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SELECT AND ULTIMATE MODELS IN MULTIPLE DECREMENT THEORY

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ABSTRACT

Usual treatments of multiple-decrement theory are on a nonselect basis, with all rates of decrement depending only on attained age. This paper develops the theory on a select basis, carried out fully in the stochastic framework. The relationship between select periods in a multiple-decrement model and those of the associated single-decrement models is investigated.

1. INTRODUCTION

In Jordan's *Life Contingencies* [3], the treatment of multiple-decrement theory is strictly deterministic. The theory is enriched considerably by the stochastic approach followed in *Actuarial Mathematics* [1], which draws on the earlier work of Hickman [2]. The approach in all these works is strictly on a *nonselect* basis; that is, the rates of all decrements depend on attained age only. In [1], the original definition suggests a select treatment, since the multiple-decrement model (T, J) is defined initially for a fixed age x. The subsequent analysis, however, is carried out on a strictly nonselect basis, and no account is taken of select models.

A nonselect model for a mortality table is often a reasonable approximation. After all, there is an obvious connection between the decrement of death and age. But this need not hold for other decrements. Withdrawal rates, for example, almost certainly depend more on duration than on attained age. It is therefore of some importance to study select models.

In this paper we consider various aspects of select and ultimate models in multiple-decrement theory. There are several goals, as follows:

1. We develop the basic concepts of multiple-decrement theory on a select basis. There are no surprises here, and the resulting formulas are intuitively obvious. We think, however, that it is useful to have a completely formal derivation of these formulas, particularly one that is carried out fully in the *stochastic* framework. Even in the single-decrement case, this provides some new insights. The treatment of this case in [1, Section

3.8] is for the most part deterministic. In fact, before discussing multiple-decrement theory in Section 3, we begin in Section 2 by reviewing the single-decrement model in this stochastic context.

2. We state explicitly the assumptions that justify the use of a nonselect model in the multiple-decrement case. In [1, Formulas 3.2.8 and 3.2.9] this is done for the single-decrement case, but the corresponding multiple-decrement conditions are not given. The basic result is that the multiple-decrement model will be nonselect if and only if each of the associated single-decrement models has this property. We prove this fact in Theorem 3.1 below.
3. In [1, Chapter 9] two different definitions are given for the force of decrement. Formula 9.2.10 states

$$\mu_{x+t}^{(j)} = \frac{f(t,j)}{{}_tp_x^{(\tau)}}. \tag{1.1}$$

In a discussion of the deterministic model, the formula above 9.4.5, states

$$\mu_{x+t}^{(j)} = \lim_{h\to 0} \frac{l_x^{(j)} - l_{x+t}^{(j)}}{hl_x^{(\tau)}}. \tag{1.2}$$

It may be intuitively clear that these definitions yield the same result, but a formal derivation is not at all obvious. In fact, the attempt to carry this out motivated the present work, as it became clear that it was necessary to consider selection in order to do so. In Section 4, we show that with the appropriate assumptions, these definitions are equivalent. In fact, we produce a more general result; see (4.4) below. Along the way we introduce some new symbols that are useful. For example, we define the quantity

$${}_tp_x^{(j)},$$

which does not seem to have appeared before; see (3.3) below. We then use this symbol to give stochastic definitions to the mortality table items $l^{(j)}$ and $d^{(j)}$, which have appeared in the deterministic case only.
4. A compromise between a nonselect model and a fully select model is a *select and ultimate* model, in which the duration of selection is measured by a select period. The nonselect model arises when there is a select period of zero. The result mentioned in item 2 above says that the multiple-decrement model will have a select period of zero, if—and

only if—each of the associated single-decrement models has a select period of zero. In Section 4, we generalize this by comparing select periods in the multiple-decrement model and the associated single-decrement models.

2. THE SINGLE-DECREMENT CASE

In preparation for the multiple-decrement material, we review the normal single-decrement case, as outlined, for example, in [1, Chapter 3]. We begin with a family of random variables $T(x)$, the time until failure of a life aged (x), defined for all x in some interval $I=[a,\omega)$. In this section, for ease of notation, the initial age a is zero. This is the usual initial age for the decrement of mortality but need not be true for all causes of failure.

We follow standard notation with the following addition. We assume that for each x in I, $T(x)$ has a probability density function defined on $[0, \infty)$, and its value at t is denoted by $f_x(t)$.

