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STATISTICAL TESTS OF THE LOGNORMAL DISTRIBUTION AS A BASIS FOR INTEREST RATE CHANGES

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ABSTRACT

In modeling the periodic change in the interest rate of a given maturity, the lognormal distribution is frequently assumed. This paper identifies several implicit assumptions underlying the use of this distribution, tests those assumptions against historical interest rates, and presents additional information from those tests.

I. INTRODUCTION

The stochastic generation of future yield curves, economic scenario testing in asset/liability management, and option-pricing techniques are some applications of the use of the lognormal distribution for modeling periodic changes in interest rates for periods up to 30 years or longer. The validity of the results, and conclusions drawn from those results, may be called into question if the application of the lognormal distribution fails to conform adequately to experience. There is strong motivation to confirm the applicability of the lognormal distribution.

The lognormal distribution is applied to the modeling of interest rates in the following manner (see [9]): Let I_t represent the interest rate at time t. The "lognormal assumption" often used is that the distribution of $\ln(I_{t+1}/I_t)$ is normal with mean zero and constant variance. Thus $I_{t+1}=I_t\times e^{\sigma\times Z_t}$, where σ is the standard deviation (volatility) and Z_t is a standard normal random variable.

The above equation is a special case of $I_{t+1}=I_t\times e^{\mu_t+\sigma_t Z_t}$, where μ_t is the mean or drift of the interest rate for period t and σ_t is the standard deviation or volatility of the rate over the same period. The often-used combination is that of mean equal to zero and constant standard deviation over time. The last equation can be written in the form

$$\ln\frac{I_{t+1}}{I_t} = \mu_t + \sigma_t Z_t.$$

Now

$$\ln \frac{I_{t+1}}{I_t} = \ln 1 + \frac{I_{t+1} - I_t}{I_t} = \ln 1 + \frac{\Delta I_t}{I_t}.$$

Thus

$$\ln \frac{I_{t+1}}{I_t} \approx \frac{\Delta I_t}{I_t}$$

as $\ln (1+x) \approx x$, when x is small. In this way one can discern the similarity of the above discrete process to the stochastic differential equation

$$\frac{dI_t}{I_t} = \mu_t \, dt + \sigma_t \, dw,$$

where W is a standard Wiener process.

Two of the major advantages of the lognormal distribution in this application are that it cannot result in negative interest rates and it is multiplicative over periods, that is, over n time periods

$$I_{t+n} = I_t \times e^{\sigma \times (Z_1 + Z_2 + \ldots + Z_n)}.$$

The lognormal distribution has many useful properties. See [4] for a thorough treatment of the lognormal distribution.

This paper examines the application of the lognormal distribution to the ratio of periodic interest rates. This is different from applying the lognormal distribution to the distribution of security or zero-coupon bond prices, as the Black-Scholes option-pricing model does.

To clarify the different implications of these two applications, consider the following illustration. Let P_t^n be the price at time t of a bond maturing at time $n+t$. The lognormal assumption applied to bond prices is the statement that $\ln(P_{t+1}^{n-1}/P_t^n)$ is normally distributed. But $1+I_t^n = P_{t+1}^{n-1}/P_t^n$; thus the lognormal assumption for bond prices implies that the $\ln(1+I_t^n)$ is normally distributed. This paper, by contrast, examines only the assumption that the ratio I_{t+1}/I_t is lognormally distributed or, equivalently, that the $\ln(I_{t+1}/I_t)$ is normally distributed.

Four assumptions are implicit in the use of the lognormal distribution with ratios of periodic interest rates. These assumptions are amenable to statistical verification. Hypotheses of these four assumptions are formulated and tested. The significance level used for these tests is 0.05. In addition to the use of the statistical tests, techniques from exploratory data analysis are employed to interpret the summary statistics of the study and to illuminate further the empirical data. These techniques are graphical.

The applicability of the lognormal distribution is examined with regard to six different maturities of Treasury securities: three month, six month, one year, three year, five year, and ten year. The tests are performed for each maturity over each of three sources of monthly Treasury security data: Federal Reserve Board, Salomon Brothers, and Moody's. The first two data sources provide traditional yield information, while the third provides derived spot-rate data. Tests are performed for the period from December 1953 through December 1988 and various subperiods. Further information on the nature of the data is provided later.

II. ASSUMPTIONS IMPLICIT IN THE USE OF THE LOGNORMAL DISTRIBUTION

Let I_t^m be the random variable representing the yield on a Treasury security of maturity m in time period t. Let i_t^m be a realization of I_t^m, that is, the actual yield observed for the security of maturity m at time t. The statement of the lognormal hypothesis is that the random variable I_{t+1}^m/I_t^m is lognormally distributed. But the random variable $J_t^m = \ln[I_{t+1}^m/I_t^m]$ is typically used, and the hypothesis is restated to be that J_t^m follows a normal distribution. Let j_t^m be a realization of J_t^m. The tests of hypotheses will be applied to the $\{j_t^m\}$ as determined from the data.

The following four assumptions are implicit in the use of the lognormal distribution:

1. $\{J_t^m\}$ are stochastically independent.
2. $\{J_t^m\}$ are normally distributed.
3. $\{J_t^m\}$ have constant variance.
4. $\{J_t^m\}$ have mean equal to zero.

III. SOURCES OF DATA

The securities used for this study are debt instruments of the U.S. Government. These securities are widely held, both by domestic and foreign owners, because of several desirable characteristics: large volumes of securities outstanding for many years; existence of a broad range of maturities; negligible credit risk; and liquidity of a large secondary market. Treasury securities of less than or equal to 10 years maturity at issue are not callable. Issues after March 1941 are fully taxable.

The analysis of econometric data is subject to various sources of error. Errors can result from the recording of data, the measurement of data, and the possibility that the definition of the data changes over time. Having several data sets for testing mitigates the presence of these effects in any one data set. Also, the use of several data sets allows for a more complete investigation, the confirmation of tests results, and the identification of differing results for further study. The data studied are the "logratios" defined by $\{j_t^m\} = \{\ln(i_{t+1}^m / i_t^m)\}$ for maturities (m) of three months, six months, one year, three years, five years, and ten years calculated, for each data set, from the monthly yield rates.

The entire period for which logratios are calculated is January 1954 through December 1988. (The December 1953 rate provides the starting point for the calculation of the logratios.) This period was chosen because it is beyond the effects of World War II, data are available for all maturities and all sources studied, and it is long enough to cover several business cycles and periods of inflation. In addition to the entire period (denoted by "All" in the paper and appendixes), several subperiods are used to test the assumptions. These subperiods are identified as follows:

A = January 1954 through December 1978
B = January 1979 through December 1988
1 = January 1954 through December 1958
2 = January 1959 through December 1963
3 = January 1964 through December 1968
4 = January 1969 through December 1973
5 = January 1974 through December 1978
6 = January 1979 through December 1983
7 = January 1984 through December 1988.

Subperiods A and B were chosen to isolate the effects of the switch of Federal Reserve Board policy from one of managing interest rates to that of managing the money supply at the end of 1978. The five-year subperiods were chosen partly for convenience in testing hypotheses and partly because business cycles range from four to seven years and five is a reasonable average. This choice facilitates the study of the period 1979 to 1983 when the Federal Reserve managed the money supply and the period 1984 to 1988 when it shifted back towards managing interest rates.

The first set of data is based on statistical summaries prepared by the Board of Governors of the Federal Reserve System. These documents are identified in [14]–[24]. The three-month and six-month data are based on the monthly auction average. The Treasury did not offer six-month securities much before 1959; therefore data for them cover the period 1959 to 1988. For the other maturities, the data are based on the "Treasury Constant Maturities." For a description of the "Treasury Constant Maturity Series," see the footnotes section of *Federal Reserve Statistical Release, G.13 (415)* from January 1990 forward. The three-month and the six-month discount rates were converted to a quarterly nominal rate for the three-month security and a bond-equivalent yield for the six-month security by application of the formula given in [5, p. 178] with the appropriate number of days until maturity. The quarterly nominal rate for the three-month security was then converted to a bond-equivalent yield. These data are referred to as FRB data, or Federal Reserve data.

The second set of data is based on *Analytical Record of Yields and Spreads*, prepared by Salomon Brothers, Inc. [12]; it is found in Part I, Table 1. Here the three-month and six-month yield data are already in bond-equivalent format. The values are derived by examining end-of-month bid prices for several of the most active, "on-the-run" securities with maturities closest to the desired maturity. Where there is a choice of coupons, Salomon follows the yields of higher coupon issues in the longer maturities. These data are referred to as SAL data, or the Salomon data.

The third set of data is based on *U.S. Treasury Yield Curves 1926–1988*, published by Moody's Investors Service Inc. [3]. Unlike the first two sources, which present traditional yield curve data, the Moody data derive spot rates associated with each security for each period. The authors use data from the U.S. Government Bond File of the Center for Research in Security Prices at the University of Chicago (CRSP). The data underlying CRSP data are Federal Reserve Board and Salomon Brothers, Inc. data. These data are referred to as the Moody data.

Unlike the first two sources, the Moody data involved significant adjustments to, and processing and smoothing of, the underlying information in order to produce the spot rates. Detailed information on this method is found in [2] and in [3, Chapter 4].

IV. "INDEPENDENCE"

This first assumption examined is that the series $\{J_t^m\}$ is stochastically independent. The series $\{j_t^m\}$ is a time series in t. Examination of the sample autocorrelation coefficients provides a means of testing independence. If the series represents random changes, then all autocorrelation coefficients are zero. Therefore, the presence of a non-zero autocorrelation coefficient demonstrates lack of independence. A given autocorrelation coefficient can be tested to determine whether it is non-zero by examining the corresponding sample autocorrelation coefficient.

Appendix A presents the sample autocorrelation coefficients, their standard error, and an indication of whether it is significantly non-zero at the 0.05 level for each combination of source of data and maturity. The period studied is the "All" period. Figures 1, 2, and 3 display the first seven sample autocorrelation coefficients (identified as LAGs) for the three-month Treasury security for FRB, Salomon, and Moody data, respectively.

Appendix B presents an analysis of the non-zero autocorrelation coefficients for each maturity by each combination of data and consecutive n year periods, where n equals 10, 15, 20, 25, 30, and 35 years. Each table (maturity) lists for each such period the number or LAG of the sample autocorrelation for which the null hypothesis, that the given autocorrelation coefficient is zero, is rejected. From these tables it is possible to determine whether a regular pattern of non-zero autocorrelation coefficients emerges, and if so, how long it takes for the effect to be recognizable and consistent.

For three-month, six-month, one-year, and three-year Treasury securities, a clear pattern of consistent non-zero autocorrelation coefficients emerges over all sources of data. This pattern relates to the LAG 1 and LAG 7 coefficients. The pattern is consistent with FRB data at consecutive 10-year periods and emerges at consecutive 15-year periods on Salomon and Moody data. The same pattern appears for longer consecutive year periods. Thus for securities of these maturities, the assumption of independence can be rejected.

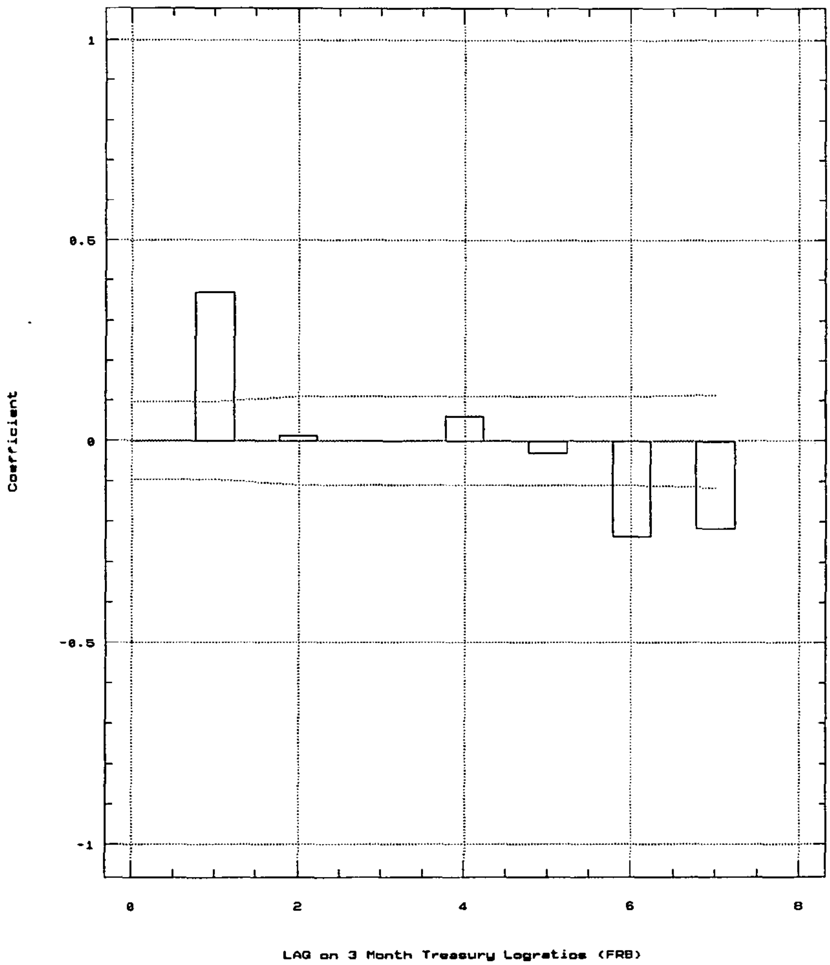
The results for five-year Treasury securities based on Salomon data are not completely consistent with FRB data for the same maturity. The results for the LAG 7 autocorrelation coefficient are consistent with FRB data, but not clearly so for the LAG 1 autocorrelation coefficient. They are also not supported by the Moody data. The hypothesis of independence can be rejected for the five-year Treasury security based on FRB and Salomon data.

For the ten-year maturity, the hypothesis can be rejected only on FRB data; the other two sources fail to support the rejection.

The results based on FRB data for three-month and six-month securities are specially significant because these yields are based on auction results. There is no averaging of end-of-month, on-the-run securities with maturities close to these two and no issue of bid versus asked. The degree of processing of data to arrive at the yields is greater for Salomon data than for FRB data

FIGURE 1

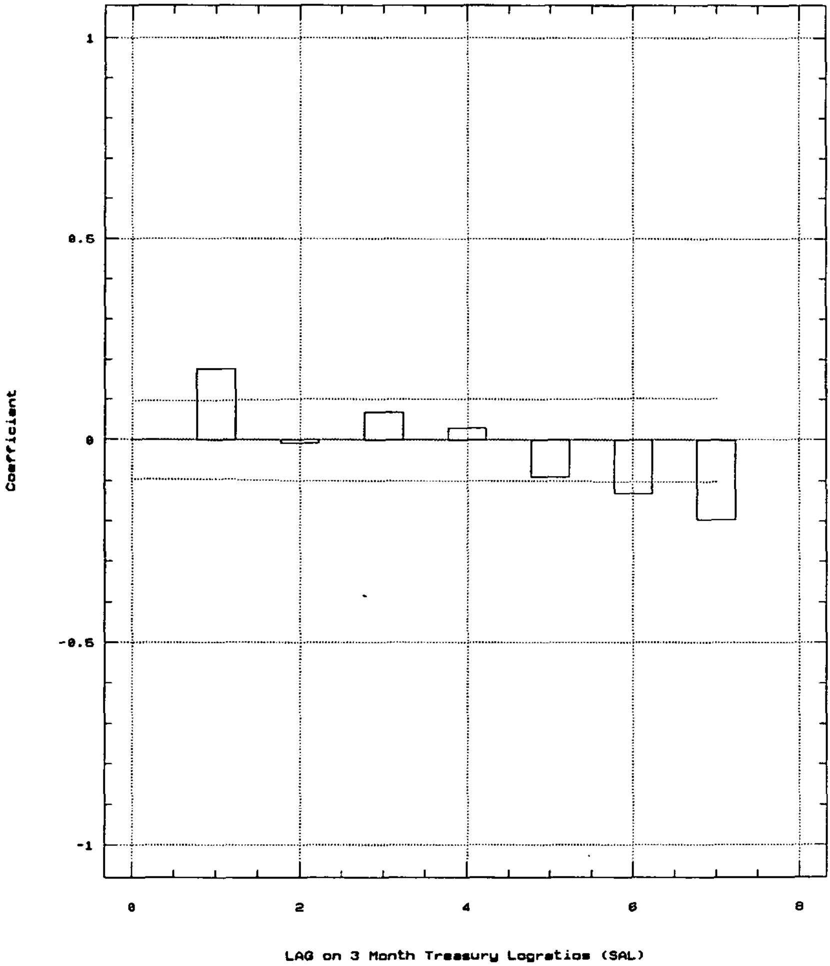
SAMPLE AUTOCORRELATIONS FOR THREE-MONTH TREASURY SECURITIES BASED ON FEDERAL RESERVE BOARD DATA



Sample autocorrelation coefficients extending beyond the dotted lines are significant at the 5% level.

