u_k \geqslant 0$, we have $\psi_k^{\theta_{k,T}^*}(u) = \hat{\psi}_k(u)$. According to (12), (13) and the definition of $\theta_{n,T}^*$, for $u_{k-1} \geqslant 0$, we have

$$\begin{aligned}
&\psi_{k-1}^{\theta_{k-1,T}^*}(u_{k-1}) \\
=&1 - F\left(\frac{u_{k-1} + c(\theta_k^*)}{\theta_k^*}\right) + \int_0^{\frac{u_{k-1}+c(\theta_k^*)}{\theta_k^*}} \psi_k^{\theta_{k,T}^*}(u_{k-1} + c(\theta_k^*) - \theta_k^* x) f(x) dx \\
=&1 - F\left(\frac{u_{k-1} + c(\theta_k^*)}{\theta_k^*}\right) + \int_0^{\frac{u+c(\theta_k^*)}{\theta_k^*}} \hat{\psi}_k(u_{k-1} + c(\theta_k^*) - \theta_k^* x) f(x) dx \\
=& \min_{0 \leqslant \theta_k \leqslant 1} \left\{ 1 - F\left(\frac{u_{k-1} + c(\theta_k)}{\theta_k}\right) + \int_0^{\frac{u_{k-1}+c(\theta_k)}{\theta_k}} \hat{\psi}_k(u_{k-1} + c(\theta_k) - \theta_k x) f(x) dx \right\} \\
=&\hat{\psi}_{k-1}(u_{k-1}).
\end{aligned}$$

Therefore, for $n \in \{0, 1, \ldots, T\}$ and $u_n \geqslant 0$, we have $\psi_n^{\theta_{n,T}^*}(u_n) = \hat{\psi}_n(u_n)$.

Next we use induction to prove that $\psi_n^{\theta}(u) \geqslant \hat{\psi}_n(u)$ for all $n \in \{0, 1, \ldots, T\}$, $\theta \in \Theta$ and $u \geqslant 0$. From (11) and (14), $\psi_T^{\theta}(u) \geqslant \hat{\psi}_T(u)$ holds for all $\theta \in \Theta$ and $u \geqslant 0$. Assume that $\psi_k^{\theta}(u) \geqslant \hat{\psi}_k(u)$ is true for all $\theta \in \Theta$ and $u \geqslant 0$. Then, according to (10) and (13), we have

$$\begin{aligned}
\psi_{k-1}^{\theta}(u) &= 1 - F\left(\frac{u + c(\theta_k)}{\theta_k}\right) + \int_0^{\frac{u+c(\theta_k)}{\theta_k}} \psi_k^{\theta}(u + c(\theta_k) - \theta_k x) f(x) dx \\
&\geqslant 1 - F\left(\frac{u + c(\theta_k)}{\theta_k}\right) + \int_0^{\frac{u+c(\theta_k)}{\theta_k}} \hat{\psi}_k(u + c(\theta_k) - \theta_k x) f(x) dx \\
&\geqslant \min_{0 \leqslant \theta_k \leqslant 1} \left\{ 1 - F\left(\frac{u + c(\theta_k)}{\theta_k}\right) + \int_0^{\frac{u+c(\theta_k)}{\theta_k}} \hat{\psi}_k(u + c(\theta_k) - \theta_k x) f(x) dx \right\} \\
&= \hat{\psi}_{k-1}(u).
\end{aligned}$$

Therefore, $\psi_n^\theta(u) \geqslant \hat{\psi}_n(u)$ holds for all $n \in \{0, 1, \ldots, T\}$, $\theta \in \Theta$ and $u \geqslant 0$.

The above results lead to the conclusion of the theorem. $\square$

In order to prove Lemma 2, we first prove the following result.

Lemma 6: Under any reinsurance strategy θ, for any time n, the ruin probability in time interval $[n, T]$, $\psi_n^\theta(u_n)$, decreases as the initial surplus at time n, u_n, increases. In particular, for any time n, the minimal ruin probability in time interval $[n, T]$, $\hat{\psi}_n(u_n)$, increases as the initial surplus at time n, u_n, increases.