The rule for defining failure can be arbitrary, and in fact, we allow for the case in which failure need not necessarily occur. We think of these as legitimate survival distributions. It simply means that $T(x)$ is not necessarily a real valued random variable but rather one that also takes ∞ as a possible value. (The associated distribution is sometimes referred to in the literature as a *defective* distribution.) The survival functions need not approach 0 at ∞ but rather an arbitrary α, with $0 \leq \alpha < 1$. The probability density function now integrates to $1-\alpha$, and there is a point mass of α at ∞. In any event, these cases cannot be avoided in our context, for they can automatically appear as the associated single-decrement distributions for a multiple-decrement model; see [1, Section 9.4]. (In fact, a familiar occurrence of this phenomenon arises in contingent function theory. Given two lives (x) and (y), consider the rule that defines failure as occurring on the death of (x), should it occur before that of (y). This is a particular example of an associated single-decrement distribution. The double-decrement model involves the joint status, (xy), rather than a single life, and the two causes of failure are death of (x), death of (y).)

Without any assumptions, we can say very little about the linkage between the infinite family of distribution $T(x)$. To reflect this situation, we need to develop various select symbols. We therefore define, for x in I and all t such that ${}_tp_x > 0$,

$$\mu_x(t) = \frac{f_x(t)}{{}_tp_x} = -\frac{1}{{}_tp_x}\frac{d}{dt}\,{}_tp_x \tag{2.1}$$

$$ {}_{s}q_{[x]+t} = \frac{{}_{s+t}q_x - {}_{t}q_x}{{}_{t}p_x} \tag{2.2} $$

$$ {}_{s}p_{[x]+t} = \frac{{}_{s+t}p_x}{{}_{t}p_x} = 1 - {}_{s}q_{[x]+t}. \tag{2.3} $$

For convenience, throughout the paper we proceed on the assumption that all distributions are unbounded, so that the definitions given in (2.1)–(2.3) apply for all $t \geq 0$. Consistent with this, we assume that $\omega = \infty$, since with a finite ω, we normally would expect that $T(x)$ is bounded by $\omega - x$. Of course, all the theory applies in the general case, with appropriate modifications.

Remarks on Notation. Note that the left-hand side of (2.1) is *not* written as μ_{x+t}. Without prior assumptions we cannot assert that this quantity depends on attained age only, and we must distinguish between the variables x and t. (For example, we cannot use μ_{50} to denote both $\mu_{30}(20)$ and $\mu_{40}(10)$ because the two quantities may be quite different.) In the *Survival Models* text [4, Formula 2.9(b)], London writes the same formula using $\mu_{[x]+t}$ for the left-hand side. This is the more traditional notation and has been used by several other authors. It is consistent with that used in Formulas (2.2) and (2.3). We prefer the subscripted x, however. It is far less awkward, and it is consistent with notation for density functions introduced above. (The subscript notation for hazard rate functions has been used for joint lives in [1, p. 253].)

The quantities defined in (2.2) and (2.3) have obvious interpretations. For example, ${}_{s}q_{[x]+t}$ denotes the probability that a life now age $x+t$, first observed at age x, will fail in the following s periods.

From (2.3) and (2.4), we easily obtain the multiplication rule

$$ {}_{r+s}p_{[x]+t} = {}_{r}p_{[x]+t}\,{}_{s}p_{[x]+t+r} \tag{2.4} $$

for $x, r, s, t, \geq 0$.

We have the usual relationship between the survival and hazard rate functions. From (2.1)

$$ {}_{t}p_x = \exp\left[- \int_0^t \mu_x(r)dr \right]. \tag{2.5} $$

The assumption that leads to a nonselect model can be stated in various equivalent ways, depending on whether one describes it in terms of distribution, survival, density, or hazard rate functions. These are given in the following theorem.

Theorem 2.1

The following are equivalent:

(a) ${}_tq_x \;=\; {}_tq_{[0]+x},$ for all $t,\ x \geq 0.$

(b) ${}_tp_x \;=\; {}_tp_{[0]+x},$ for all $t,\ x \geq 0.$

(c) $f_x(t) \;=\; \dfrac{f_0(x+t)}{{}_xp_0},$ for all $t,\ x \geq 0.$

(d) $\mu_x(t) \;=\; \mu_0(x+t),$ for all $t,\ x \geq 0.$

Proof

We sketch the proof briefly, because the more general multiple-decrement case appears in the next section. However, we will need to use this single-decrement result in the course of that proof.