FIGURE 2

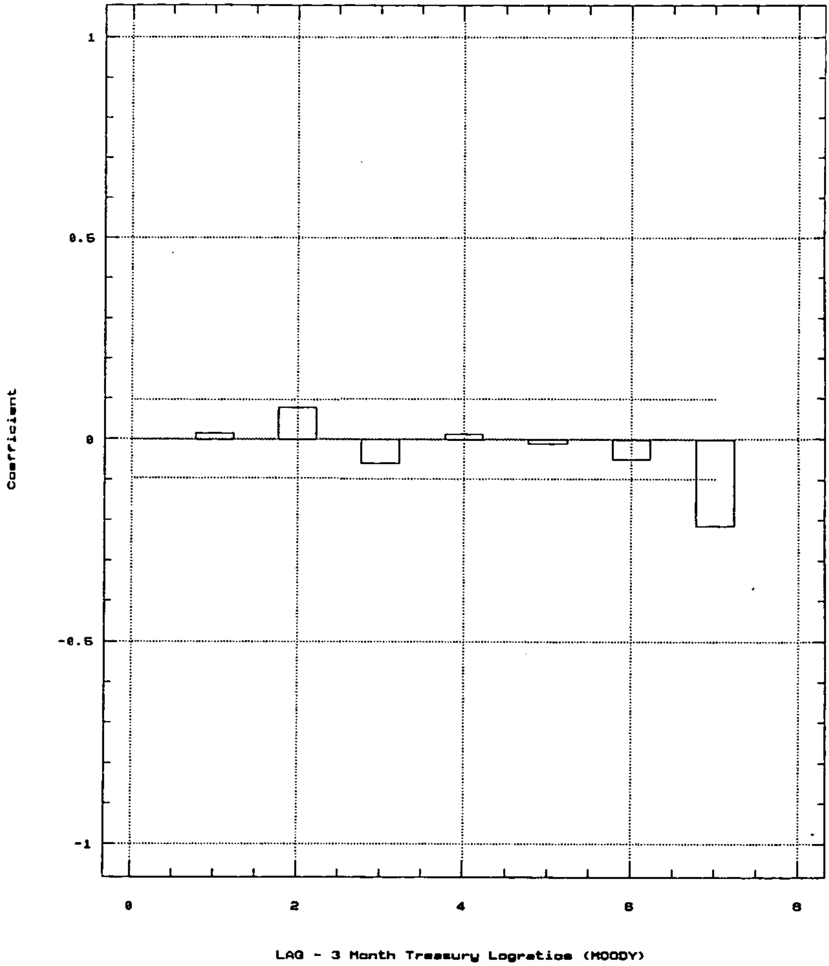
SAMPLE AUTOCORRELATIONS FOR THREE-MONTH TREASURY SECURITIES
BASED ON SALOMON DATA



Sample autocorrelation coefficients extending beyond the dotted lines are significant at the 5% level.

FIGURE 3

SAMPLE AUTOCORRELATIONS FOR THREE-MONTH TREASURY SECURITIES
BASED ON MOODY'S DATA



Sample autocorrelation coefficients extending beyond the dotted lines are significant at the 5% level.

and greater still for Moody data. The presence of averaging or other techniques may reduce the "signal" in the underlying data. This could account for the tendency for longer periods to be required for rejection on Salomon and Moody data, for the LAG 1 coefficient not to be significant on Moody data, and for the LAG 6 coefficient not to appear significant on Salomon or Moody data.

There is important information beyond the rejection of the assumption of independence. For securities of maturity less than or equal to three years, the algebraic sign of the LAG 1 coefficient is always positive, the LAG 6 always negative, and the LAG 7 always negative. The information being analyzed is the logratios of the yields of various maturities. The positive LAG 1 coefficient indicates the tendency for an increase (decrease) in yields to be followed by an increase (decrease). The negative LAG 6 and LAG 7 coefficients indicate the tendency for a decrease (increase) in yield in a given month if six and/or seven months prior there occurred an increase (decrease). The negative LAG 6 and LAG 7 autocorrelation coefficients suggest the presence of mean reversion.

V. "THE DISTRIBUTION IS NORMAL"

The second assumption examined is that the distribution is normal. Three tests are employed: the chi-square goodness-of-fit test, the standardized skewness test, and the standardized kurtosis test. The first test is an overall test for the equality of observed and expected frequencies. The second and third tests are targeted at explicit characteristics of the normal distribution.

A graphical technique referred to as "hanging histobars," or just "histobars," is a graph of the data with the best fitting normal distribution imposed. Bars, representing the actual frequency in each class, are attached to the normal curve at the appropriate point and "hang down" rather than being plotted from the horizontal axis. The distance of the bottom of the bar to the horizontal axis indicates the degree of deviation from the expected frequency. If the data are normally distributed, then the bottoms of the bars should be randomly scattered about the horizontal axis. Any pattern in how the bars vary around the horizontal axis indicates deviation from a normal distribution.

The data used for these tests and graphical techniques are the actual data; no attempt has been made to "smooth" the data or remove "outliers." Removing outliers is often supported because there may have been errors in measurement or recording of data of an experiment. In this case, that is less likely because these data are not from an isolated experiment; there are standardized techniques for collecting and recording the data and there are several sets of data for examination. To smooth or remove data would eliminate information, and such information is important because any model built by using "smoothed" data could seriously underestimate the extremes of predicted behavior.

Figures 4, 5, and 6 present the histobars of the ALL period for the three-month Treasury security for FRB, Salomon and Moody data sources, respectively. Each figure has the same pattern. The bars closest to the mean extend well below the horizontal axis; the next group is well above the horizontal axis on each side; and there are many outliers. This means the frequency of data near the mean is too high; the frequency away from the mean is too low; and, significantly, many points are more than three standard deviations from the mean. This pattern holds for the other maturities and sources of data. The indication is that the hypothesis that the distribution is normal over the ALL period should not be accepted.

The chi-square distribution is used to test the hypothesis that the underlying distribution is normal. Expected frequencies for the best fitting normal distribution are compared to the observed frequencies. The significance level is 0.05. The number of classes used and the endpoints for the intervals were chosen by algorithms in a statistical software package. Intervals are automatically grouped to avoid intervals with too few actual observations. For completeness and a check of the software, other choices for intervals were made manually, but no differences in the results of the tests were found. Note that the chi-square has low power at the smaller sample sizes. The p values for the chi-square tests are presented in Appendix C for all combinations of source of data, maturity, and period.

The results of the chi-square tests show that, for long periods, the hypothesis of normality should be rejected. For the shorter periods, B and the five-year periods, it is not generally possible to reject the hypothesis using this test. The evidence would support rejecting the hypothesis for the three-year maturity, based on all sources of data, and the five-year maturity, based on the Salomon and Moody data.

Skewness represents any deviation from a symmetric distribution. Positive skewness indicates a longer right tail and negative skewness a longer left tail. A normal distribution is symmetric, and therefore, the skewness should be zero. Negative kurtosis indicates a flat distribution with short tails compared to a normal distribution. Positive kurtosis indicates either a very peaked distribution, one with long tails, or both, compared to a normal distribution. The standardized skewness and kurtosis tests are approximately standard

FIGURE 4

HISTOBARS OF LOGRATIOS FOR THREE-MONTH TREASURY SECURITIES
BASED ON FEDERAL RESERVE BOARD DATA

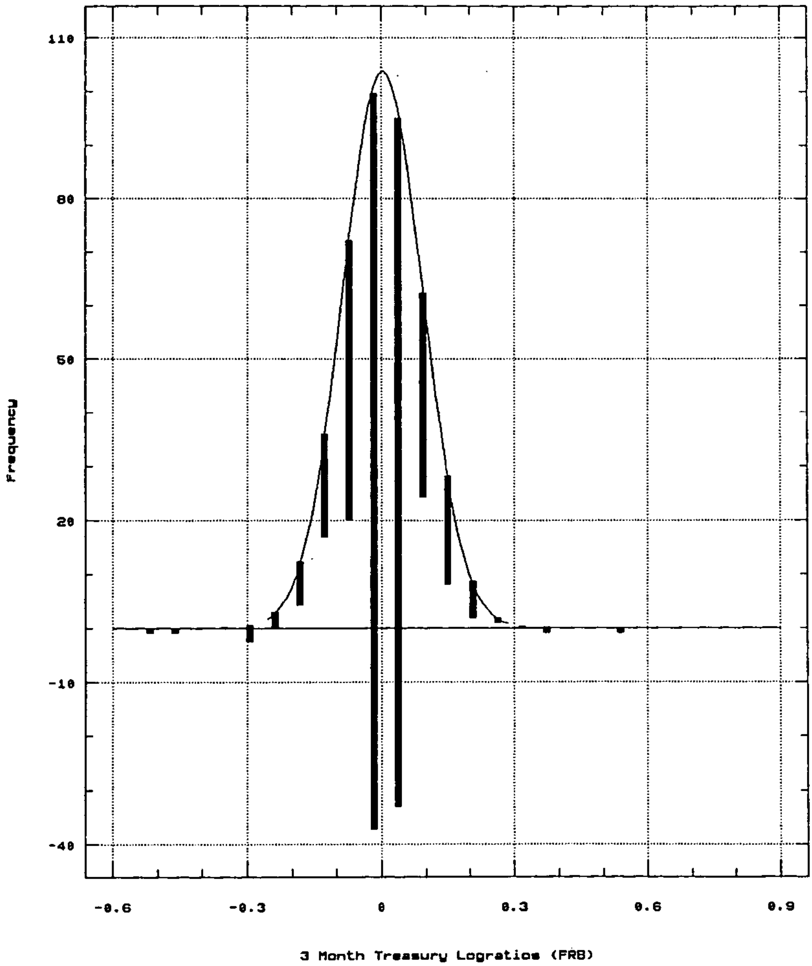


FIGURE 5

HISTOBARS OF LOGRATIOS FOR THREE-MONTH TREASURY SECURITIES
BASED ON SALOMON DATA

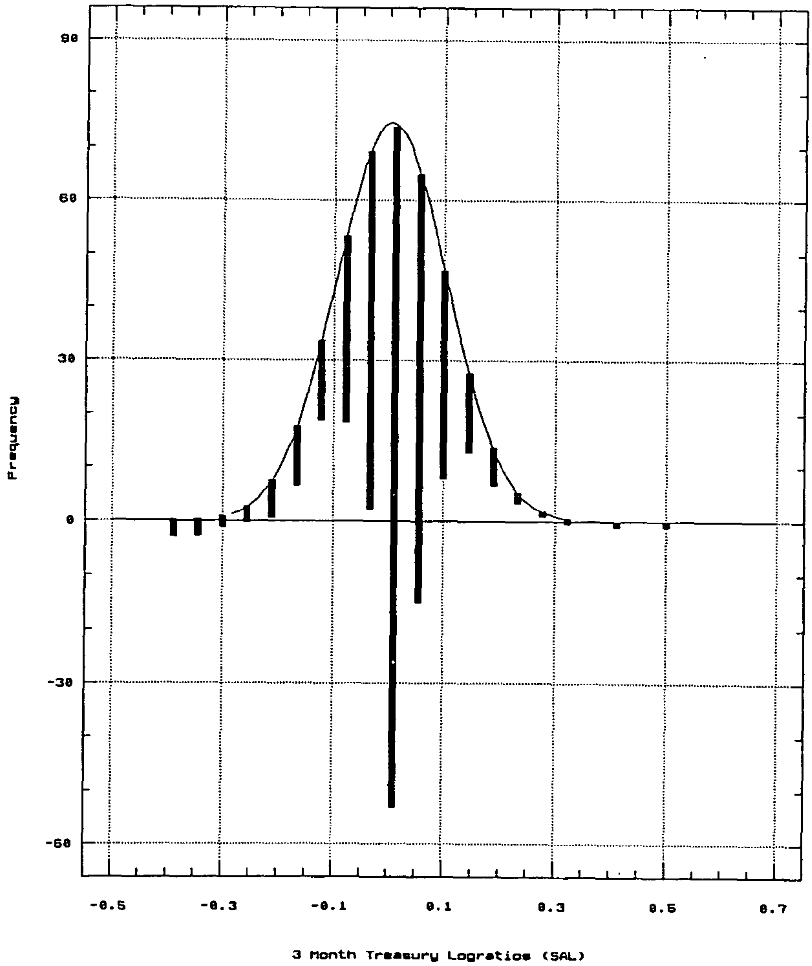
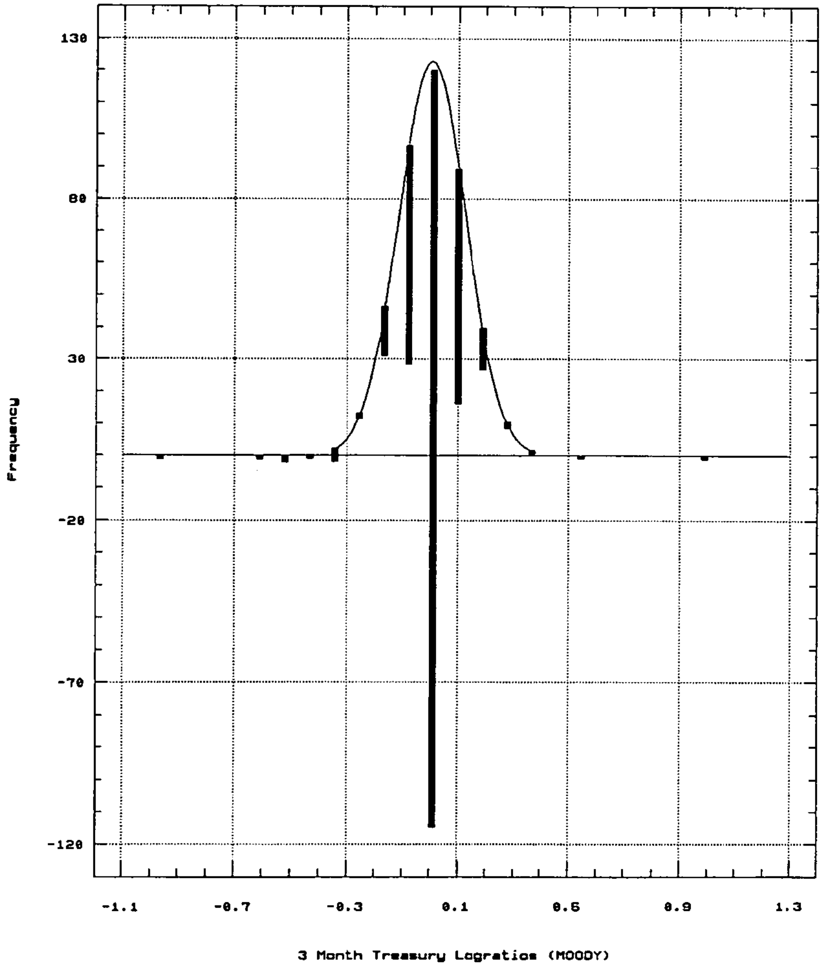


FIGURE 6

HISTOBARS OF LOGRATIOS FOR THREE-MONTH TREASURY SECURITIES
BASED ON MOODY'S DATA



normal for large sample sizes. Note that this in no way requires the data examined to be normally distributed, because this is precisely what the two tests are trying to determine. The standardized skewness and kurtosis tests are supplemented by direct tests of skewness and kurtosis that are valid at small sample sizes. If the skewness test results in rejection, then the data are asymmetric and unlikely to be a realization of a normal distribution. If the kurtosis results in rejection, then the data are again unlikely to be a realization of a normal distribution. The significance level for these tests is 0.05.

Appendix D presents the "summary statistics" for each combination of maturity, period, and source of data. The summary statistics include: mean, standard deviation, minimum, maximum, range, standardized skewness, and standardized kurtosis. With probability 0.95 both the standardized skewness and standardized kurtosis are between -2.0 and $+2.0$ for samples from a normally distributed population (because the standardized skewness and kurtosis tests are approximately standard normal). Standardized skewness or kurtosis values falling outside that range would result in the rejection of the underlying distribution being normal at the 0.05 level. Please note the extreme values obtained for the standardized kurtosis (shown in Appendix D) compared to the five percent rejection limits of -2.0 and $+2.0$.

The results of the chi-square goodness-of-fit test, the standardized skewness test, and the standardized kurtosis test are summarized in Appendix E. For each combination of source of data, maturity, and period, an entry is made to indicate failure of any of the tests. In the tables the failure of the chi-square test, the standardized skewness test, and the standardized kurtosis test is indicated by the presence of the letters C, S and K, respectively. The three tables indicate that the assumption that the underlying distribution is normal should be rejected for the ALL, A, and B periods and for periods 1–6. Note the high frequency of rejections of the standardized kurtosis test.

These statistical tests require the data to be independent. This is not usually a problem in experimental situations because randomization can be designed into the experiment. In the present case, dependence is present when periods of ten or more years are considered. This has the effect of slightly shrinking the confidence interval around the value of the null hypothesis when testing the longer periods, thus making it more likely to reject the null hypothesis when it should not be rejected. But the extreme values obtained for nearly all of the chi-square, standardized skewness, and kurtosis statistics indicate that the decision to reject would not change at the five percent level.

For the five-year periods, it is not possible to consistently reject the hypothesis of independence. This fact and the magnitude of the test results also indicate that the decision to reject the hypothesis of normality for the five-year periods would not change.

An important implication of the high positive kurtosis of the actual data is that data can be found much farther from the mean more commonly than in the case of a normal distribution. For a normal distribution the probability that a point is five or more standard deviations from the mean is on the order of two in a million. Using Tchebycheff's inequality (for any random variable with finite mean and standard deviation), the upper bound on the probability of a point being five or more standard deviations from the mean is on the order of two in fifty.

V. "VARIANCE IS CONSTANT"

The third assumption examined is that the variance is constant. To test this, consider the seven five-year periods. If each five-year period is viewed as a sample from the identical distribution, then consider the null hypothesis that variances over all five-year periods are equal. The alternative is that for at least two five-year periods the variances are different. The significance level is 0.05.