Proof: Let strategy $\theta \in \Theta$ and time n be given, and assume that $u^{(2)} > u^{(1)} > 0$. According to the definition of survival probability, we have

$$\phi_n^\theta(u^{(1)}) = \Pr\left(U_\theta(n) \geqslant 0, U_\theta(n+1) \geqslant 0, \ldots, U_\theta(T) \geqslant 0 | U_\theta(n) = u^{(1)}\right).$$

Applying (6), we have

$$\begin{aligned}
\phi_n^\theta(u^{(2)}) &= \Pr\left(U_\theta(n) \geqslant 0, U_\theta(n+1) \geqslant 0, \ldots, U_\theta(T) \geqslant 0 | U_\theta(n) = u^{(2)}\right) \\
&= \Pr\Big(U_\theta(n) + u^{(2)} - u^{(1)} \geqslant 0, U_\theta(n+1) + u^{(2)} - u^{(1)} \geqslant 0, \ldots, \\
&\qquad U_\theta(T) + u^{(2)} - u^{(1)} \geqslant 0 | U_\theta(n) = u^{(1)}\Big) \\
&\geqslant \phi_n^\theta(u^{(1)}).
\end{aligned}$$

Notice that $\psi_n^\theta(u) = 1 - \phi_n^\theta(u)$, we have $\psi_n^\theta(u^{(1)}) \geqslant \psi_n^\theta(u^{(2)})$. This shows that $\psi_n^\theta(u_n)$ decreases with u_n for any strategy θ. The minimal ruin probability $\hat{\psi}_n(u_n)$, as the ruin probability under the optimal strategy, is therefore decreases with u_n. $\square$

The proof of Lemma 2:

Assume that $\theta = \{\theta_1, \theta_2, \ldots, \theta_T\}$ is a reinsurance strategy. For all $n \in \{1, 2, \ldots, T\}$, we assume that $0 \leqslant u_{n-1} < \frac{\mu - \lambda}{\alpha}$. According to (5) and (6), we have

$$U_\theta(n) = u_{n-1} + c(\theta_n) - \theta_n X_n = u_{n-1} - \frac{\mu - \lambda}{\alpha} + \left(\frac{1+\mu}{\alpha} - X_n\right)\theta_n.$$

As $u_{n-1} \geqslant 0$, according to (7), we have

$$\begin{aligned}
&\phi_{n-1}^\theta(u_{n-1}) \\
=& \Pr\left(U_\theta(n-1) \geqslant 0, U_\theta(n) \geqslant 0, \ldots, U_\theta(T) \geqslant 0 | U_\theta(n-1) = u_{n-1}\right) \\
=& \Pr\left(U_\theta(n) \geqslant 0, U_\theta(n+1) \geqslant 0, \ldots, U_\theta(T) \geqslant 0 | U_\theta(n-1) = u_{n-1}\right)
\end{aligned}$$

Define events A and B as

$$\begin{aligned}
A =& \{U_\theta(n) \geqslant 0 | U_\theta(n-1) = u_{n-1}\}, \\
B =& \{U_\theta(n+1) \geqslant 0, \ldots, U_\theta(T) \geqslant 0 | U_\theta(n-1) = u_{n-1}\}.
\end{aligned}$$

Then, we have $\phi_{n-1}^{\theta}(u_{n-1}) = \Pr(A \cap B) = \Pr(A)\Pr(B|A)$. It is easy to derive that

$$\begin{aligned}
P(A) =& \Pr\left(u_{n-1} + c(\theta_n) - \theta_n X_n \geqslant 0\right) \\
=& \Pr\left(X_n \leqslant \frac{1}{\theta_n}\left(u_{n-1} - \frac{\mu-\lambda}{\alpha}\right) + \frac{1+\mu}{\alpha}\right), \\
P(B|A) =& \Pr\left(U_\theta(n+1) \geqslant 0, \ldots, U_\theta(T) \geqslant 0 | U_\theta(n-1) = u_{n-1}, U_\theta(n) \geqslant 0\right) \\
=& \Pr\Bigg(U_\theta(n) \geqslant 0, U_\theta(n+1) \geqslant 0, \ldots, U_\theta(T) \geqslant 0 | U_\theta(n) = u_{n-1} - \frac{\mu-\lambda}{\alpha} \\
&+ \left(\frac{1+\mu}{\alpha} - X_n\right)\theta_n \geqslant 0\Bigg).
\end{aligned}$$