Obviously (a) and (b) are equivalent. (In fact, in view of (2.2) and (2.3), they are just restatements of [1, Formulas 3.2.8 and 3.2.9].) Formula (a) implies (c) by differentiating, while (c) implies (a) by integrating and making a change of variable. If (b) and (c) hold, then (d) follows from (2.1) using (2.3) and (2.4). Finally, (2.5) can be used to show that (d) implies (c) after an obvious change of variable. □

It is of interest to picture (c) geometrically. It says that, for any age z, the graph of the density function of $T(z)$ is essentially obtained from that of $T(0)$ by chopping off everything to the left of the line $x=z$. Of course, we must also divide by the area of the region to the right of this line, which is ${}_zp_0$, in order to obtain a total area of one.

When any of the equivalent conditions of Theorem 2.1 hold, we have the usual nonselect model. Part (d) of the theorem shows that we can use the symbol μ_{x+t} to unambiguously stand for $\mu_x(t)$. (Referring again to our previous example, we now *can* use μ_{50} to denote each of μ_{30} (20) and μ_{40} (10), since they are *both* equal to μ_0 (50).) From part (b) and (2.3), we have

$$
{}_tp_x = \frac{{}_{x+t}p_0}{{}_xp_0},
$$

and using (2.3) again

$$
{}_sp_{[x]+t} = \frac{{}_{x+t+s}p_0}{{}_{x+t}p_0},
$$

so we can unambiguously denote this latter quantity by ${}_sp_{x+t}$. Similarly, we can remove the square bracket from ${}_sq_{[x]+t}$.

3. THE MULTIPLE-DECREMENT MODEL

We now consider the multiple-decrement model. As outlined in [1, Section 9.1], a multiple-decrement survival distribution is a joint distribution (T, J), where T is the time until failure and $J=\{1, 2, ..., m\}$ is the cause of failure. We assume that we have a family of multiple-decrement distributions, $(T(x), J)$, $x \geq a$, where $T(x)$ is the time until failure of (x). (We now write our formulas with the more general initial age a.) As in the single-decrement case, we do not *a priori* postulate any connection between the distributions of $(T(x), J)$ for various values of the entry age x.

We will follow the notation in [1], except that, as in Section 2, we will take care to distinguish all quantities by the entry age, using either a subscript or the select symbol [ ]. Hence, we let $f_x(t,j)$ = the joint density function of $(T(x), J)$ and $h_x(j)$ = the marginal probability function for J, with respect to $(T(x), J)$.

We then define, analogously to (2.1) and (2.2),

$$\mu_x^{(j)}(t) = \frac{f_x(t,j)}{{}_tp_x^{(\tau)}}, \tag{3.1}$$

and

$${}_sq_{[x]+t}^{(j)} = \frac{{}_{s+t}q_x^{(j)} - {}_tq_x^{(j)}}{{}_tp_x^{(\tau)}}. \tag{3.2}$$

Recall that, associated to our multiple-decrement model is the single-decrement distribution related to total decrement (denoted by τ). Functions related to the τ-distribution are generally obtained by summing the corresponding function over all causes j.

We find it convenient to introduce a completely new symbol. Let

$${}_tp_x^{(j)} = h_x(j) - {}_tq_x^{(j)} = \int_t^{\infty} f_x(s,j)ds. \tag{3.3}$$

Note that we do *not* define

$${}_tp_x^{(j)} = 1 - {}_tq_x^{(j)}.$$

The clue to the proper definition is to realize that, while we often interpret the general symbol ${}_tp$ as the probability of surviving t periods, we can alternatively think of it as the probability of failure after t periods. (In the usual case of nondefective distributions, the two interpretations are obviously equivalent.) The symbol ${}_tp_x^{(j)}$, as defined, gives the probability that (x) will terminate from cause j, after t periods.

Suppose that the τ-distribution is nondefective. Summing (3.3) over all j and using the fact that

$$\sum_j h_x(j) = 1$$

yields

$$\sum_j {}_tp_x^{(j)} = 1 - {}_tq_x^{(\tau)} = {}_tp_x^{(\tau)}$$

as we would expect. In the case in which the τ-distribution has a point mass of α at ∞, the above formula becomes

$$\sum_j {}_tp_x^{(j)} = {}_tp_x^{(\tau)} - \alpha.$$

This is easily verified intuitively. We must deduct the probability of surviving forever from the right-hand side because it is not included in the left-hand side.

We now define the corresponding select survivorship function

$${}_sp_{[x]+t}^{(j)} = \frac{{}_{s+t}p_x^{(j)}}{{}_tp_x^{(\tau)}}, \tag{3.4}$$

the probability that in the model for age x, an individual succumbs to cause j after time $s+t$ given that he or she has survived all causes up to time t. We similarly define

$${}_sp_{[x]+t}^{(\tau)}$$

by summation over all j.