These tests are performed for each combination of maturity and source of data. Common tests for the equality of variance are: Bartlett's test, Cochran's C test, and Hartley's test. These three tests are sensitive to deviations of the underlying data from normality, especially positive kurtosis. Thus these three tests are not reliable because the underlying data are unlikely to be normally distributed and, in fact, have extreme positive kurtosis. Instead, a robust test, that is, not requiring the underlying data to be normally distributed, developed by Layard [10] is employed. The test statistics and p values (where significant) for all combinations are shown in Appendix F.

The null hypothesis can be rejected for all but one combination of maturity and source of data. Thus the constancy of variance must be rejected. This will not surprise most people familiar with financial data. The volatility (standard deviation) of the change in interest rates is believed to change over time. But there are implications of this fact that should be considered.

If each series of logratios is viewed as a time series, then, in the language of time series analysis, this result would be termed as showing that the series is heteroscedastic; that is, the variance is changing over time. A consequence of this in modeling future interest rates is that the assumption of a constant variance is unrealistic and the results are less credible than for a model that incorporated the heteroscedasticity.

Consider an application of exploratory data analysis. This graphically displays the disparity of variances over the five-year periods and illustrates other features of the data. These other features provide confirming information about the lack of normality of the underlying distribution.

The technique is known as a "notched box-and-whisker plot." A box-and-whisker plot is a method that effectively: displays summary statistics graphically; detects outliers, or data unusually far from the median; detects asymmetric behavior; and reveals the degree of concentration of the data. A box-and-whisker plot has the following characteristics:

- It divides the data into four regions of equal frequency;
- The top and bottom of the box are the 75th and 25th percentiles, respectively;
- The central line of the box is the median (50th percentile);
- The whiskers normally extend below the 25th percentile to the lower limit and above the 75th percentile to the upper limit of the data;
- The length of the whisker is limited to 1.5 times the interquartile range;
- Any data more than 1.5 times the interquartile range from the median is identified by a separate point beyond the end of the whisker (these are called "outliers");
- The width of the box is proportional to the square root of the number of observations in the data set; and
- A "notch" corresponds to the width of a confidence interval for the median. The confidence is set to allow a pairwise comparison of two such intervals at the 95 percent level by examining if they overlap.

Figures 7, 8, and 9 display notched box-and-whisker plots for the logratios of three-month Treasury securities based on FRB, Salomon, and Moody data, respectively. Each figure shows all seven five-year periods. The character of the graphs for Treasury securities of other maturities is similar and not shown.

Based on the definition of a box-and-whisker plot, the figures show the wide variations in the dispersion of data about the median across the five-year periods. These results are consistent with the rejection of the null hypothesis of the constancy of variance. The figures also demonstrate the concentration of the data toward the median with a significant number of

FIGURE 7

SUBPERIOD NOTCHED BOX-AND-WHISKER PLOT
FOR THREE-MONTH TREASURY SECURITIES
BASED ON FEDERAL RESERVE BOARD DATA

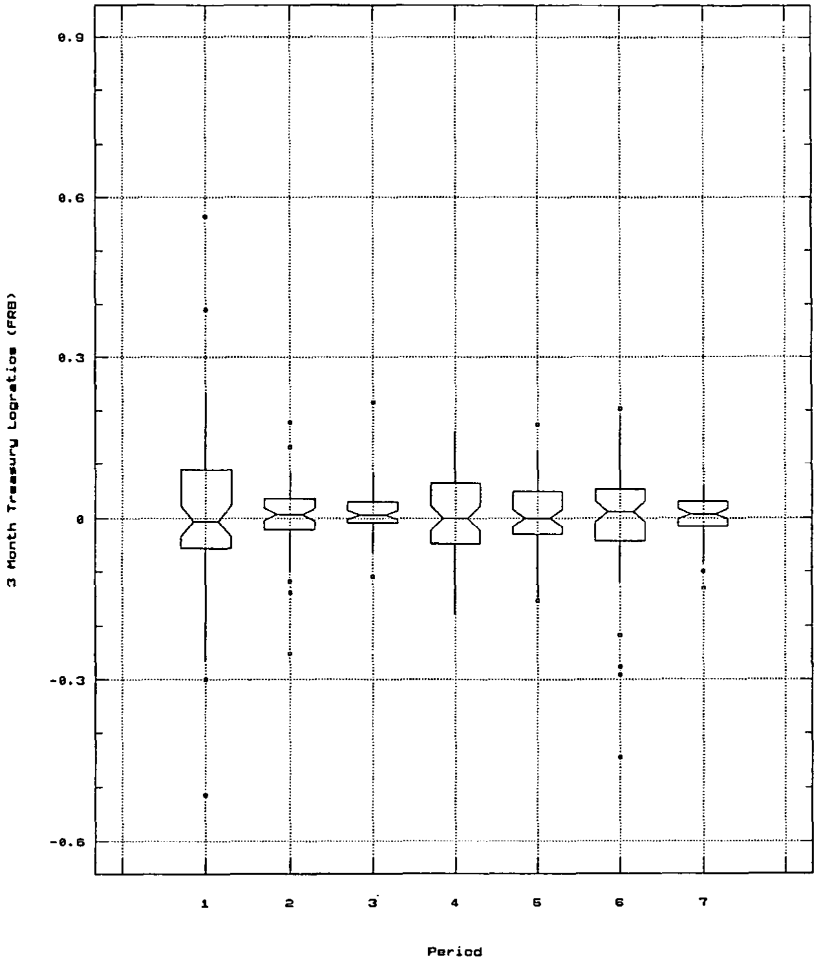


FIGURE 8

SUBPERIOD NOTCHED BOX-AND-WHISKER PLOT
FOR THREE-MONTH TREASURY SECURITIES
BASED ON SALOMON DATA

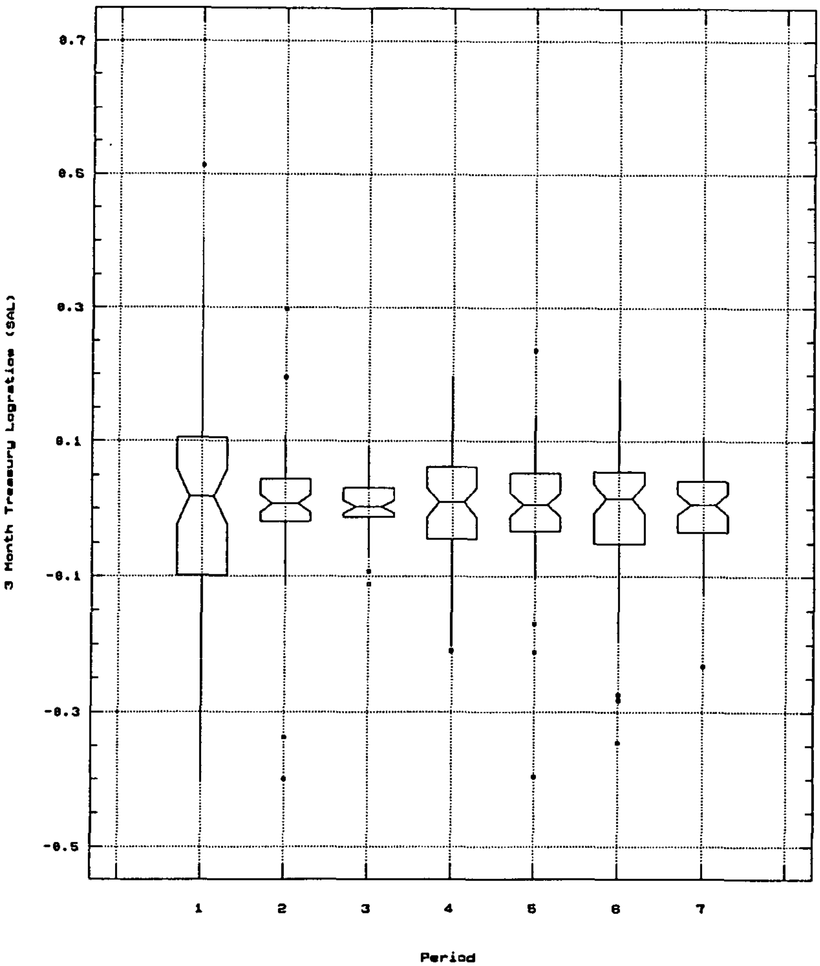
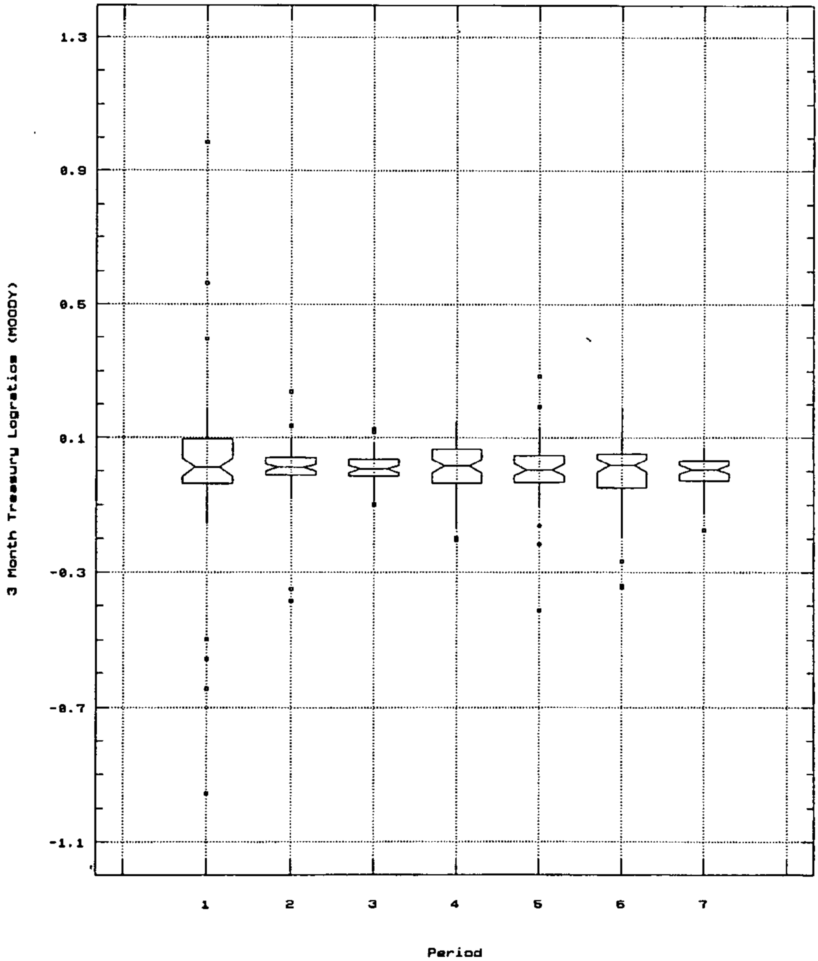


FIGURE 9

SUBPERIOD NOTCHED BOX-AND-WHISKER PLOT
FOR THREE-MONTH TREASURY SECURITIES
BASED ON MOODY'S DATA



outliers, which are indicated numerically by the extremely positive standardized kurtosis values. This would not be expected if the underlying distribution were normal. The lengths of the upper and lower halves of the boxes suggest there may be some asymmetry. Note that these observations apply to all three figures, that is, all sources of data. Graphs for other maturities and sources of data are similar.

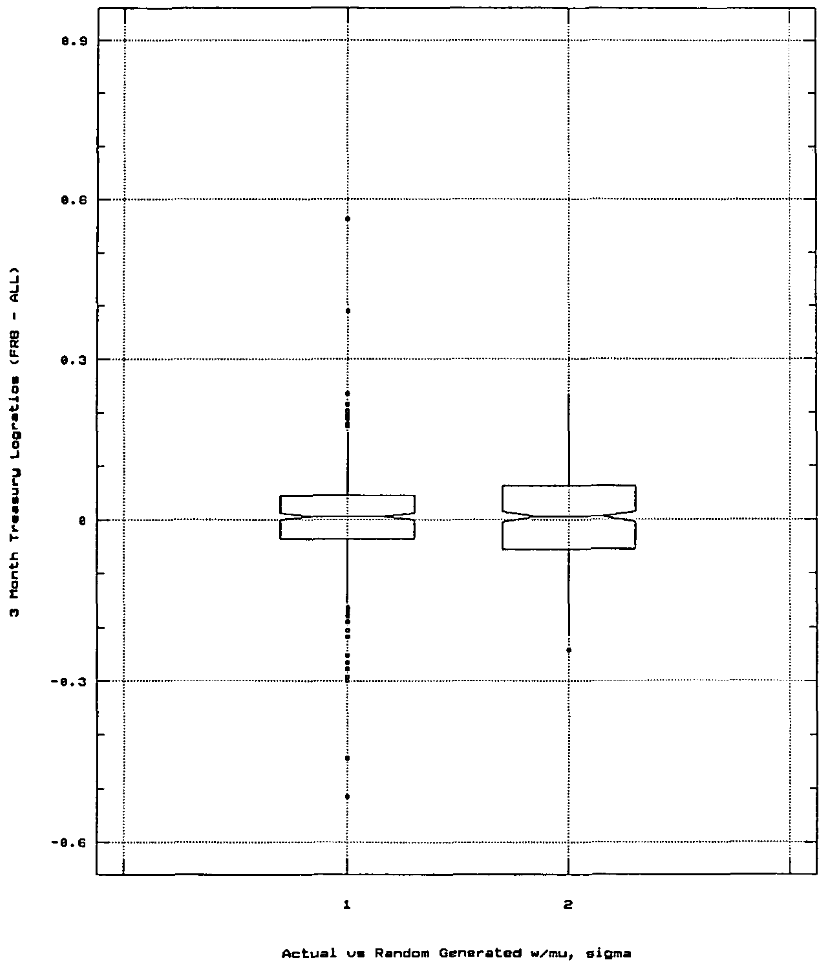
Figure 10 presents a notched box-and-whisker plot for the logratios of the three-month Treasury security over the period 1954 to 1988 (FRB data). The actual data are represented by the left box. The right box is another box-and-whisker plot based on a randomly generated sample of the same size as the actual data, where the distribution was assumed to be normal with the same sample mean and sample variance as the actual data. Comparison of the two plots shows the greater concentration of data about the median and the significantly larger number of outliers for the actual data versus the randomly generated data. This could be accounted for by the variance changing over time, as it was shown to be. But if this explanation is valid, then a similar comparison of actual to randomly generated data with same mean and standard deviation over a shorter time interval, where the variance is reasonably constant, should result in a graph in which the two plots are similar in size, symmetry, concentration of data, and presence of outliers, if any.

Figure 11 shows a comparison of the notched box-and-whisker plots for two new data sets. The first set is the period 1 logratios for three-month securities based on FRB data. The second data set is a random sample of 60 items in which each item was independently drawn from a normal population with mean and variance equal to that of the data in the first period. Figure 12 is a similar comparison, but uses the period 6 three-month security, FRB data. In each case the actual data still exhibit more concentration towards the median and more outliers and show some skewness than the randomly generated data based on a normal distribution with the same mean and variance as the actual data. This indicates that the characteristic concentration and frequency of outliers of the actual data are not due only to variance changing over long periods. If changing variance is the explanation, then it must change over short periods.

Layard's test is again employed to determine whether the variance is constant over short periods. In this case it is used to test whether the variance is constant within the five-year subperiods. Each data source, each maturity, and each five-year subperiod are tested. The results are shown in Appendix F. The hypothesis that the variance is constant over the five-year subperiods

FIGURE 10

NOTCHED BOX-AND-WHISKER PLOT OF ACTUAL VERSUS RANDOMLY GENERATED LOGRATIOS BASED ON FEDERAL RESERVE BOARD DATA FOR ALL PERIODS



is clearly rejected based on FRB and Moody data. The result is less amply demonstrated for the Salomon data, where the hypothesis is rejected in only 38 percent of the cells. Note that if the null hypothesis were true, "rejection" would be expected in only five percent of the cells.

The finding that the variance is not constant over the five-year subperiods suggests that the variance may not exist, that is, a non-finite second moment. In either case (an extremely volatile variance or non-existence of the variance), a normal assumption with constant variance cannot be supported for projections over any significant length of time.

VII. "MU EQUALS ZERO"

The fourth assumption examined is that the mean of the distribution is equal to zero. The alternative is that the mean is not equal to zero. A two-tailed test is used. The significance level is 0.05.

The test of the null hypothesis is performed for all sources of data, all maturities, and all periods. The p values for these tests are presented in Appendix G.

In general, the test results indicate that the hypothesis that the mean of the distribution is equal to zero at the 0.05 level cannot be rejected. In fact, for three-month, six-month, one-year, three-year, and five-year maturities, the hypothesis was not rejected for any combination of data source and time period. For ten-year maturities the hypothesis was not rejected for the ALL, B, and all five-year periods no matter what source of data was used. The A period for the ten-year maturity was rejected under FRB data and Salomon data, but not under Moody data. The Moody result was slightly larger than 0.05. The hypothesis for the ten-year maturity was not rejected for any combination of five-year periods and source of data.

These tests were performed using the t statistic. Although this test is "robust" for large samples (which frees the test from requiring the underlying data to be normally distributed), it does assume that the random variables are independent. But the p values are conservative because the effect of the dependence is to narrow the confidence interval. Thus, if the test is not rejected with the original critical region, then it will not be rejected if the region is enlarged.