In the light of $u_{n-1} < \frac{\mu-\lambda}{\alpha}$ and the expressions above, we can see that $P(A)$ is strictly increasing with θ_n. Also, taking advantage of

$$u_{n-1} < \frac{\mu-\lambda}{\alpha}, U_\theta(n) = u_{n-1} - \frac{\mu-\lambda}{\alpha} + \left(\frac{1+\mu}{\alpha} - X_n\right)\theta_n \geqslant 0,$$

we can easily derive that $\frac{1+\mu}{\alpha} - X_n > 0$, namely $U_\theta(n)$ increases as θ_n increases. It is clear that $P(B|A)$ is the survival probability in time interval $[n, T]$ under strategy θ, given the initial surplus $U_\theta(n) \geqslant 0$ at time n. Therefore, according to Lemma 6, $P(B|A)$ increases as θ_n increases. Accordingly, $\phi_{n-1}^{\theta}(u_{n-1})$ strictly increases as θ_n increases. Hence, the optimal reinsurance strategy $\theta^* = (\theta_1^*, \theta_2^*, \ldots, \theta_T^*)$ satisfies that $\theta_n^* = 1$. The proof is completed. □

The proof of Lemma 3:
We prove the lemma by apagoge. Suppose that $\theta = \{\theta_1, \theta_2, \ldots, \theta_T\}$ is an optimal reinsurance strategy. Then, according to Lemma 1 and its proof, the substrategy $\theta_{k-1,T} = (\theta_k, \theta_{k+1}, \ldots, \theta_T)$ minimizes the insurer's ruin probability in time interval $[k-1, T]$.

Applying Lemma 2 to θ, which is supposed to be the optimal strategy, and noticing that $U_\theta(k) = U_\theta(k-1) + c(\theta_k) - \theta_k X_k < \frac{\mu-\lambda}{\alpha}$, we have $\theta_{k+1} = 1$. For brevity, we denote $u = U_\theta(k-1) \geqslant 0$. By the definition of $\phi_{k-1}^{\theta_{k-1,T}}(u)$ and $\theta_{k+1} = 1$, we have

$$\begin{aligned}
&\phi_{k-1}^{\theta_{k-1,T}}(u) \\
=& \Pr(U_\theta(k-1) \geqslant 0, U_\theta(k) \geqslant 0, \ldots, U_\theta(T) \geqslant 0 | U_\theta(k-1) = u) \\
=& \Pr\Bigg(u + c(\theta_k) - \theta_k X_k \geqslant 0, u + \frac{1+\lambda}{\alpha} + c(\theta_k) - \theta_k X_k - X_{k+1} \geqslant 0, \\
&\qquad U_\theta(k+2) \geqslant 0, \ldots, U_\theta(T) \geqslant 0\Bigg) \\
\leqslant& \Pr\left(u + \frac{1+\lambda}{\alpha} + c(\theta_k) - \theta_k X_k - X_{k+1} \geqslant 0, U_\theta(k+2) \geqslant 0, \ldots, U_\theta(T) \geqslant 0\right) \\
=& \Pr\Bigg(u + \frac{1+\lambda}{\alpha} + c(\theta_k) \geqslant X_{k+1}, u + \frac{1+\lambda}{\alpha} + c(\theta_k) - \theta_k X_k - X_{k+1} \geqslant 0, \\
&\qquad U_\theta(k+2) \geqslant 0, \ldots, U_\theta(T) \geqslant 0\Bigg),
\end{aligned} \tag{22}$$

where the last equation holds because

$$\{u+\frac{1+\lambda}{\alpha}+c(\theta_k)-\theta_k X_k - X_{k+1}\geqslant 0\}\subseteq\{u+\frac{1+\lambda}{\alpha}+c(\theta_k)\geqslant X_{k+1}\}.$$

In order to derive a contradiction, we construct a different reinsurance strategy $\hat{\theta}_{k-1,T}=\{\hat{\theta}_k,\hat{\theta}_{k+1},\ldots,\hat{\theta}_T\}$ as follows:

$$\hat{\theta}_k=1,\ \hat{\theta}_i=\theta_i\ (k+2\leqslant i\leqslant T),\ \hat{\theta}_{k+1}=\begin{cases}\theta_k & \text{if} \quad X_k\leqslant U_\theta(k-1)+\dfrac{1+\lambda}{\alpha}+c(\theta_k),\\ 1 & \text{if} \quad X_k> U_\theta(k-1)+\dfrac{1+\lambda}{\alpha}+c(\theta_k).\end{cases}$$

We are going to prove that in time interval $[k-1,T]$, the insurer's survival probability under the strategy $\hat{\theta}_{k-1,T}$ is larger than that under the strategy $\theta_{k-1,T}$.