From (3.4), we can easily obtain the analog of the multiplication rule (2.4),

$${}_{r+s}p_{[x]+t}^{(j)} = {}_rp_{[x]+t}^{(\tau)}\,{}_sp_{[x]+t+r}^{(j)} \tag{3.5}$$

Recall that for each cause j, we have an associated family of single-decrement distributions indexed by x. These are, namely, the distributions with hazard rate function given by $\mu_x^{(j)}(t)$. In other words, in the associated single-decrement distribution for cause j, the probability of surviving t periods is given by

$$\exp\left[-\int_0^t \mu_x^{(j)}(r)dr\right]. \tag{3.6}$$

We can now prove equivalent conditions for a nonselect model analogous to those of Theorem 2.1.

Theorem 3.1

The following are equivalent:

(a) ${}_tq_x^{(j)} = {}_tq_{[a]+x-a}^{(j)}$, for all $x \geq a$, $t \geq 0$, $j = 1, 2, \ldots, m$.

(b) ${}_tp_x^{(j)} = {}_tp_{[a]+x-a}^{(j)}$, for all $x \geq a$, $t \geq 0$, $j = 1, 2, \ldots, m$.

(c) $f_x(t,j) = \dfrac{f_a(x-a+t,j)}{{}_{x-a}p_a^{(\tau)}}$, for all $x \geq a$, $t \geq 0$, $j = 1, 2, \ldots, m$.

(d) $\mu_x^{(j)}(t) = \mu_a^{(j)}(x+t-a)$, for all $x \geq a$, $t \geq 0$, $j = 1, 2, \ldots, m$.

(e) For each j, the conditions of Theorem 2.1 hold in the family of associated single-decrement distributions for cause j, as given by (3.6).

Proof

(a) $\Rightarrow$ (c). This follows directly by differentiating with respect to t.

(c) $\Rightarrow$ (b). Assuming (c), then

$$
\begin{aligned}
{}_tp_x^{(j)} &= \int_t^\infty f_x(s,j)ds = \frac{1}{{}_{x-a}p_a^{(\tau)}}\int_t^\infty f_a(x-a+s,j)ds \\
&= \frac{1}{{}_{x-a}p_a^{(\tau)}}\int_{x-a+t}^\infty f_a(r,j)dr = \frac{{}_{x-a+t}p_a^{(j)}}{{}_{x-a}p_a^{(\tau)}} = {}_tp_{[a]+x-a}^{(j)}.
\end{aligned}
$$

(b) $\Rightarrow$ (d). Suppose (b) holds. First, note that (c) also holds by differentiation with respect to t. By summation over all j in (b) the same statement holds with j replaced by τ. Hence, part (b) of

Theorem (2.1) holds for the τ-family. Taking the multiplication rule (2.4) for this family and dropping the square brackets from the subscripts, we can write

$$_{x-a+t}p_a^{(\tau)} = {}_{x-a}p_a^{(\tau)}\ {}_tp_x^{(\tau)}. \tag{3.7}$$

Using (3.1), part (c), and (3.7) implies

$$\mu_x^{(j)}(t) = \frac{f_x(t,j)}{{}_tp_x^{(\tau)}} = \frac{1}{{}_tp_x^{(\tau)}}\frac{f_a(x-a+t,j)}{{}_{x-a}p_a^{(\tau)}} = \mu_a^{(j)}(x+t-a).$$

(d) $\Leftrightarrow$ (e). For each j, the statement in (d) of the present theorem is simply statement (d) of Theorem 2.1 applied to the distributions given in (3.6), so it is clear that (d) and (e) say exactly the same thing.

(d) $\Rightarrow$ (a). From (d), by summation over all j, we again use Theorem 2.1 to derive (3.7). Now

$$_{x+t-a}q_a^{(j)} - {}_{x-a}q_a^{(j)} = \int_{x-a}^{x-a+t} {}_rp_a^{(\tau)}\ \mu_a^{(j)}(r)\,dr.$$

We now change the variable to $s=r-(x-a)$ and invoke (d) and (3.7). The integral above then becomes

$$_{x-a}\,p_a^{(\tau)}\int_0^t {}_sp_x^{(\tau)}\ \mu_x^{(j)}\,(s)\,ds = {}_{x-a}p_a^{(\tau)}\ {}_tq_x^{(j)}.$$

We now see from (3.1) that (a) is established.