VIII. SUMMARY AND FURTHER INVESTIGATIONS

In general, the only assumption that could not be rejected was that the mean of the distribution equals zero.

FIGURE 11

NOTCHED BOX-AND-WHISKER PLOT OF ACTUAL VERSUS RANDOMLY GENERATED LOGRATIOS BASED ON FEDERAL RESERVE BOARD DATA FOR PERIOD 1

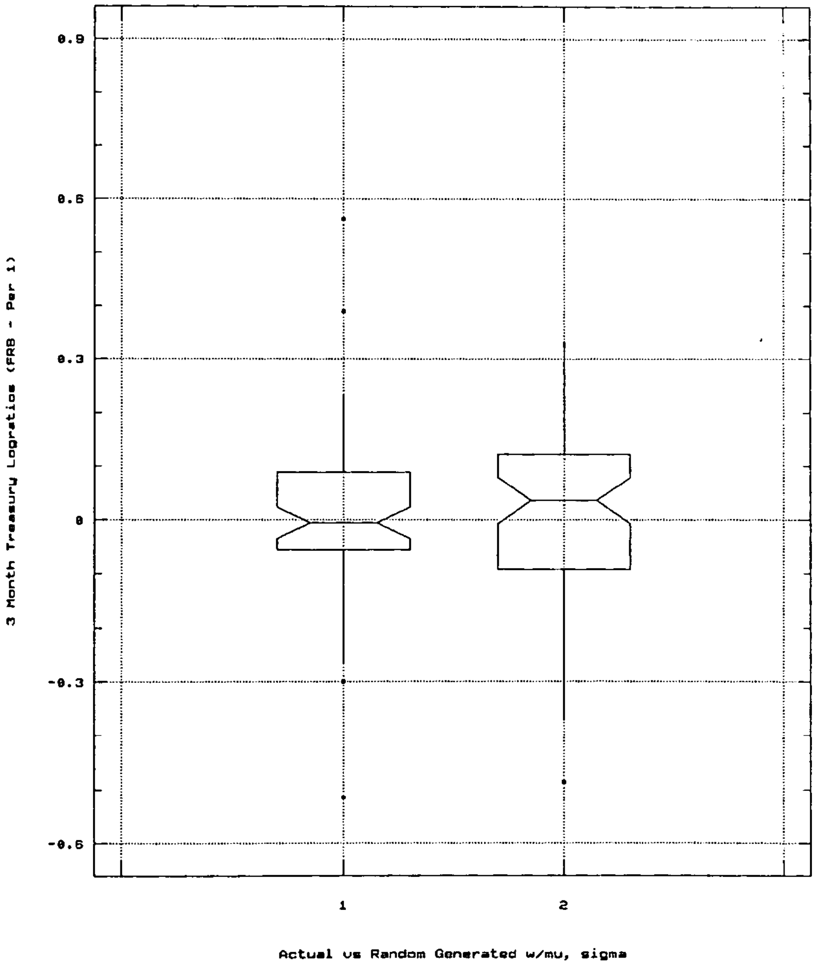
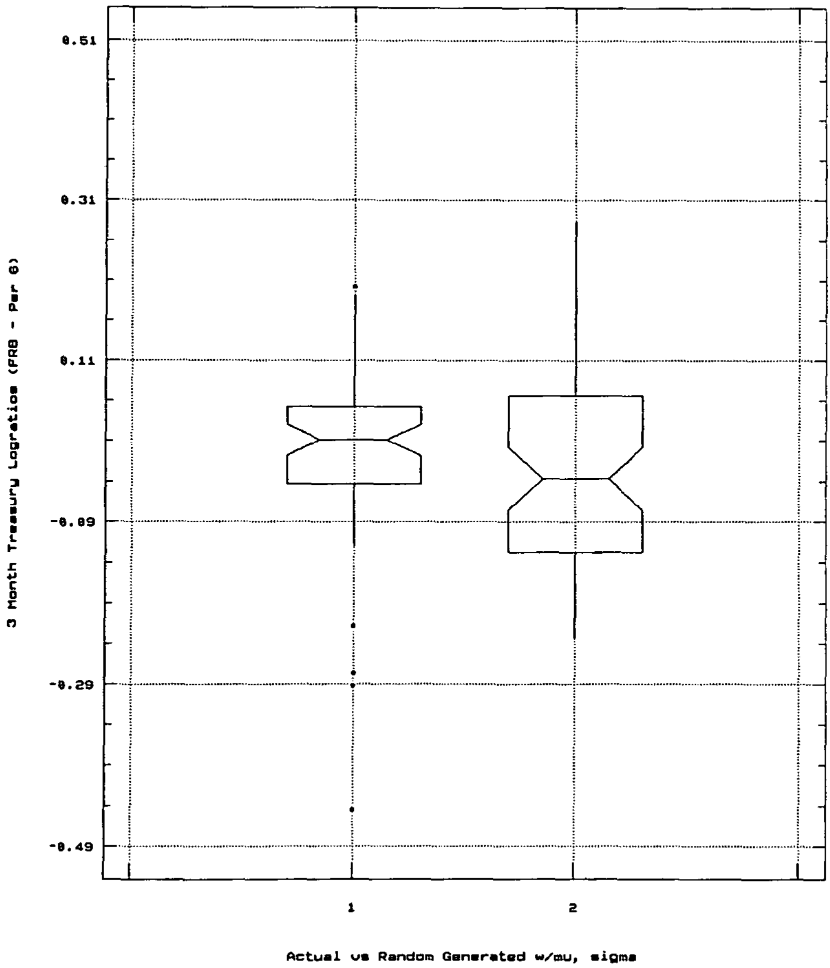


FIGURE 12

NOTCHED BOX-AND-WHISKER PLOT OF ACTUAL VERSUS RANDOMLY GENERATED LOGRATIOS BASED ON FEDERAL RESERVE BOARD DATA FOR PERIOD 6



The assumption of a constant variance was rejected based on Layard's test for both the ALL period and the five-year subperiods.

The assumption of normality was rejected based on the chi-square, standardized skewness, and standardized kurtosis tests. The striking frequency of high kurtosis values indicates a distribution with data heavily concentrated near the mean with significantly many outliers in nearly all periods studied. The various graphical analyses demonstrate this visually. The volatility or possible non-existence of the variance also casts doubt on the applicability of the normal assumption.

The assumption of independence must be rejected for all securities less than or equal to five years in maturity. Consistency of non-zero autocorrelation coefficients emerges with consecutive ten-year periods on FRB data and fifteen-year periods on Salomon and Moody data. The consistently positive LAG 1 autocorrelation coefficient indicates a greater probability that an increase (decrease) in rates will be followed by another increase (decrease). The consistently negative LAG 6 and/or LAG 7 coefficients are consistent with the existence of a mean reversion process occurring within less than a twelve-month period. This negative character indicates that a decrease (increase) is more likely in a given month if an increase (decrease) occurred six and/or seven months prior. The magnitude of the LAG 1 autocorrelation coefficient (0.37 FRB data) on three-month Treasury securities indicates a narrowing of the confidence interval for predictions in future logratios by 7 percent. If the LAG 1, LAG 6, and LAG 7 information is used, the narrowing is by 10 percent. The narrowing is less for other maturities and data sources.

One possible explanation for the presence of heteroscedasticity and extreme positive kurtosis could be the situation in which the underlying distribution is a stable distribution. Stable distributions can have characteristics of concentration of data, a significant amount of data far from the mean, and non-finite second moment, that is, no standard deviation. Without a finite standard deviation any test for homogeneity of variance (or normality) would likely result in rejection. One can think of this type of situation as one in which the data cluster about the mean with greater frequency than in a normal distribution, but the dispersion of the data about the mean is materially larger than in a normal distribution. Work has been done in this area with regard to the movement of stock prices with some success. (This is the lognormal assumption applied to security prices.) See [1], [6]–[8] and [11] for investigations of stable distributions and applications.

A second possible explanation for the characteristics of the observed data is a combination of a random change based on the lognormal assumption coupled with a mean reversion adjustment. The mean reversion adjustment can take different functional forms (see, for example, [9, p. 430 and pp. 439–441]). The concept of a mean reversion is that for a given maturity there is a "long-run" yield rate to which the actual yield rates tend over time. Once this long-run rate and the functional form are specified, then the next period's rate is determined from a random shock to last period's rate coupled with a reversion that adjusts the "shocked rate" back towards the long-run rate. Often the reversion is larger the greater the difference between the shock rate and the long-run rate. Algebraically, this might be written as:

$$I_{t+1} = I_t \times e^{\sigma \times z_t} \times f(a_1, a_2, \ldots, a_n, i, I_t, \sigma, z_t),$$

where $a_1, \ldots, a_n$ are parameters in the functional form, i is the long-run rate, σ is the volatility, and Z_t is a standard normal random variable.

This mechanism provides for the observed pattern of logratios. In the absence of mean reversion, there is a normal distribution of logratios with a given volatility. But with mean reversion, the effect is to reduce the size of the change in the rate. Thus the logratios would exhibit a more concentrated distribution and have a volatility less than that without mean reversion. Together these could account for the presence of outliers with regard to the lower volatility of the rates after mean reversion.

This mechanism can be tested in the following way: Having chosen a functional form and a given parameterization, adjust the $\{i_t\}$ by the mean reversion and then apply the statistical tests to the resulting series. This approach is dependent on the presence of mean reversion.

A third possible explanation is the combination of the lognormal process with a "jump" process, for example, a Poisson process. This model has the jump superimposed upon the lognormal process, much as the mean reversion was superimposed above. The jumps can be either positive or negative. This type of approach has been used to better model the movement of security prices where the jump is applied after the lognormal process is applied to the security price. For an application refer to Merton [12].

Any of these alternatives or other models must still deal with the presence of autocorrelation. More research on models of interest rate movement is clearly needed and will be welcomed by the actuarial and financial communities.

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APPENDIX A

This appendix presents the first seven sample autocorrelation coefficients and their standard errors for each maturity and each data source for the period ALL.

TABLE A1

THREE-MONTH MATURITY

		Sample Autocorrelation Coefficient		
LAG		Federal Reserve Board	Salomon	Moody's
1	Value	0.3693‡	0.1763‡	0.0146
	Standard Error	0.0488	0.0488	0.0488
2	Value	0.0117	−0.0092	0.0776
	Standard Error	0.0551	0.0503	0.0488
3	Value	−0.0017	0.0681	−0.0604
	Standard Error	0.0551	0.0503	0.0491
4	Value	0.0599	0.0307	0.0127
	Standard Error	0.0551	0.0505	0.0493
5	Value	−0.0316	−0.0908	−0.0113
	Standard Error	0.0552	0.0506	0.0493
6	Value	−0.2362‡	−0.1310†	−0.0510
	Standard Error	0.0553	0.0509	0.0493
7	Value	−0.2178‡	−0.1976‡	−0.2167‡
	Standard Error	0.0576	0.0517	0.0494

*Significant at the 0.05 level.
†Significant at the 0.01 level.
‡Significant at the 0.001 level.

TABLE A2

SIX-MONTH MATURITY

		Sample Autocorrelation Coefficient		
LAG		Federal Reserve Board	Salomon	Moody's
1	Value	0.2941‡	0.1011[a]	0.1257*
	Standard Error	0.0527	0.0527	0.0527
2	Value	−0.0147	−0.0284	−0.0292
	Standard Error	0.0571	0.0532	0.0535
3	Value	−0.0628	0.0312	0.0377
	Standard Error	0.0571	0.0532	0.0536
4	Value	−0.0030	0.0289	−0.0606
	Standard Error	0.0573	0.0533	0.0537
5	Value	0.0053	−0.0412	−0.0064
	Standard Error	0.0573	0.0534	0.0538
6	Value	−0.1721†	−0.0754	−0.0892
	Standard Error	0.0573	0.0535	0.0538
7	Value	−0.1682†	−0.1541†	−0.1656†
	Standard error	0.0587	0.0551	0.0542

[a]This sample autocorrelation coefficient has a p value of 0.057.
*Significant at the 0.05 level.
†Significant at the 0.01 level.
‡Significant at the 0.001 level.

TABLE A3

ONE-YEAR MATURITY

LAG		Sample Autocorrelation Coefficient		
		Federal Reserve Board	Salomon	Moody's
1	Value	0.4300‡	0.2454‡	0.0771
	Standard Error	0.0488	0.0488	0.0488
2	Value	0.0777	−0.0212	0.0029
	Standard Error	0.0571	0.0516	0.0491
3	Value	0.0074	0.0002	0.0297
	Standard Error	0.0574	0.0517	0.0491
4	Value	0.0099	0.0379	0.0052
	Standard Error	0.0574	0.0517	0.0491
5	Value	−0.0310	−0.0062	−0.0204
	Standard Error	0.0574	0.0517	0.0491
6	Value	−0.1721†	−0.0880	−0.0230
	Standard Error	0.0574	0.0517	0.0492
7	Value	−0.1558†	−0.1678‡	−0.2052‡
	Standard Error	0.0586	0.0521	0.0492

*Significant at the 0.05 level.
†Significant at the 0.01 level.
‡Significant at the 0.001 level.

TABLE A4

THREE-YEAR MATURITY

LAG		Sample Autocorrelation Coefficient		
		Federal Reserve Board	Salomon	Moody's
1	Value	0.3899‡	0.1360†	0.1293†
	Standard Error	0.0488	0.0488	0.0488
2	Value	−0.0204	−0.0001	0.0043
	Standard Error	0.0557	0.0497	0.0496
3	Value	−0.0252	−0.0391	−0.0418
	Standard Error	0.0557	0.0497	0.0496
4	Value	0.0231	0.0643	0.0142
	Standard Error	0.0558	0.0498	0.0497
5	Value	0.0272	0.0202	0.0379
	Standard Error	0.0558	0.0500	0.0497
6	Value	−0.1158*	−0.0266	−0.0690
	Standard Error	0.0558	0.0500	0.0498
7	Value	−0.1661‡	−0.1268*	−0.1074*
	Standard Error	0.0564	0.0500	0.0500

*Significant at the 0.05 level.
†Significant at the 0.01 level.
‡Significant at the 0.001 level.

TABLE A5

FIVE-YEAR MATURITY

LAG		Sample Autocorrelation Coefficient		
		Federal Reserve Board	Salomon	Moody's
1	Value	0.3741‡	0.1043*	0.0946
	Standard Error	0.0488	0.0488	0.0488
2	Value	−0.0617	−0.0282	−0.0157
	Standard Error	0.0552	0.0493	0.0492
3	Value	−0.0194	−0.0396	−0.0476
	Standard Error	0.0554	0.0494	0.0492
4	Value	0.0368	0.0625	0.0551
	Standard Error	0.0554	0.0494	0.0494
5	Value	0.0319	−0.0095	0.0123
	Standard Error	0.0554	0.0496	0.0495
6	Value	−0.1091*	−0.0262	−0.0624
	Standard Error	0.0555	0.0496	0.0495
7	Value	−0.1450*	−0.1056*	−0.0765
	Standard Error	0.0560	0.0496	0.0497

*Significant at the 0.05 level.
†Significant at the 0.01 level.
‡Significant at the 0.001 level.

TABLE A6

TEN-YEAR MATURITY

LAG		Sample Autocorrelation Coefficient		
		Federal Reserve Board	Salomon	Moody's
1	Value	0.3282‡	0.0754	0.0653
	Standard Error	0.0488	0.0488	0.0488
2	Value	−0.0999	−0.0159	−0.0347
	Standard Error	0.0538	0.0491	0.0490
3	Value	−0.0060	−0.0702	−0.0948
	Standard Error	0.0542	0.0491	0.0491
4	Value	0.0565	0.0733	0.1111*
	Standard Error	0.0542	0.0493	0.0495
5	Value	0.0420	−0.0074	−0.0198
	Standard Error	0.0544	0.0496	0.0501
6	Value	−0.0847	−0.0184	−0.0320
	Standard Error	0.0544	0.0496	0.0501
7	Value	−0.0990	−0.0790	−0.0643
	Standard Error	0.0548	0.0496	0.0502

*Significant at the 0.05 level.
†Significant at the 0.01 level.
‡Significant at the 0.001 level.

APPENDIX B

This appendix presents the LAG of the sample autocorrelation coefficients for which the null hypothesis, that such coefficient was zero, could be rejected for various maturities, sources of data, and consecutive 10-, 15-, 20-, 25-, 30-, and 35-year periods.

The tables show the LAG number of the coefficients that are non-zero. Note that, unless otherwise stated, all occurrences of certain LAG coefficients have the same algebraic sign; that is, they are all positive or all negative. The following LAG is always positive unless otherwise stated: LAG 1. The following LAGs are always negative: LAGs 6 and 7. For any other LAGs or unusual occurrences, the sign of the LAG coefficient ("+" or "−") is inserted behind the LAG number.

This appendix also includes data for the six-month Treasury security. For FRB data the auction yields were used, not the secondary market, and they were converted to bond-equivalent yields according to the same formula as the three-month Treasury yields (also auction yields) based on FRB data, with allowance for the different number of days to maturity.