According to the definition of $\phi_{k-1}^{\hat{\theta}_{k-1,T}}(u)$ and $\hat{\theta}_{k-1,T}$, we have

$$\begin{aligned}&\phi_{k-1}^{\hat{\theta}_{k-1,T}}(u)\\ =&\Pr(U_{\hat{\theta}}(k-1)\geqslant 0,U_{\hat{\theta}}(k)\geqslant 0,\ldots,U_{\hat{\theta}}(T)\geqslant 0|U_{\hat{\theta}}(k-1)=u)\\ =&\Pr\bigg(u+\frac{1+\lambda}{\alpha}-X_k\geqslant 0,u+\frac{1+\lambda}{\alpha}-X_k+c(\hat{\theta}_{k+1})-\hat{\theta}_{k+1}X_{k+1}\geqslant 0,\\ &\qquad U_{\hat{\theta}}(k+2)\geqslant 0,\ldots,U_{\hat{\theta}}(T)\geqslant 0\bigg).\end{aligned}$$

Applying the law of total probability to the right hand side of the above equation, we have

$$\begin{aligned}\phi_{k-1}^{\hat{\theta}_{k-1,T}}(u)=&\Pr\bigg(u+\frac{1+\lambda}{\alpha}+c(\theta_k)\geqslant X_k,u+\frac{1+\lambda}{\alpha}+c(\theta_k)-\theta_k X_{k+1}-X_k\geqslant 0,\\ &\qquad U_\theta(k+2)\geqslant 0,\ldots,U_\theta(T)\geqslant 0\bigg)\\ &+\Pr\bigg(u+\frac{1+\lambda}{\alpha}+c(\theta_k)<X_k\leqslant u+\frac{1+\lambda}{\alpha},u+\frac{2(1+\lambda)}{\alpha}-X_k-X_{k+1}\geqslant 0,\\ &\qquad U_\theta(k+2)\geqslant 0,\ldots,U_\theta(T)\geqslant 0\bigg).\end{aligned}$$

Since $f(x)>0$ for $x\in\mathbb{R}$, the second term of the right hand side of the above equation is strictly positive. Thus

$$\begin{aligned}\phi_{k-1}^{\hat{\theta}_{k-1,T}}(u)>&\Pr\bigg(u+\frac{1+\lambda}{\alpha}+c(\theta_k)\geqslant X_k,u+\frac{1+\lambda}{\alpha}+c(\theta_k)-\theta_k X_{k+1}-X_k\geqslant 0,\\ &\qquad U_\theta(k+2)\geqslant 0,\ldots,U_\theta(T)\geqslant 0\bigg).\end{aligned}\tag{23}$$

Noting that $\{X_i\}_{i=1}^T$ are independent and identically distributed random variables, we have

$$
\begin{aligned}
&\Pr\bigg(u+\frac{1+\lambda}{\alpha}+c(\theta_k)\geqslant X_k, u+\frac{1+\lambda}{\alpha}+c(\theta_k)-\theta_k X_{k+1}-X_k\geqslant 0,\\
&\qquad U_\theta(k+2)\geqslant 0,\dots,U_\theta(T)\geqslant 0\bigg)\\
=&\Pr\bigg(u+\frac{1+\lambda}{\alpha}+c(\theta_k)\geqslant X_{k+1}, u+\frac{1+\lambda}{\alpha}+c(\theta_k)-\theta_k X_k-X_{k+1}\geqslant 0,\\
&\qquad U_\theta(k+2)\geqslant 0,\dots,U_\theta(T)\geqslant 0\bigg).
\end{aligned}
\tag{24}
$$

Expressions (22), (23) and (24) implies that

$$\phi_{k-1}^{\hat{\theta}_{k-1,T}}(u)>\phi_{k-1}^{\theta_{k-1,T}}(u),$$

which is equivalent to

$$\psi_{k-1}^{\theta_{k-1,T}}(u)>\psi_{k-1}^{\hat{\theta}_{k-1,T}}(u).$$

This contradicts with that the substrategy $\theta_{k-1,T}=(\theta_k,\theta_{k+1},\dots,\theta_T)$ minimizes the insurer's ruin probability in time interval $[k-1,T]$. The lemma is proven. $\square$

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