The proof of the theorem is now complete. □

Suppose the conditions of the theorem hold. Arguing similarly to the remarks following Theorem 2.1 and using (3.2) and (3.4), we easily show we have a nonselect model, with all quantities depending only on attained age. That is, we can let

$$\mu_{x+t}^{(j)},\ {}_sq_{x+t}^{(j)},\ \text{and}\ {}_sp_{x+t}^{(j)}$$

unambiguously denote

$$\mu_x^{(j)}\,(t),\ {}_sq_{[x]+t}^{(j)},\ \text{and}\ {}_sp_{[x]+t}^{(j)}$$

respectively.

One of the main consequences of the theorem is shown by statement (e), which says that the multiple-decrement model will be nonselect if and only if each of the associated single-decrement models is nonselect.

Example 3.1

It is possible for the nonselect assumption to apply to the τ-decrement without applying to the individual decrements. Consider the case of two decrements, each of which depend solely on duration (which, as indicated, may be typical of withdrawal). Suppose our model for each age x is given by

$$f_x(t,1) = e^{-t}\frac{1}{1+t}, \qquad f_x(t,2) = e^{-t}\frac{t}{1+t},$$

It is then easy to calculate that

$$\mu_x^{(1)}(t) = \frac{1}{1+t}, \text{ for all } x,$$

and

$$\mu_x^{(2)}(t) = \frac{t}{1+t}, \text{ for all } x.$$

We see immediately that statement (d) of Theorem 3.1 fails for both decrements. However, since

$$\mu_x^{(\tau)}(t) = 1, \quad \text{for all } t,\, x \geq 0,$$

part (d) of Theorem (2.1) applies for the τ-distributions.

4. MULTIPLE-DECREMENT TABLE SYMBOLS

We now define the select table symbols in a stochastic way.

Choose $l_{[x]}^{(\tau)}$ arbitrarily for each x, and define for $t \geq 0$ and $j = 1, 2, \ldots, m$

$$l_{[x]+t}^{(j)} = l_{[x]}^{(\tau)}\, {}_tp_x^{(j)}. \tag{4.1}$$

We then define for $t \geq 0$

$$l_{[x]+t}^{(\tau)} = \sum_{j=1}^{m} l_{[x]+t}^{(j)} = l_{[x]}^{(\tau)}\, {}_tp_x^{(\tau)}. \tag{4.2}$$

Clearly, $l^{(j)}_{[x]+t}$ denotes the *expected* number of individuals who will terminate from cause j, at age $x+t$ or *higher*, out of an original group of $l^{(\tau)}_{[x]}$ people age x. In order to so terminate, (x) must *first* survive to age $x+t$ and then terminate from cause j. This suggests the identity

$$l^{(j)}_{[x]+t} = l^{(\tau)}_{[x]+t}\ {}_{\infty}q^{(j)}_{[x]+t}. \tag{4.3}$$

This is easily verified formally as follows.

$${}_{\infty}q^{(j)}_{[x]+t} = \lim_{s\to\infty} {}_{s}q^{(j)}_{[x]+t},$$

which from (3.2)

$$= \frac{h_x(j) - {}_tq^{(j)}_x}{{}_tp^{(\tau)}_x} = \frac{{}_tp^{(j)}_x}{{}_tp^{(\tau)}_x}.$$

Now (4.3) follows immediately from (4.1) and (4.2).

We can express the forces of decrement in terms of l,'s as follows.

$$\mu^{(j)}_{[x]+t} = \frac{f_x(t,j)}{{}_tp^{(\tau)}_x} = \frac{\frac{d}{dt}\,{}_tq^{(j)}_x}{{}_tp^{(\tau)}_x} = \lim_{k\to 0}\frac{{}_{t+k}q^{(j)}_x - {}_tq^{(j)}_x}{k\,{}_tp^{(\tau)}_x}$$

$$= \lim_{k\to 0}\frac{{}_tp^{(j)}_x - {}_{t+k}p^{(j)}_x}{k\,{}_tp^{(\tau)}_x} = \lim_{k\to 0}\frac{l^{(j)}_{[x]+t} - l^{(j)}_{[x]+t+k}}{k\ l^{(\tau)}_{[x]+t}}. \tag{4.4}$$

(We obtain the last equality by multiplying all terms by $l^{(\tau)}_{[x]}$.)

We indicate at the end of the section how this achieves the goal of showing that the two different definitions of the force of decrement, given in [1], are equivalent.