TABLE B1

THREE-MONTH MATURITY

Consecutive Period	LAG Number of Sample Autocorrelation Coefficient		
	Federal Reserve Board	Salomon	Moody's
10-Year Periods			
1954–1963	1, 6, 7	1, 7	7
1959–1968	1, 7	3+, 7	3+, 7
1964–1973	1		1
1969–1978	1		
1974–1983	1, 6	6	
1979–1988	1	1	1
15-Year Periods			
1954–1968	1, 6, 7	1, 7	7
1959–1973	1, 7	3+, 7	3+, 7
1964–1978	1, 6	6	
1969–1983	1, 6, 7	6	
1974–1988	1, 6, 7	6	
20-Year Periods			
1954–1973	1, 6, 7	1, 7	7
1959–1978	1, 6, 7	3+, 7	3+
1964–1983	1, 6, 7	6	6
1969–1988	1, 6, 7	6	6
25-Year Periods			
1954–1978	1, 6, 7	1, 3+, 6, 7	7
1959–1983	1, 6, 7	7	6
1964–1988	1, 6, 7	6	
30-Year Periods			
1954–1983	1, 6, 7	1, 6, 7	7
1959–1988	1, 6, 7	7	7
35-Year Period			
1954–1988	1, 6, 7	1, 6, 7	7

TABLE B2

Six-Month Maturity

Consecutive Period	LAG Number of Sample Autocorrelation Coefficient		
	Federal Reserve Board	Salomon	Moody's
10-Year Periods			
1959–1968	1,	1, 7	3+, 7
1964–1973	1, 7	1	1, 7
1969–1978	1, 6		
1974–1983	1, 6		
1979–1988	1		1
15-Year Periods			
1959–1973	1, 7	7	3+, 7
1964–1978	1, 6, 7	7	7
1969–1983	1, 6, 7		1, 7
1974–1988	1, 6		1
20-Year Periods			
1959–1978	1, 6, 7	3, 7	3+, 7
1964–1983	1, 6, 7	1, 7	1, 6, 7
1969–1988	1, 6, 7	1 7	1, 7
25-Year Periods			
1959–1983	1, 6, 7	7	1, 7
1964–1988	1, 6, 7	1, 7	1, 6, 7
30-Year Periods			
1959–1988	1, 6, 7	[1] 7	1, 7

Note that six-month Treasury securities were not issued until 1958–1959. Therefore there is no auction information for them. To keep the comparison on the same basis, the Salomon and Moody data were utilized for the same periods.

The "[1]" represents the LAG 1 sample autocorrelation coefficient that had a p value of 0.057, as shown in Appendix A.

TABLE B3

ONE-YEAR MATURITY

Consecutive Period	LAG Number of Sample Autocorrelation Coefficient		
	Federal Reserve Board	Salomon	Moody's
10-Year Periods			
1954–1963	1, 2+	1	7
1959–1968	1	3+, 7	3+, 7
1964–1973	1		1
1969–1978	1, 6	7	
1974–1983			
1979–1988	1, 7	1, 3−	1, 3−
15-Year Periods			
1954–1968	1, 2+, 6	1, 7	7
1959–1973	1	1, 7	
1964–1978	1, 6, 7	7	7
1969–1983	1, 6, 7	1, 7	1 7
1974–1988	1, 7	7	1
20-Year Periods			
1954–1973	1, 2+, 6	1, 7	7
1959–1978	1, 6	7	7
1964–1983	1, 6, 7	1, 7	1, 7
1969–1988	1, 6, 7	1, 7	1
25-Year Periods			
1954–1978	1, 6, 7	1, 7	7
1959–1983	1, 6, 7	1, 7	7
1964–1988	1, 6, 7	1, 7	1, 7
30-Year Periods			
1954–1983	1, 6, 7	1, 7	7
1959–1988	1, 6, 7	1, 7	1, 7
35-Year Period			
1954–1988	1, 6, 7	1, 7	7

TABLE B4

THREE-YEAR MATURITY

Consecutive Period	LAG Number of Sample Autocorrelation Coefficient		
	Federal Reserve Board	Salomon	Moody's
10-Year Periods			
1954–1963	1	1, 4+	1
1959–1968	1	7	
1964–1973	1	5−	
1969–1978	1		
1974–1983	1, 7		
1979–1988	1		1
15-Year Periods			
1954–1968	1, 7	1, 4+, 7	1
1959–1973	1		
1964–1978	1	5−	7
1969–1983	1, 7		
1974–1988	1		1
20-Year Periods			
1954–1973	1, 7	1, 7	
1959–1978	1, 7	7	7
1964–1983	1, 7	7	7
1969–1988	1, 7		
25-Year Periods			
1954–1978	1, 6, 7	1, 7	7
1959–1983	1, 7	7	7
1964–1988	1, 7		
30-Year Periods			
1954–1983	1, 7	1, 7	1, 7
1959–1988	1, 7	7	7
35-Year Period			
1954–1988	1, 6, 7	1, 7	1, 7

TABLE B5

FIVE-YEAR MATURITY

Consecutive Period	LAG Number of Sample Autocorrelation Coefficient		
	Federal Reserve Board	Salomon	Moody's
10-Year Periods			
1954–1963	1	1	1, 6
1959–1968	1		
1964–1973	1	5–	
1969–1978	1		
1974–1983	1		
1979–1988	1		
15-Year Periods			
1954–1968	1, 6	1, 7	
1959–1973	1		
1964–1978	1, 7	5–	
1969–1983	1, 7		
1974–1988	1		
20-Year Periods			
1954–1973	1, 6, 7		
1959–1978	1, 7		
1964–1983	1, 7	7	
1969–1988	1		
25-Year Periods			
1954–1978	1, 6, 7	7	7
1959–1983	1, 7		
1964–1988	1		
30-Year Periods			
1954–1983	1, 6, 7	1, 7	1, 7
1959–1988	1, 7		
35-Year Period			
1954–1988	1, 6, 7	1, 7	

TABLE B6

TEN-YEAR MATURITY

Consecutive Period	LAG Number of Sample Autocorrelation Coefficient: Federal Reserve Board	Salomon	Moody's
10-Year Periods			
1954–1963	1, 4+, 6	1, 6	4+
1959–1968	1		
1964–1973	1		
1969–1978	1		
1974–1983	1, 7		
1979–1988	1		
15-Year Periods			
1954–1968	1, 6	7	4+
1959–1973	1		3−
1964–1978	1		
1969–1983	1	3−	3−
1974–1988	1		
20-Year Periods			
1954–1973	1		4+
1959–1978	1		3−
1964–1983	1, 7		3−
1969–1988	1		
25-Year Periods			
1954–1978	1, 6	7	4+
1959–1983	1, 2−, 7		3−
1964–1988	1		
30-Year Periods			
1954–1983	1, 7	7	3−
1959–1988	1, 2−		3−
35-Year Period			
1954–1988	1		4+

APPENDIX C

This appendix presents the p values of the test for normality based on the chi-square distribution. The significance level is 0.05. The results are reported for all combinations of sources of data, maturities, and periods.

TABLE C1

p VALUES FOR CHI-SQUARE TESTS OF NORMALITY

	p Value					
Period	3 Month	6 Month	1 Year	3 Year	5 Year	10 Year
Data Source: Federal Reserve Board						
All	<0.01*	<0.01*	<0.01*	0.03*	0.03*	0.13
A	<0.01*	0.02*	<0.01*	0.16	0.14	0.24
B	<0.01*	<0.01*	0.09	0.53	0.76	0.51
1	<0.01*	n/a	0.12	0.69	0.03*	0.17
2	0.02*	<0.01*	0.02*	0.11	0.31	0.15
3	0.24	0.25	0.11	0.57	0.09	0.08
4	0.74	0.11	0.99	0.20	0.61	0.47
5	0.16	0.31	0.36	0.16	0.05	0.35
6	0.02*	0.09	0.19	0.55	0.57	0.40
7	0.02*	0.12	0.23	0.65	0.29	0.28
Data Source: Salomon Brothers, Inc.						
All	<0.01*	<0.01*	<0.01*	<0.01*	<0.01*	<0.01*
A	<0.01*	<0.01*	<0.01*	<0.01*	<0.01*	<0.01*
B	<0.01*	0.08	0.40	0.27	0.52	0.44
1	0.25	n/a	0.13	0.30	0.17	0.39
2	<0.01*	<0.01*	0.08	0.01*	<0.01*	0.20
3	0.15	<0.01*	<0.01*	<0.01*	<0.01*	0.03*
4	0.50	0.26	0.37	0.08	0.37	0.49
5	0.27	0.37	0.42	0.45	0.32	0.36
6	0.03*	0.43	0.51	0.27	0.69	0.64
7	0.15	0.26	0.06	0.18	<0.01*	0.37
Data Source: Moody's Investors Service Inc.						
All	<0.01*	<0.01*	<0.01*	<0.01*	<0.01*	<0.01*
A	<0.01*	<0.01*	<0.01*	<0.01*	<0.01*	<0.01*
B	<0.01*	0.01*	0.15	0.15	0.76	0.54
1	<0.01*	n/a	<0.01*	0.09	0.07	0.11
2	<0.01*	<0.01*	0.06	0.09	<0.01*	0.04*
3	0.03*	0.07	0.02*	0.02*	<0.01*	0.30
4	0.12	0.79	0.04*	0.17	0.02*	0.04*
5	0.03*	0.35	0.13	0.43	0.03*	0.35
6	0.04*	0.38	0.19	0.11	0.60	0.89
7	0.64	0.07	0.39	0.05	0.08	0.17

Note: A "<" indicates a p value materially less than 0.01.
*Significant at the 0.05 level.

APPENDIX D

SUMMARY STATISTICS FOR LOGRATIOS

TABLE D1

DATA SOURCE: FEDERAL RESERVE BOARD

Statistic	Period									
	All	A	B	1	2	3	4	5	6	7
					Three-Month Treasury Security					
Mean	0.003877	0.005843	−0.001036	0.009183	0.003811	0.008806	0.003730	0.003684	−0.003161	−0.001756
Std. Deviation	0.089557	0.090991	0.086043	0.157710	0.062467	0.046008	0.081839	0.065845	0.114076	0.043794
Minimum	−0.514592	−0.514592	−0.444579	−0.514592	−0.252953	−0.109896	−0.178879	−0.155139	−0.444579	−0.130952
Maximum	0.562527	0.562527	0.202134	0.562527	0.177952	0.215292	0.161138	0.173610	0.202134	0.058664
Range	1.07712	1.07712	0.64671	1.07712	0.43090	0.32519	0.34002	0.32875	0.64671	0.18962
Std. Skewness	−2.416*	0.806	−6.815*	0.420	−3.180*	3.828*	−0.489	−0.460	−4.129*	−2.954*
Std. Kurtosis	35.328*	31.533*	15.477*	5.570*	7.723*	9.884*	−0.561	0.523	5.805*	0.407
					Six-Month Treasury Security					
Mean	0.002820	0.004800	−0.001141		0.003030	0.008443	0.003671	0.004056	−0.000473	−0.001809
Std. Deviation	0.069989	0.064370	0.080202		0.061804	0.048893	0.078617	0.066143	0.103952	0.046548
Minimum	−0.421438	−0.242356	−0.421438		−0.242356	−0.094968	−0.174659	−0.144451	−0.421438	−0.117434
Maximum	0.233720	0.233720	0.184979		0.189850	0.233720	0.137706	0.177926	0.184979	0.070653
Range	0.65516	0.47608	0.60642		0.43221	0.32869	0.31236	0.32238	0.60642	0.18809
Std. Skewness	−6.469*	−1.180	−6.460*		−2.831*	4.632*	−0.720	−0.332	−4.189*	−1.840
Std. Kurtosis	17.691*	4.992*	14.359*		7.665*	11.006*	−0.671	0.150	6.182*	−0.610
					One-Year Treasury Security					
Mean	0.003962	0.006084	−0.001346	0.011401	0.002446	0.008088	0.002680	0.005806	−0.000310	−0.002381
Std. Deviation	0.073321	0.073851	0.072009	0.119529	0.053966	0.038573	0.071823	0.061119	0.091238	0.046173
Minimum	−0.348119	−0.286425	−0.348119	−0.286425	−0.225522	−0.090514	−0.161104	−0.132904	−0.348119	−0.104342
Maximum	0.440057	0.440057	0.168742	0.440057	0.123298	0.111813	0.156468	0.122541	0.168742	0.087130
Range	0.78818	0.72648	0.51686	0.72648	0.34882	0.20233	0.31757	0.25544	0.51686	0.191472
Std. Skewness	−0.868	1.737	−4.810*	1.173	−4.680*	−0.363	0.114	−0.649	−3.395*	−0.913
Std. Kurtosis	20.890*	18.392*	9.671*	3.754*	8.187*	1.220	−0.408	−0.538	4.454*	−0.865

TABLE D1—*Continued*

Statistic	Period: All	A	B	1	2	3	4	5	6	7
Three-Year Treasury Security										
Mean	0.003528	0.005019	−0.000199	0.009770	0.001251	0.007155	0.001672	0.005247	0.002940	−0.003338
Std. Deviation	0.053656	0.051249	0.059312	0.071930	0.041512	0.034460	0.056530	0.044458	0.070527	0.045840
Minimum	−0.241616	−0.173710	−0.241616	−0.173710	−0.106358	−0.088589	−0.119263	−0.083622	−0.241616	−0.103990
Maximum	0.234647	0.234647	0.165639	0.234647	0.116751	0.100682	0.148420	0.130590	0.165639	0.106576
Range	0.47626	0.40836	0.40725	0.40836	0.22311	0.18927	0.26768	0.21421	0.40726	0.21056
Std. Skewness	−1.495	0.740	−2.660*	0.246	−0.964	−0.522	0.271	0.909	−2.514*	−0.122
Std. Kurtosis	8.822*	6.246*	5.223*	1.872†	1.750†	1.760†	0.066	0.041	3.482*	−0.404
Five-Year Treasury Security										
Mean	0.003251	0.004548	0.000009	0.008311	0.000933	0.006922	0.001756	0.004819	0.003996	−0.003977
Std. Deviation	0.045268	0.041790	0.053032	0.057035	0.034607	0.029267	0.047335	0.035516	0.061059	0.043725
Minimum	−0.188794	−0.188794	−0.173911	−0.188794	−0.086000	−0.069526	−0.106160	−0.066586	−0.173911	−0.111508
Maximum	0.172040	0.172040	0.159722	0.172040	0.093475	0.078138	0.122218	0.090316	0.159722	0.108742
Range	0.36083	0.36083	0.33363	0.36083	0.17947	0.14766	0.22838	0.15690	0.33363	0.22025
Std. Skewness	−1.741	−0.849	−1.081	−0.882	−0.790	−0.498	−0.015	0.595	−1.384	0.120
Std. Kurtosis	7.940*	7.284*	2.901*	3.289*	1.607†	0.920	0.326	−0.569	2.145*	0.134
Ten-Year Treasury Security										
Mean	0.002995	0.004156	0.000092	0.006650	0.001126	0.006308	0.001855	0.004838	0.004538	−0.004354
Std. Deviation	0.035312	0.030318	0.045486	0.039191	0.023927	0.024791	0.035834	0.024995	0.050068	0.040324
Minimum	−0.147453	−0.147453	−0.123492	−0.147453	−0.062132	−0.064022	−0.074502	−0.059339	−0.123492	−0.111767
Maximum	0.138956	0.100976	0.138956	0.100976	0.054898	0.056281	0.091717	0.062677	0.138956	0.100937
Range	0.28641	0.24843	0.26245	0.24843	0.11703	0.12030	0.16622	0.12202	0.26245	0.21270
Std. Skewness	−3.327*	−2.906*	−1.101	−2.501*	−1.076	−1.020	−0.346	−0.034	−1.514	0.018
Std. Kurtosis	8.408*	7.365*	2.052*	4.880*	0.870	0.953	0.344	0.543	2.041*	0.358

*Significant at the 0.05 level.
†Significant at the 0.05 level using Pearson's small sample table.

TABLE D2

DATA SOURCE: SALOMON BROTHERS, INC.