We now define for all $x \geq a$, $t \geq 0$, $k \geq 0$, $j = 1, 2, \ldots, m$,

$${}_kd^{(j)}_{[x]+t} = l^{(j)}_{[x]+t} - l^{(j)}_{[x]+t+k}.$$

From (4.1) and (3.3)

$$\begin{aligned} {}_kd^{(j)}_{[x]+t} &= l^{(\tau)}_{[x]}\,[{}_{t+k}q^{(j)}_x - {}_tq^{(j)}_x] \\ &= l^{(\tau)}_{[x]}\ {}_tp^{(\tau)}_x\ {}_kq^{(j)}_{[x]+t} \\ &= l^{(\tau)}_{[x]+t}\ {}_kq^{(j)}_{[x]+t}. \end{aligned} \tag{4.5}$$

This is the select analog of [1, formula 9.3.4].

So, ${}_kd^{(j)}_{[x]+t}$ is just the expected number of the original $l^{(\tau)}_{[x]}$ individuals age x, who survive t years and then succumb to cause j within the subsequent k years.

Formula (4.5) is important, because it allows us to compare this stochastic definition of l and d with the deterministic one given in [1, Section 9.4]. A natural question that arises is whether these agree. This is not as trivial a point as it may seem. In fact, the issue is already present in the single-variable nonselect case discussed, for example, in [1, Section 3.4]. We briefly review this. Suppose, for example, that $q_0 = 0.10$ and that we start with $l_0 = 1000$. In the stochastic model, the number of survivors at age 1 is a binomial random variable, and l_1 represents its expected value of 900. In the deterministic model, we assume that *exactly* 10 percent of those age 0 will die within 1 year. There is still randomness present, but only due to the fact that we don't know which particular individuals will fail. In this case, l_1 represents the exact number of lives who remain alive at age 1. Despite two very different interpretations, the two values of l_1 are the same. Is this true for l_x for all integers x? This is not completely obvious, but it can be readily established by induction. The same considerations apply to the multiple-decrement select case.

We temporarily use $\bar{l}$ to denote deterministic l's. For a natural deterministic construction of the multiple-decrement table, choose $\bar{l}_{[x]}^{(\tau)}$ arbitrarily for each integer x, and then, thinking of the q's as exact rates of decrement, define inductively for integers x and t

$$\bar{d}_{[x]+t}^{(j)} = \bar{l}_{[x]+t}^{(\tau)}\, q_{[x]+t}^{(j)},\ j = 1, 2, \ldots, m$$

$$\bar{l}_{[x]+t+1}^{(\tau)} = \bar{l}_{[x]+t}^{(\tau)} - \sum_{j=1}^{m} \bar{d}_{[x]+t}^{(j)} .$$

Now, provided we have

$$\bar{l}_{[x]}^{(\tau)} = l_{[x]}^{(\tau)}, \text{ for all } x,$$

we can use induction, taking $k = 1$ and reversing the steps in the derivation of (4.5) to show that

$$\bar{l}_{[x]+t}^{(j)} = l_{[x]+t}^{(j)} \tag{4.6}$$

for all integers $x \geq a$, all positive integers t, and $j = 1, 2, \ldots, m$ or τ. We leave the details to the reader.

Suppose now that the conditions of Theorem 3.1 hold, and suppose moreover that we choose

$$l_{[x]}^{(\tau)} = l_{[a]}^{(\tau)}\ {}_{x-a}p_a^{(\tau)}.$$

Then

$$l_{[x]+t}^{(j)} = l_{[a]}^{(\tau)}\ {}_{x-a}p_a^{(\tau)}\ {}_tp_x^{(j)},$$

which, substituting from statement (b) of Theorem 3.1 and using (3.5),

$$= l_{[a]}^{(\tau)}\ {}_{x-a}p_a^{(\tau)}\ {}_tp_{[a]+x+a}^{(j)}$$

$$= l_{[a]}^{(\tau)}\ {}_{x+t-a}p_x^{(j)} = l_{[a]+x+t-a}^{(j)}.$$

We can therefore remove the select brackets and write

$$l_{[x]+t}^{(j)} = l_{x+t}^{(j)}. \tag{4.7}$$

Consider now formulas (1.1) and (1.2), in which the latter is to be interpreted deterministically. Assume the conditions of Theorem 3.1 and that the deterministic model is constructed as noted above, so that (4.6) and (4.7) hold. Then we see from (4.4) that (1.1) and (1.2) are indeed equivalent.