Statistic	Period									
	All	A	B	1	2	3	4	5	6	7
Three-Month Treasury Security										
Mean	0.003711	0.005668	−0.001181	0.008670	0.003742	0.007575	0.004594	0.003759	−0.000252	−0.002109
Std. Deviation	0.099715	0.104807	0.085884	0.167373	0.096287	0.042411	0.091634	0.090364	0.107052	0.058441
Minimum	−0.401626	−0.401626	−0.345862	−0.401626	−0.399473	−0.110945	−0.210652	−0.395865	−0.345862	−0.232059
Maximum	0.512824	0.512824	0.192803	0.512824	0.297732	0.091708	0.196042	0.237059	0.192803	0.106838
Range	0.91445	0.91445	0.53866	0.91445	0.69721	0.20265	0.40669	0.63292	0.53866	0.33890
Std. Skewness	−3.711*	−2.177*	−5.098*	0.540	−4.536*	−1.080	−1.814	−4.451*	−3.225*	−4.018*
Std. Kurtosis	19.812*	17.032*	7.014*	1.785†	12.231*	0.774	0.312	9.805*	2.825*	4.793*
Six-Month Treasury Security										
Mean	0.002935	0.005053	−0.001302		0.003582	0.007361	0.006083	0.003189	−0.000447	−0.002156
Std. Deviation	0.076530	0.072990	0.083318		0.076334	0.047158	0.086052	0.078187	0.104425	0.055636
Minimum	−0.361694	−0.318165	−0.361694		−0.318165	−0.106556	−0.188111	−0.269040	−0.361694	−0.147636
Maximum	0.190934	0.162411	0.190934		0.160343	0.131521	0.162411	0.122148	0.190934	0.081401
Range	0.55263	0.48058	0.55263		0.47851	0.23808	0.35052	0.39119	0.55263	0.22904
Std. Skewness	−7.420*	−5.831	−4.354*		−5.496*	0.832	−1.182	−3.202*	−2.934*	−2.195*
Std. Kurtosis	11.297*	8.331*	7.028*		10.017*	1.777†	−0.371	3.126*	3.097*	0.156
One-Year Treasury Security										
Mean	0.003636	0.005612	−0.001305	0.008849	0.002434	0.007027	0.005169	0.004581	−0.000267	−0.002344
Std. Deviation	0.085232	0.087448	0.079557	0.142696	0.060440	0.047860	0.082083	0.075410	0.099849	0.052857
Minimum	−0.341844	−0.340759	−0.341844	−0.340759	−0.215751	−0.134753	−0.199796	−0.237166	−0.341844	−0.155006
Maximum	0.380549	0.380549	0.214666	0.380549	0.099330	0.154713	0.174041	0.149143	0.214666	0.091216
Range	0.72239	0.72131	0.55651	0.72131	0.31508	0.28947	0.37384	0.38631	0.55651	0.246222
Std. Skewness	−1.794	−0.559	−3.190*	0.416	−4.905*	0.966	−0.708	−2.770*	−2.104*	−2.096*
Std. Kurtosis	13.546*	11.741*	6.302*	1.104	6.388*	4.260*	0.001	2.794*	2.582*	0.589

TABLE D2—*Continued*

Statistic	Period: All	A	B	1	2	3	4	5	6	7
Three-Year Treasury Security										
Mean	0.003388	0.004836	−0.000231	0.008483	0.001520	0.005970	0.003098	0.005108	0.002938	−0.003401
Std. Deviation	0.063140	0.062748	0.064235	0.085801	0.048982	0.046541	0.072769	0.051980	0.075592	0.050864
Minimum	−0.240014	−0.206921	−0.240014	−0.206921	−0.160722	−0.130053	−0.203456	−0.148711	−0.240014	−0.102593
Maximum	0.243256	0.243256	0.211910	0.220587	0.109199	0.156321	0.243256	0.133311	0.211910	0.096511
Range	0.48327	0.45018	0.45192	0.42751	0.26992	0.28637	0.44671	0.28202	0.45192	0.19910
Std. Skewness	−0.894	− 0.828	−0.346	−0.418	−3.448*	−0.154	0.341	−0.353	−0.529	−0.013
Std. Kurtosis	7.958*	7.241*	3.818*	0.909	3.301*	4.786*	2.865*	1.685†	2.585*	−1.021
Five-Year Treasury Security										
Mean	0.003261	0.004532	0.000084	0.008188	0.001171	0.005572	0.003133	0.004595	0.004239	−0.004070
Std. Deviation	0.053734	0.051123	0.059879	0.065111	0.044492	0.040525	0.060057	0.041815	0.068670	0.049811
Minimum	−0.210954	−0.156060	−0.210954	−0.153685	−0.152721	−0.130425	−0.156060	−0.107359	−0.210954	−0.108423
Maximum	0.195163	0.182652	0.195163	0.171626	0.133219	0.126616	0.182652	0.114893	0.195163	0.089933
Range	0.40612	0.33871	0.40612	0.32531	0.28594	0.25704	0.33871	0.22225	0.40612	0.19836
Std. Skewness	−0.780	−0.728	−0.152	−0.088	−2.512*	−1.070	0.278	−0.348	−0.321	−0.279
Std. Kurtosis	6.406*	5.708*	2.892*	0.357	4.773*	5.468*	1.980†	1.696†	2.309*	−0.847
Ten-Year Treasury Security										
Mean	0.003036	0.004142	0.000270	0.006714	0.001226	0.005088	0.003064	0.004619	0.004575	−0.004035
Std. Deviation	0.040242	0.035216	0.050745	0.040856	0.028173	0.032109	0.043682	0.029414	0.054127	0.047184
Minimum	−0.160653	−0.117461	−0.160653	−0.117461	−0.080232	−0.107420	−0.114832	−0.070769	−0.160653	−0.104869
Maximum	0.134857	0.100245	0.134857	0.100245	0.071120	0.076961	0.091567	0.086029	0.134857	0.093839
Range	0.29551	0.21771	0.29551	0.21771	0.15135	0.18438	0.20640	0.15680	0.29551	0.19871
Std. Skewness	−3.013*	−3.409*	−0.649	−1.358	−1.646	−3.226*	−1.779	0.980	−0.590	−0.564
Std. Kurtosis	5.384*	5.763*	0.728†	2.574*	1.054	4.900*	1.097	1.194	1.169	−0.525

*Significant at the 0.05 level.
†Significant at the 0.05 level using Pearson's small sample table.

TABLE D3

DATA SOURCE: MOODY'S INVESTORS SERVICE INC.

Statistic	Period: All	A	B	1	2	3	4	5	6	7
Three-Month Treasury Security										
Mean	0.004337	0.006504	−0.001080	0.011370	0.004657	0.009386	0.003304	0.003801	−0.000207	−0.001952
Std. Deviation	0.121158	0.132929	0.085126	0.251973	0.087142	0.047024	0.084868	0.094852	0.109360	0.051521
Minimum	−0.955511	−0.955511	−0.346107	−0.955511	−0.385104	−0.100335	−0.204902	−0.411855	−0.346107	−0.174215
Maximum	0.986495	0.986495	0.191219	0.986495	0.239951	0.127026	0.148966	0.285275	0.191219	0.070452
Range	1.94201	1.94201	0.53733	1.94201	0.62506	0.22736	0.35387	0.69713	0.53733	0.24467
Std. Skewness	−5.949*	−4.611*	−5.743*	−1.152	−7.343*	−0.394	−2.304*	−3.496*	−3.641*	−3.337*
Std. Kurtosis	99.584*	79.980*	8.938*	11.615*	17.086*	0.961	0.229	10.342*	3.435*	1.991†
Six-Month Treasury Security										
Mean	0.004768	0.007137	−0.001152		0.003842	0.009542	0.002815	0.004381	−0.000751	−0.001556
Std. Deviation	0.202446	0.234033	0.081472		0.076851	0.050946	0.086049	0.078995	0.103614	0.051494
Minimum	−2.484910	−2.484910	−0.333224		−0.265318	−0.108214	−0.207030	−0.236119	−0.333224	−0.132578
Maximum	2.360850	2.360850	0.189813		0.178356	0.154151	0.179341	0.168066	0.189813	0.095177
Range	4.84576	4.84576	0.52304		0.44367	0.26236	0.38637	0.40418	0.52304	0.22776
Std. Skewness	−4.932*	−3.850*	−4.888*		−3.847*	0.832	−0.792	−2.554*	−3.218*	−2.105*
Std. Kurtosis	415.527*	274.790*	8.390*		5.147*	1.805†	−0.073	2.462*	3.460*	0.473
One-Year Treasury Security										
Mean	0.004197	0.006297	−0.001053	0.010463	0.004164	0.008740	0.002323	0.005797	−0.000502	−0.001603
Std. Deviation	0.101357	0.108955	0.079409	0.195435	0.073181	0.053341	0.090459	0.074415	0.099043	0.053928
Minimum	−0.607018	−0.607018	−0.317142	−0.607018	−0.243166	−0.124737	−0.230178	−0.198356	−0.317142	−0.121326
Maximum	0.617510	0.617510	0.226066	0.617510	0.167574	0.222650	0.211119	0.177421	0.226066	0.101754
Range	1.22453	1.22453	0.54321	1.22453	0.41074	0.34739	0.44130	0.37578	0.54321	0.22308
Std. Skewness	−0.174	0.463	−3.415*	0.244	−3.601*	2.526*	−0.036	−1.323	−2.487*	−0.650
Std. Kurtosis	35.321*	29.459*	7.609*	3.883*	4.969*	6.687*	0.475	1.213	3.640*	−0.646

TABLE D3—*Continued*

Statistic	Period									
	All	A	B	1	2	3	4	5	6	7
Three-Year Treasury Security										
Mean	0.003607	0.005050	0.000000	0.010132	0.001002	0.007394	0.001799	0.004922	0.003489	−0.003489
Std. Deviation	0.064054	0.063812	0.064783	0.088657	0.053801	0.045992	0.072108	0.050069	0.076074	0.051506
Minimum	−0.269410	−0.208015	−0.269410	−0.208015	−0.164378	−0.114297	−0.188645	−0.125323	−0.269410	−0.107566
Maximum	0.372404	0.372404	0.229122	0.372404	0.091249	0.132700	0.220543	0.136759	0.229122	0.119238
Range	0.64181	0.58041	0.49853	0.58042	0.25563	0.24700	0.40919	0.262082	0.49853	0.22680
Std. Skewness	0.831	1.580	−0.876	1.857	−4.040*	0.183	0.269	0.127	−1.157	0.416
Std. Kurtosis	16.105*	15.425*	6.146*	6.413*	3.534*	2.583*	2.126*	1.522†	4.468*	−0.634
Five-Year Treasury Security										
Mean	0.003407	0.004659	0.000276	0.009627	0.000503	0.007058	0.001547	0.004559	0.004587	−0.004035
Std. Deviation	0.052896	0.051035	0.057389	0.068348	0.044681	0.039797	0.057888	0.039220	0.064109	0.049950
Minimum	−0.214194	−0.212039	−0.214194	−0.212039	−0.137139	−0.107889	−0.151318	−0.094009	−0.214194	−0.108649
Maximum	0.260210	0.260210	0.192777	0.260210	0.069376	0.126548	0.164828	0.114238	0.192777	0.108960
Range	0.47440	0.47225	0.40697	0.47225	0.20652	0.23444	0.31615	0.20825	0.40697	0.21761
Std. Skewness	−0.700	−0.677	−0.127	0.273	−3.941*	−0.084	−0.189	0.283	−0.545	0.324
Std. Kurtosis	11.472*	12.288*	3.475*	5.528*	3.130*	2.847*	2.150*	1.570†	3.392*	−0.598
Ten-Year Treasury Security										
Mean	0.003090	0.004185	0.000352	0.007683	0.000777	0.006637	0.002114	0.003714	0.005150	−0.004447
Std. Deviation	0.040672	0.037106	0.048522	0.049534	0.032260	0.031692	0.042710	0.024899	0.049743	0.047196
Minimum	−0.171045	−0.171045	−0.143780	−0.171045	−0.099192	−0.088262	−0.114410	−0.051224	−0.143780	−0.122785
Maximum	0.163924	0.163924	0.120507	0.163924	0.062370	0.082888	0.106117	0.074724	0.120507	0.105880
Range	0.33497	0.33497	0.26429	0.33497	0.16156	0.17115	0.22053	0.12595	0.26429	0.22866
Std. Skewness	−2.889*	−2.764*	−0.939	−1.352	−2.905*	−1.149	−0.938	0.574	−0.674	−0.802
Std. Kurtosis	8.051*	10.981*	0.824	5.206*	2.362*	1.853†	1.781†	0.530	1.106	0.166

*Significant at the 0.05 level.
†Significant at the 0.05 level using Pearson's small sample table.

APPENDIX E

This appendix presents summary results for tests of normality for each combination of source of data, maturity, and period. In the tables, "S" represents failure of the standardized skewness test; "K" represents failure of the standardized kurtosis test; and "C" represents failure of the chi-square goodness-of-fit test. An "S" or "K" also represents a failure of the direct tests of the skewness or kurtosis. The skewness and kurtosis results are in the tables in Appendix B and the chi-square results are in Appendix C.

TABLE E1

SUMMARY OF FAILURES OF TESTS OF NORMALITY

	Test Failures					
Period	3 Month	6 Month	1 Year	3 Year	5 Year	10 Year
			Data Source: Federal Reserve Board			
All	S,K,C	S,K,C	K,C	K,C	K,C	S,K
A	K,C	K,C	K,C	K	K	S,K
B	S,K,C	S,K,C	S,K	S,K	K	K
1	K,C	n/a	K	K,C	K	S,K
2	S,K,C	S,K,C	S,K,C	K	K	
3	S,K	S,K		K		
4						
5						
6	S,K,C	S,K	S,K	S,K	K	K
7	S, C					
			Data Source: Salomon Brothers, Inc.			
All	S,K,C	S,K,C	K,C	K,C	K,C	S,K,C
A	S,K,C	S,K,C	K,C	K,C	K,C	S,K,C
B	S,K,C	S,K	S,K	K	K	K
1	K	n/a				K
2	S,K,C	S,K,C	S,K	S,K,C	S,K,C	
3		K,C	K,C	K,C	K,C	S,K,C
4				K	K	
5	S,K	S,K	S,K	K	K	
6	S,K,C	S,K	S,K	K	K	
7	S,K	S	S		C	
			Data Source: Moody's Investors Service Inc.			
All	S,K,C	S,K,C	K,C	K,C	K,C	S,K,C
A	S,K,C	S,K,C	K,C	K,C	K,C	S,K,C
B	S,K,C	S,K,C	S,K	K	K	
1	K,C	n/a	K,C	K	K	K
2	S,K,C	S,K,C	S,K	S,K	S,K,C	S,K,C
3	C	K	S,K,C	K,C	K,C	K
4	S		C	K	K,C	K,C
5	S,K,C	S,K		K	K,C	
6	S,K,C	S,K	S,K	K	K	
7	S,K	S				

APPENDIX F

This appendix presents the scores and *p* values for the tests of a constant variance of the series of logarithms of ratios of successive interest rates. Each maturity across each data source is tested. The null hypothesis is that the variance of each five-year period is equal to a common value. The alternative is that the variance is different between at least two periods out of period 1 through period 7. In addition, a similar hypothesis is tested within each five-year period; that is, is the variance constant within each five-year period? The test used is Layard's test [10].

For each maturity/data source/test combination, the value of the test statistic and its *p* value (where significant) are shown. The significance level is 0.05.

TABLE F1

RESULTS OF TESTS FOR A CONSTANT VARIANCE

Period		Chi-Square Values					
		3 Month	6 Month	1 Year	3 Year	5 Year	10 Year
Data Source: Federal Reserve Board							
All	Value	23.57	11.56	22.57	15.54	15.29	17.24
	p Value	< 0.01*	0.04*	< 0.01*	0.02*	0.02*	0.01*
1	Value	15.84		12.63	9.62	10.62	8.39
	p Value	< 0.01*		0.02*		0.03*	
2	Value	13.51	12.20	8.17	22.81	23.70	21.46
	p Value	0.01*	0.02*		< 0.01*	< 0.01*	< 0.01*
3	Value	7.03	8.14	14.51	13.16	17.77	29.28
	p Value			< 0.01*	0.01*	< 0.01*	< 0.01*
4	Value	16.14	12.00	12.68	11.53	11.85	8.87
	p Value	< 0.01*	0.02*	0.02*	0.02*	0.02*	
5	Value	10.10	12.75	20.41	13.89	15.90	6.56
	p Value	0.04*	0.01*	< 0.01*	0.01*	< 0.01*	
6	Value	14.21	11.03	10.16	9.49	10.59	9.48
	p Value	0.01*	0.03*	0.04*		0.04*	0.05
7	Value	4.13	1.88	3.69	2.44	3.00	3.94
	p Value						

TABLE F1—*Continued*

Period		Chi-Square Values					
		3 Month	6 Month	1 Year	3 Year	5 Year	10 Year
Data Source: Salomon Brothers, Inc.							
All	Value	28.82	11.78	25.68	12.95	6.34	14.44
	p Value	< 0.01*	0.04*	< 0.01*	0.04*		0.03*
1	Value	15.74		17.63	5.49	11.48	8.98
	p Value	< 0.01*		0.02*		0.03*	
2	Value	13.97	10.83	9.25	19.57	15.47	11.99
	p Value	< 0.01*	0.04*		< 0.01*	< 0.01*	0.02*
3	Value	23.43	17.50	11.86	17.65	18.35	24.90
	p Value	< 0.01*	< 0.01*	0.02*	< 0.01*	< 0.01*	< 0.01*
4	Value	5.59	9.03	8.11	5.98	5.71	19.96
	p Value						< 0.01*
5	Value	6.33	7.04	11.15	6.71	7.12	4.06
	p Value			0.03*			
6	Value	8.89	6.98	6.88	5.70	5.69	3.47
	p Value						
7	Value	2.27	0.91	1.50	1.39	1.69	3.52
	p Value						
Data Source: Moody's Investors Service Inc.							
All	Value	23.91	11.50	38.11	38.24	44.33	61.69
	p Value	< 0.01*	0.04*	< 0.01*	< 0.01*	< 0.01*	< 0.01*
1	Value	15.58		17.22	4.62	6.54	12.59
	p Value	< 0.01*		< 0.01*			0.02
2	Value	11.41	19.97	12.43	26.71	42.62	39.29
	p Value	0.02*	< 0.01*	0.02*	< 0.01*	< 0.01*	< 0.01*
3	Value	14.06	10.03	5.18	8.95	18.54	46.87
	p Value	0.01*	0.05*			< 0.01*	< 0.01*
4	Value	9.49	14.74	7.17	5.73	8.83	18.28
	p Value	0.05*	< 0.01*				< 0.01*
5	Value	8.63	15.11	21.33	14.30	11.50	9.43
	p Value		< 0.01*	< 0.01*	0.01*	0.05*	0.05
6	Value	10.58	8.26	8.17	6.35	5.33	4.43
	p Value	0.05*					
7	Value	3.05	1.76	0.86	1.65	1.97	5.01
	p Value						

Note: A "<" indicates a *p* value materially less than 0.01.
*Significant at the 0.05 level.

APPENDIX G

This appendix presents the p values for tests of the following hypotheses:

$$H_0: \text{mu} = 0; \text{ versus}$$
$$H_1: \text{mu not equal to } 0.$$

The alpha level of the test is 0.05; the test is two-tailed. This means that a p value of less than 0.05 implies rejection of the null hypothesis. This hypothesis is tested for all maturities (3 month, 6 month, 1 year, 3 year, 5 year, and 10 year), across all periods (entire period, A, B, 1, 2, 3, 4, 5, 6, and 7), and across all data sets (Federal Reserve Board, Salomon Brothers, Inc., and Moody's Investors Service Inc.). Note that results for the six-month Treasury security are based on auction data. This means that the data begin with 1959, when these securities were first auctioned. This means the ALL refers to 1959–1988, A to 1959–1978, and there are no period 1 values. Results for Salomon and Moody data for the six-month security will be based on the same time periods.

TABLE G1

p VALUES FOR TESTS OF THE MEAN

Period	p Values					
	3 Month	6 Month	1 Year	3 Year	5 Year	10 Year
	Data Source: Federal Reserve Board					
All	0.38	0.44	0.27	0.18	0.14	0.08
A	0.27	0.25	0.16	0.09	0.06	0.02*
B	0.90	0.88	0.84	0.97	0.99	0.98
1	0.65	n/a	0.46	0.30	0.26	0.19
2	0.64	0.70	0.73	0.82	0.84	0.72
3	0.14	0.19	0.11	0.11	0.07	0.05
4	0.73	0.72	0.77	0.82	0.78	0.69
5	0.67	0.64	0.47	0.36	0.30	0.14
6	0.98	0.97	0.98	0.75	0.61	0.49
7	0.76	0.76	0.69	0.58	0.48	0.41

TABLE G1—*Continued*

Period	p Values					
	3 Month	6 Month	1 Year	3 Year	5 Year	10 Year
Data Source: Salomon Brothers, Inc.						
All	0.45	0.46	0.38	0.27	0.21	0.12
A	0.35	0.28	0.27	0.18	0.13	0.04*
B	0.88	0.86	0.86	0.97	0.99	0.95
1	0.69	n/a	0.63	0.45	0.33	0.21
2	0.76	0.72	0.76	0.81	0.84	0.74
3	0.17	0.23	0.26	0.32	0.29	0.22
4	0.70	0.59	0.63	0.74	0.69	0.59
5	0.75	0.75	0.64	0.45	0.40	0.23
6	0.98	0.97	0.98	0.76	0.63	0.52
7	0.78	0.77	0.73	0.61	0.53	0.51
Data Source: Moody's Investors Service Inc.						
All	0.46	0.63	0.40	0.25	0.19	0.12
A	0.40	0.60	0.32	0.17	0.12	0.05
B	0.89	0.88	0.88	0.99	0.96	0.94
1	0.73	n/a	0.68	0.38	0.28	0.23
2	0.68	0.76	0.66	0.89	0.93	0.85
3	0.13	0.15	0.21	0.22	0.18	0.11
4	0.76	0.80	0.84	0.85	0.84	0.70
5	0.76	0.67	0.55	0.45	0.37	0.25
6	0.99	0.96	0.97	0.72	0.58	0.43
7	0.77	0.82	0.82	0.60	0.53	0.47

*Significant at the 0.05 level. A value of "0.05" not marked by an asterisk indicates a value greater than 0.05 that rounds to 0.05.

DISCUSSION OF PRECEDING PAPER

THOMAS N. HERZOG:

In this paper, Dr. Becker questions the use of the lognormal distribution for modeling periodic changes in interest rates.

My main quibble with Dr. Becker follows from a well-known statement of Professor George Box of the University of Wisconsin: "All models are wrong, but some are useful." The point is that if you have a large amount of data, you will almost always reject the null hypothesis in a real-life modeling problem. Some questions that should be asked are: (1) How robust is the model to misspecification? and (2) What alternative models can be used? Moreover, the modeler needs to be concerned with the specific use at hand. The assumption of the lognormal distribution may be useful for one application but not for another. The alternative models suggested in Section VIII may be of use here as well.

I wondered (1) what the algorithms were that Dr. Becker referred to in an unnamed "statistical software package" in Section V and (2) what was the pseudo-random number generator referred to in Section V. Finally, I would like Dr. Becker to supply us with a reference in the statistical literature to the *notched* box-and-whisker plot discussed in Section V.

DOUGLAS C. DOLL AND DANIEL W. TUCKER, II*:

Dr. Becker has presented a timely and interesting analysis of the lognormal distribution assumption for interest rate changes. His conclusions about the appropriateness of assuming independence and that the distribution is normal will be useful to anyone who is concerned about the distribution of future interest rates.

The analysis in Dr. Becker's paper was all on monthly interest rate changes. It would have been useful to have had analysis of interest rate changes over periods longer than one month. As Dr. Becker notes in his beginning sentence, the lognormal distribution is being used in scenario testing for periods up to 30 years or longer. In life insurance company asset-liability projections, significant fluctuations in results generally occur, not from month-to-month interest rate changes, but from major interest rate movements that occur over periods of a year or more. Over these periods, Mr. Becker's conclusions are not necessarily applicable.

*Mr. Tucker, not a member of the Society, is an associate consultant with Tillinghast/Towers Perrin in Atlanta, Ga.

We have tested a data base of Treasury interest rates from January 1970 to July 1991. The source is the *Annual Statistical Digest 1980–1989*, published by the Board of Governors of the Federal Reserve System (Washington D.C., March 1991), except that the 1991 rates were taken from the Federal Reserve Statistical Release dated January 2, 1990, "Selected Interest Rates." We analyzed the ratio of interest rate changes over periods of 1 month, 3 months, 12 months, and 36 months. We analyzed 90-day rates and 10-year rates.

Our first analysis was calculation of the standard deviations of the log-ratios (the log of the ratio of i_t/i_{t-1}). If the interest rate changes are independent and the volatility constant over time, the standard deviation should increase as the square root of the time period. Table 1 shows the calculated standard deviations compared with the "predicted" standard deviations based on the shorter term results.

TABLE 1

Time Period (Months)	Actual Standard Deviation	"Predicted" Standard Deviation Based on		
		1 Month	3 Months	12 Months
90-Day Rates				
1	0.077			
3	0.157	0.133		
12	0.295	0.267	0.314	
36	0.463	0.462	0.544	0.511
10-Year Rates				
1	0.038			
3	0.076	0.066		
12	0.155	0.132	0.152	
36	0.254	0.228	0.263	0.268

The volatility of the 3-month period is larger than predicted by the 1-month volatility. This is consistent with Dr. Becker's conclusion that the monthly changes in interest rates are not independent.

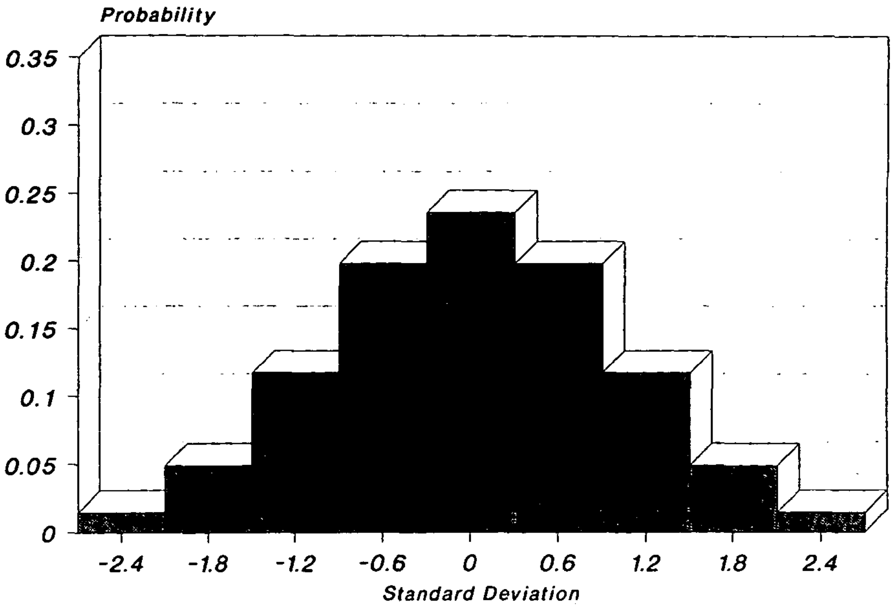
However, when the analysis is extended to longer time frames, we find that the 12-month volatility is closer to that predicted by the 3-month rate, and the 36-month volatility is less than that predicted by the 3-month and 12 month volatilities. Dr. Becker noted a possible "mean reversion" effect in his monthly statistics. Our statistics also indicate a mean reversion effect.

Our second analysis was of the distribution of the logratios compared with a normal distribution. This was done by preparing histograms of the actual logratios and graphing the differences from the normal distribution. The graphs are shown using increments of 0.6 standard deviation.

Figure 1 shows the normal distribution. Figures 2A and 2B show the distributions of logratios for the 90-day rates and 10-year rates, respectively. Figures 3A and 3B show the differences between the actual distributions and the normal distributions for the 90-day rates and the 10-year rates, respectively.

FIGURE 1

NORMAL DISTRIBUTION



In Figure 3A, the one-month logratios clearly show the characteristic noted by Dr. Becker; that is, the logratio values close to the mean are significantly more frequent than the normal distribution. The 3-month logratios show the same effect, but to a lesser degree. The 12-month logratios, on the other hand, show the opposite result, with logratios between −0.6 and +0.6 standard deviations occurring less frequently than normal.