5. SELECT PERIODS IN THE SINGLE-DECREMENT MODEL

We do not always have the simple situation, in which rates of decrement depend on attained age only. A compromise between this and a fully select model is the select and ultimate model, in which we postulate that the effect of selection wears off after a certain time interval, known as the *select period*. We wish to compare select periods in the multiple-decrement model and the associated single-decrement models. The goal is to show that in order to keep select periods small in the multiple-decrement model, it is necessary and sufficient to have all the associated single-decrement select periods small. The first step is to give precise definitions. In the multiple-decrement case, it is not completely obvious how to define the select period. We will motivate this by first carrying out a complete discussion of this concept in the single-decrement case.

Suppose that we have the family $T(x)$, $x \geq a$, as given in Section 2.

We define the *select period* of this family as the infimum (that is, greatest lower bound) of the set of all r such that

$$ {}_tp_{[x]+r'} = {}_tp_{[a]+r'+x-a}, \text{ for all } x \geq a,\ t \geq 0,\ r' > r. \tag{5.1}$$

If no such r exists, the select period will be ∞.

(The last statement requires some modification in the general case of bounded random variables. We would then postulate that (5.1) holds if both sides of the equation are undefined. Suppose, for example, that there is a limiting age ω, and each $T(x)$ is bounded by $\omega - x$. We can show that with this definition the select period will always be less than or equal to $\omega - a$. This is clear intuitively. Selection has worn off at time $\omega - a$, in a vacuous way, because nobody is surviving at this time.)

If there is a finite select period r_0, one can show by the right continuity of survival functions that (5.1) will hold for $r = r_0$, so the infimum is in fact a minimum in this case.

Often it will be more convenient to write the equation in (5.1) as

$$\frac{{}_{r'+t}p_x}{{}_{r'}p_x} = \frac{{}_{r'+t+x-a}p_a}{{}_{r'+x-a}p_a}. \tag{5.2}$$

The form of the definition of select period has been chosen with the multiple-decrement case in mind. In this single-decrement model, some simplification is possible.

Proposition 5.1

The select period is the infimum of the set of all r such that,

$${}_tp_{[x]+r} = {}_tp_{[a]+r+x-a}, \text{ for all } x \geq a,\ t \geq 0. \tag{5.3}$$

Proof

Suppose that (5.3) holds for r. Given any $r' > r$,

$$\frac{{}_{r'+t}p_x}{{}_{r'}p_x} = \frac{{}_{r+t+(r'-r)}p_x}{{}_rp_x} \frac{{}_rp_x}{{}_{r+(r'-r)}p_x}$$

and applying (5.3) to both fractions in the product yields the right-hand side of (5.2). □

Proposition (5.1) is useful in the following way. Suppose we want to show that the select period is less than or equal to r. Arguing directly from the definition, we would have to illustrate (5.1) for all $r' \geq r$. The proposition says that it is sufficient to do so for the single value, r.

It is immediate from Proposition (5.1) that the conditions of Theorem 2.1 hold if and only if the select period is 0. Simply look at part (b) of the theorem.

We can compare our definition with the criteria for r given in [1, bottom of page 73], that

$$q_{[x-j]+j+r} = q_{[x]+r}, \quad j > 0.$$

This is (5.3) with q replacing p, $x-j=a$, and $t=1$. This last simplification is possible in the context of the mortality table, where one is essentially interested only in integer values of t.

It is convenient to have the equivalent hazard rate or density function formulation for select periods. We can take the select period to be the infimum of the set of all r, such that

$$\frac{f_x(r+t)}{{}_rp_x} = \frac{f_a(r+t+x-a)}{{}_{r+x-a}p_a}, \text{ for all } x \geq a,\ t \geq 0 \tag{5.4}$$

or

$$\mu_x(r') = \mu_a(r'+x-a), \text{ for all } r' \geq r. \tag{5.5}$$

To see this, we note that if (5.3) holds for r, then we can obtain (5.4) by differentiating with respect to t. Moreover, by Proposition (5.1) we know that (5.4) is true for all $r'>r$. We now can derive (5.5) by simply setting $t=0$ in (5.4). Finally, we can use (2.5) to derive (5.3) from (5.5).

6. SELECT PERIODS IN THE MULTIPLE-DECREMENT MODEL

Given the multiple-decrement model of Section 3, we define the *select period for cause j*, which we will denote by *sel*(j) as the infimum of the set of all r satisfying

$${}_tp^{(j)}_{[x]+r'} = {}_tp^{(j)}_{[a]+r'+x-a} \text{ for all } x \geq a,\ t \geq 0,\ r' \geq r. \tag{6.1}$$

That is,

$$\frac{{}_{r'+t}p^{(j)}_x}{{}_{r'}p^{(\tau)}_x} = \frac{{}_{r'+t+x-a}p^{(j)}_a}{{}_{r'+x-a}p^{(\tau)}_a}, \text{ for all } x \geq a,\ t \geq 0,\ r' \geq r. \tag{6.2}$$

When no such r exists, *sel*(j)$=\infty$. (In the bounded case we have the same modifications as discussed in Section 5.)