FIGURE 2A

HISTOGRAM OF LOGRATIO OF INTEREST RATES
90-DAY TREASURIES

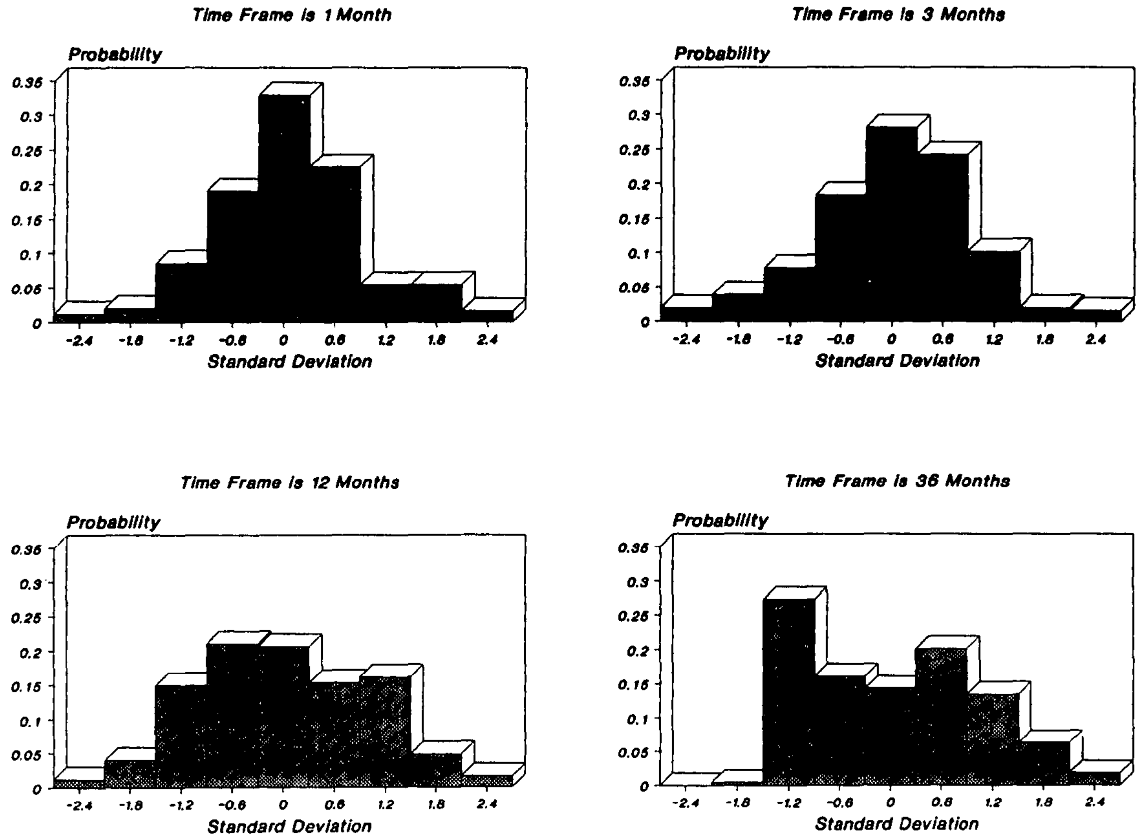


FIGURE 2B

HISTOGRAM OF LOGRATIO OF INTEREST RATES
10-YEAR TREASURIES

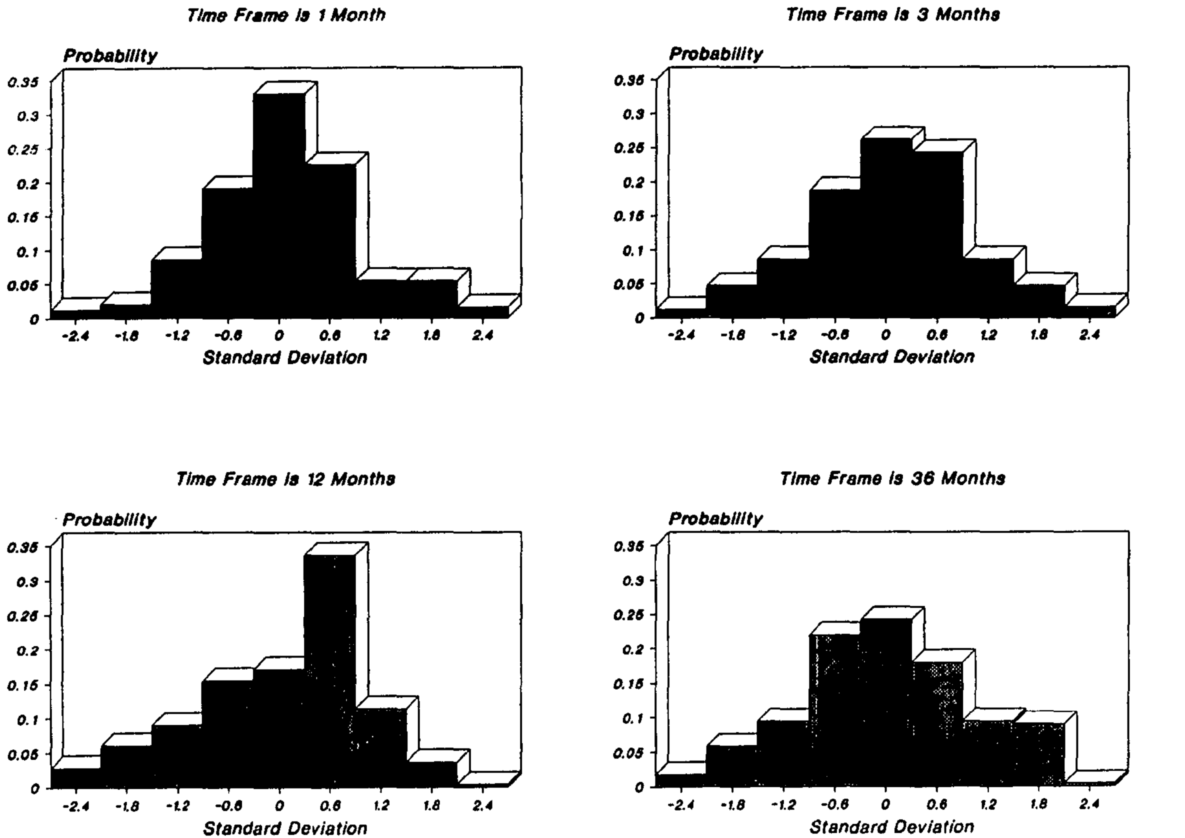


FIGURE 3A

ACTUAL DISTRIBUTION LESS NORMAL DISTRIBUTION
90-DAY TREASURIES

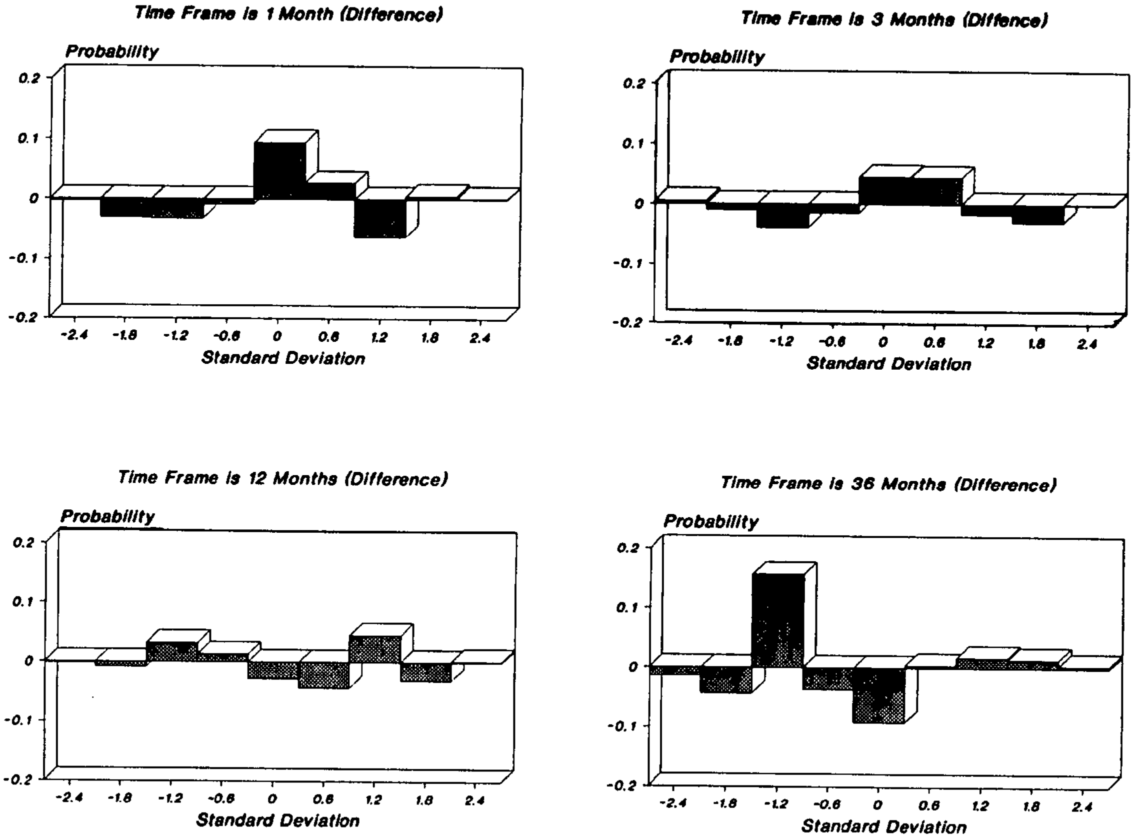
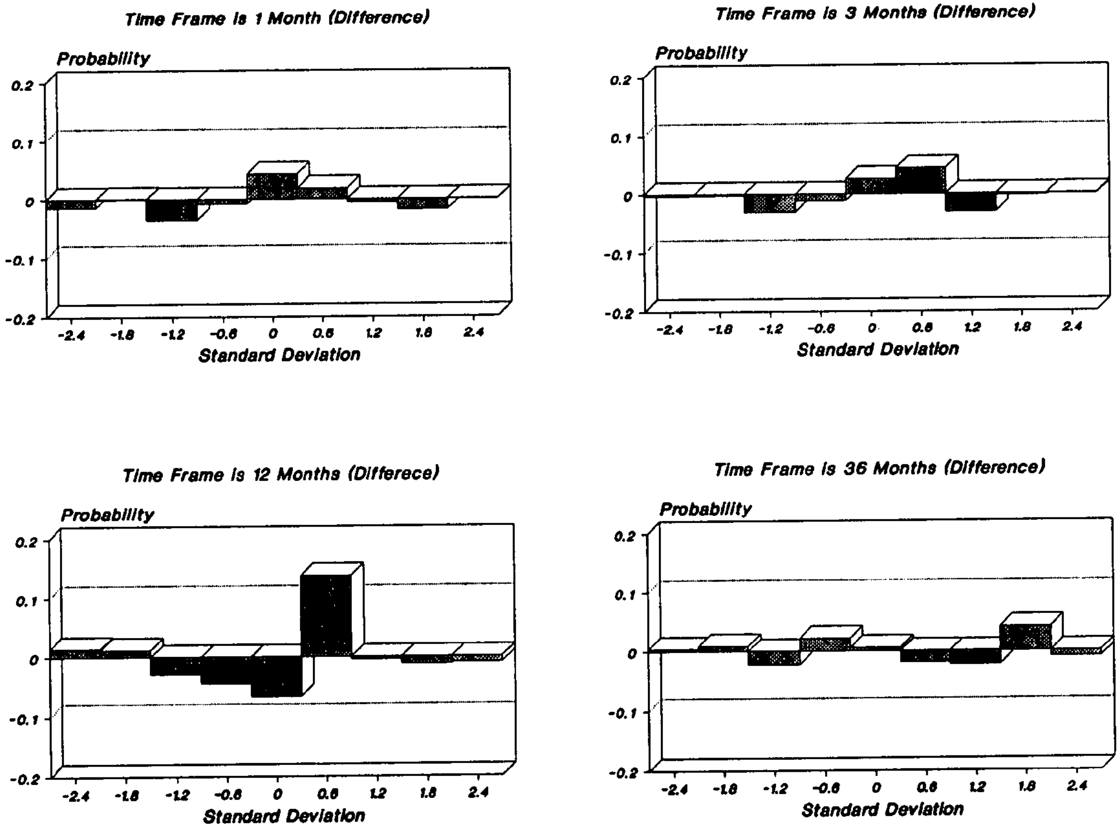


FIGURE 3B

ACTUAL DISTRIBUTION LESS NORMAL DISTRIBUTION
10-DAY TREASURIES



The 36-month logratios in Figure 3A show a significantly non-normal distribution; however, the limited amount of data (20 years' worth) gives a less significant credibility to distribution of the three-year results.

Figure 3B shows less clear results for the 10-year rates. Dr. Becker's paper shows that the 10-year Treasury monthly logratios in his "B" period (1979–1988) are much closer to normal than in his "A" period (1954–1978), although not completely normal. Comparison of the 3-month and 12-month logratios in Figure 3B indicates the same amount of non-normality, but is this enough non-normality to discard the lognormal model as a practical tool?

Dr. Becker demonstrates that the variance of the logratios is not constant over the five-year subperiods and concludes that "a normal assumption with constant variance cannot be supported for projections over any significant length of time." All the characteristics of interest rates that Dr. Becker has analyzed (that is, non-independence, non-normality, nonconstant variance) indicate that, at best, much more work is needed to develop a model that gives us accurate future distributions of interest rates. Perhaps it is impossible to develop such a model, because all our statistics come from an historic environment unlikely to be duplicated in the future. If a statistical solution to scenario generation is not available, practical considerations may move to the forefront.

We suggest that a practical model for long-term scenario projections should satisfy the following two conditions:

1. Produce the desired long-term volatility and distribution of rates. (This will be based partly on judgment and partly on historical patterns.)
2. Preserve a short-term volatility and distribution of rates that is close to historic patterns. (It is presumed that the user will assume continuation of historic patterns.)

A lognormal model, perhaps combined with mean reversion factors, may satisfy the above two conditions within the practical limitations of the asset-liability projection being performed.

(AUTHOR'S REVIEW OF DISCUSSION)

DAVID N. BECKER:

In response to Dr. Herzog's inquiries, the pseudo-random numbers were generated by the software package "Statgraphics" by the Statistical Graphics Corporation, and four references on box-and-whisker plots are as follows:

1. FRIGGE, M., HOAGLAND, D.C., AND IGLEWICZ, B. "Some Implementations of the Boxplot," *American Statistician* 43 (1989): 50–54.
2. MCGILL, R., TUKEY, J.W., AND LARSEN, W.A. "Variation of Box Plots," *American Statistician* 32 (1978): 12–16.
3. TUKEY, J. W. *Exploratory Data Analysis*. Reading, Massachusetts: Addison-Wesley, 1977.
4. VELLEMAN, P.F., AND HOAGLAND, D.C. *Applications, Basics, and Computing of Exploratory Data Analysis*. Belmont, California: Duxbury Press, 1981.

To facilitate the response to Dr. Herzog's other observation, consider a population and a random variable defined on that population that is known to have a given distribution. The goal is to test the null hypothesis that some parameter of the distribution has a specific value, a_0, versus an alternative hypothesis that it is not equal to a_0. Suppose that a_1, the correct value of the desired population parameter, is a value such that the difference between a_1 and a_0 is small. If the sample size of the test of the null hypothesis is sufficiently large, then the null hypothesis can be rejected, even though the true value is only slightly different. Note that the conclusion to reject is exactly correct. (In fact, a strength of the procedure is that the sample size can be increased so that the test can be more discriminating.) But even though the value a_0 is incorrect, because it is very close to the correct value a_1, *useful* predictions about the population *might* have been made with the null hypothesis. Of course, the degree of usefulness depends on the application at hand and the degree of accuracy required. In general, the caution is that the modeler may just entirely discard the null hypothesis and miss an opportunity.

This raises the question, "How large is sufficiently large?" In the above example this could be investigated in principle because the prior distribution was assumed known. When the prior distribution is unknown, which is often the case in applications (and which is specifically under examination in this paper), it is difficult to determine when the sample size is so large that this concern is material. The data for this paper are time series data. On page 18 of *Time Series Analysis: Forecasting and Control* (Oakland, Calif.: Holden-Day, 1976), authors Box and Jenkins state: "If possible, at least 50 and preferably 100 observations or more should be used." Thus the sample sizes in the paper do not seem unduly large for the application at hand. In fact, the sample sizes used in this paper or larger are often employed in analyzing econometric data.

It is also useful to consider the nature of the tests being made to assess their potential for this problem. For example, in the test of independence the autocorrelation coefficients must be zero to conclude independence. In this case "close, but different" still implies non-zero coefficients and independence would still be rejected. And the data do not even support "close." "Close, but different" subperiod variances still imply the variance is not constant. A nonconstant variance may have a significant impact in modeling future interest rates. Standardized kurtosis and skewness tests require a value of zero to support the null hypothesis, that is, that the distribution is normal. Again, "close, but different" implies they are non-zero, and one would still reject. And so forth. Thus it does not appear that these tests are at risk.

Doll and Tucker ask an extremely interesting question, that is, What happens if movements other than monthly are examined? The motivation is based on the assumption that for insurance company asset-liability projections, significant fluctuations in results generally occur, not from month-to-month interest rate changes, but from major changes that occur over periods of one year or more. It seems reasonable that periods longer than one month are needed for an insurance enterprise to experience adverse selection from policyholder exercise of embedded options; but it is less clear that it takes a year or longer before policyholders react. As policyholders and producers become more attuned to the economic environment and the options in their contracts, periods even less than one year will be required for behavior to change. Therefore modeling the impacts of fluctuations that occur in less than a year is important from a liability perspective. An insurance enterprise will likely incur the effects of exercise of embedded options in its asset portfolio due to interest rate changes that occur in periods much less than one year. Thus month-to-month fluctuations are clearly a significant issue from the asset perspective. When asset-liability modeling is performed, both asset and liability behavior must be adequately reflected; this suggests that accurate modeling of frequent fluctuations is important in determining liability and asset cash flows and the asset side of the balance sheet.

I have two concerns about Doll and Tucker's analysis: the limited data and the use of descriptive statistics and informal relationships instead of statistical tests on which observations are made. The data used for their computations cover a period of approximately 20 years. This means, roughly, that there are 80 data points for quarterly changes, 40 data points for semi-annual changes, 20 data points for annual changes, and so on. As noted, these are time series data, and the caution from Box and Jenkins is that at least 50 and preferably 100 or more data points should be used. Therefore there is a borderline amount of data for quarterly changes and an insufficient amount for changes occurring over longer periods. To determine whether real effects are present, actual statistical tests should be used to confirm impressions gathered from descriptive statistics based on, and relations derived from, the observational data. Conclusions should not be drawn from either statistical tests or descriptive statistics that are based on data sets that are too small.

The three data sets for the interest rates used in this paper can be parsed to obtain quarterly, semiannual, and annual rates and the corresponding logratios. The FRB data set was chosen for the analysis presented below. Maturities investigated were: 3 month, 6 month, 1 year, 3 year, 5 year, and 10 year. Note that for the quarterly logratios, three distinct series could be examined, depending on the starting month chosen, that is, beginning with December, January, and February. For semiannual changes there are 6, and for annual, 12. Each such series was examined. Each quarterly series had 140 data points (120 for the 6-month maturity); each semiannual, 70 (60); and each annual, 35 (30). Even for the 35-year history, the data are not abundant when the longer time intervals for change are examined.

The following briefly summarizes the results. Independence of the data for quarterly changes was generally rejected by the statistical tests, but could not be rejected for semiannual and annual changes. Normality of the data was generally rejected by the chi-square goodness-of-fit test for quarterly and semiannual changes but not for annual changes. Normality by kurtosis tests was rejected regularly for quarterly changes but not for semiannual and annual changes. Thus the major conclusions of the paper for monthly changes are replicated for quarterly changes by using statistical tests. For changes over periods longer than quarterly, the tests do not reject the null hypotheses. Note that the amount of data for semiannual and annual changes is either borderline insufficient or insufficient for making these tests.

The quantity on which Doll and Tucker based their findings was the standard deviation of the logratios, that is, the volatility. This was compared in various ways to what theory would have suggested if the data were truly independent for changes more frequent than monthly.

In a similar analysis of the historic FRB data in the paper, the actual volatilities for quarterly, semiannual, and annual changes were computed; the volatilities for periods based on the same lengths of change were estimated from the *actual* monthly data using the assumption of independence; for each corresponding pair the difference was computed; and for each pair the ratio of that difference to the volatility expected from *actual* monthly volatility using the independence assumption was determined. This was done for each of the above-described maturities and for each distinct series for each period of changes, that is, three series for quarterly changes, and so on.

If the data are dependent, then these ratios will tend to be negative because the dependence results in a standard deviation (volatility) smaller than that if the data were independent. If the data are independent, both positive and negative ratios clustering around the value zero would be expected. The table below shows the results of averaging the ratios obtained for each distinct series within each maturity and each interval of change combination.

ACTUAL-TO-EXPECTED VOLATILITY RATIOS
EXPECTED BASED ON ASSUMPTION OF INDEPENDENCE

Maturity	Quarterly Changes	Semiannual Changes	Annual Changes
3 Month	−16.5%	−17.8%	− 8.5%
6 Month	−14.7	−15.6	− 8.8
1 Year	−20.7	−24.2	−15.9
3 Year	−17.8	−20.7	−15.0
5 Year	−16.7	−19.4	−15.4
10 Year	−14.5	−17.1	−17.2

Note that the ratio of actual-to-expected value for each series of each maturity (3 for quarterly changes, 6 for semiannual, and 12 for annual) used in the determination of the average ratios shown in the table was negative for each combination of maturity and basis for change, except the three-month and three-year maturities for annual changes.

Note the uniformity of negative ratios and their consistent magnitude. Although this is not a statistical test, it confirms the statistical test performed on quarterly logratios that rejected independence; and it certainly suggests the dependence of interest rate changes for semiannual and annual intervals of change for which the usual time series tests lack power due to insufficient data.

The purpose of this paper was to test the implicit assumptions supporting the generic lognormal methodology used by both actuaries and by investment personnel. The tests have demonstrated that these assumptions are not borne out in fact. This does not mean that a statistical solution is impossible; it means the current methodology is too primitive. But in the process certain features of the global character of the changes in rates have been revealed.

The three key characteristics are: dependence as exhibited by the non-zero autocorrelations; positive kurtosis; and nonconstant standard deviation, or volatility.

A superior model would be one in which these global characteristics are inherent within the design. The practical solution proposed is actually an enhancement of the base lognormal model tested in the paper. This enhancement incorporates mean reversion and changing variance. Both of these topics are discussed in the paper. The author has constructed data sets of rates using a lognormal assumption coupled with a mean reversion to a long run rate. Examination of the logratios of the rates in those data sets did not demonstrate any consistent evidence of dependence, much less the historical pattern discovered in the paper. This may not, perhaps, be so unexpected, because the dependence is based on the measure of the change in the rates, not on the rates themselves. The mean reversion was made directly on the rates, not on the change in rates. Using another form of mean reversion, that is, mean reversion to the implied forward rates of a given term structure, did result in data sets displaying some evidence of dependence. But it required a significant degree of mean reversion and resulted only in a negative LAG 1 sample autocorrelation coefficient; the other coefficients were not significantly different from zero. Again, this is entirely different from the historical pattern. Direct mechanisms for including mean reversion may not adequately describe the historical data. Because the dependence in the changes of rates is significant, the practical solution does not represent an advance.

Based on information revealed by the statistical tests, a very straightforward modification to the lognormal model might result in a more satisfactory description of the change process. This new model would be a lognormal model with direct provision for dependence on prior movements (changes in the rates), for example, as demonstrated in this paper employing the historical, statistically significant non-zero autocorrelation coefficients. The volatility assumption could be replaced with several alternatives: volatility proportional to the absolute level of the rate or the square root of the rate (it can be arranged that initial volatility matches the environment at the time of projection); and volatility allowed to vary over time in a manner that short-term volatility matches the current environment at the time of projection but grades into a long-term volatility target.

Incorporating dependence eliminates two failures of the current lognormal model to be consistent with historic data. First, the interest rate paths so constructed will demonstrate historical dependence. Second, the paths will likely also show the characteristic positive kurtosis seen in the historic data.

This is due to recent research in the statistical analysis of data with slowly decaying correlations. Data of this type typically fail the common tests for normality, including the kurtosis test. (Note that failing the kurtosis test would likely result in failing the chi-square test as well). This research would then suggest that the least drastic change in the current approach to modeling interest rate changes would be the incorporation of the dependence as described in the paper. This approach also provides a more natural manner to account for mean reversion than is done by specifying reversion to some arbitrarily chosen "long run" value. Also, a nonconstant volatility can result in the generation of data with positive kurtosis.

In many models the period-to-period mean (or drift) is solved so that it ensures that the totality of paths is arbitrage-free. Although the test of the hypothesis that the mean was equal to zero was not rejected, the adjustments needed to ensure arbitrage-free paths in a continuous model may not be significantly different from zero. If so, it seems reasonable to incorporate non-zero mean (drift) values for the benefit of arbitrage-free paths without imposing undue restrictions on the movements of rates or distorting the model's ability to capture the global character of interest rate movement.

These changes may represent the most straightforward modification of the current approach that will result in a model whose results better conform to the global characteristics of the historical data.