Equivalently, *sel*(j) will equal the infimum of the set of all r satisfying

$$\frac{f_x(r'+t,j)}{{}_{r'}p^{(\tau)}_x} = \frac{f_a(r'+t+x-a,j)}{{}_{r'+x-a}p^{(\tau)}_a}, \text{ for all } x \geq a,\ t \geq 0,\ r' \geq r. \tag{6.3}$$

We differentiate with respect to t to show that (6.2) implies (6.3) and integrate to show the converse.

If (6.3) holds, we can take $t=0$ to show that

$$\mu_x^{(j)}(r') = \mu_a^{(j)}(r' + x - a), \text{ for all } x \geq a,\ r' \geq r.$$

As we will see later, we cannot go back the other way. Condition (6.4) is not equivalent to (6.3).

The following proposition shows that, as in the single-decrement case, our multiple-decrement definition of select period works as we expect in the nonselect model.

Proposition 6.1

The conditions of Theorem (3.1) hold if and only if

$$sel(j) = 0, \text{ for all } j.$$

Proof

If (6.1) holds for all j, with $r=0$, we immediately obtain statement (b) of Theorem 3.1. Conversely, assume (b) of Theorem 3.1. By summation, the same statement holds for τ. Then, for any non-negative r' and t

$$\frac{{}_{r'+t+x-a}p_a^{(j)}}{{}_{r'+x-a}p_a^{(\tau)}} = \frac{{}_{r'+t+x-a}p_a^{(j)}}{{}_{x-a}p_a^{(\tau)}} \frac{{}_{x-a}p_a^{(\tau)}}{{}_{r'+x-a}p_a^{(\tau)}}$$

$$= \frac{{}_{r'+t}p_{[a]+x-a}^{(j)}}{{}_{r'}p_{[a]+x-a}^{(\tau)}} = \frac{{}_{r'+t}p_x^{(j)}}{{}_{r'}p_x^{(\tau)}},$$

so that (6.2) holds with $r=0$. □

We now let $sel'(j)$ denote the select period in the associated single-decrement model for cause j, and let $sel(\tau)$ denote the select period in the τ-distribution. The main result of this section gives the relationship between these select periods.

Theorem 6.1

For all $x \geq a$

(a) $sel(\tau) \leq \max\{sel'(j): j = 1, 2, \ldots, m\}$
(b) $sel(\tau) \leq \max\{sel(j): j = 1, 2, \ldots, m\}$
(c) For all j, $sel(j) = \max\{sel'(j), sel(\tau)\}$.

Proof

Fix any $x \geq a$.

(a) Take any $r' \geq \max\{sel(j): j=1, 2, \ldots, m\}$. Then from (5.5) applied to the associated single-decrement distributions, (6.4) holds for all j. By summation it holds for τ, showing that $sel(\tau) \leq r'$. Since r' was arbitrary, we have

$$sel(\tau) \leq \max\{sel'(j): j = 1, 2, \ldots, m\}.$$

(b) Take any $r \geq \max\{sel(j=1, 2, \ldots, m\}$. We sum the equation in (6.1) over all j to conclude that $sel(\tau) \leq r$. Since r is arbitrary, we have

$$sel(\tau) \leq \max\{sel(j): j = 1, 2, \ldots, m\}.$$

(c) Fix any j. Let $r_1 = \max\{sel(\tau), sel'(j)$ and let $r_0 = sel(j)$. Apply (5.2) to the τ-family with $u = r + t$, to obtain

$$_up_x^{(\tau)} = \frac{_rp_x^{(\tau)}}{_{r+x-a}p_a^{(\tau)}}\, {}_{u+x-a}p_a^{(\tau)}, \text{ for all } u,\ r > r_1. \tag{6.5}$$

Choose any $r > r_1$. Then

$$_rp_x^{(j)} = \int_r^{\infty} {}_up_x^{(\tau)}\, \mu_x^{(j)}(u)du,$$

and substituting from (6.5) and (6.4),

$$_rp_x^{(j)} = \frac{_rp_x^{(\tau)}}{_{r+x-a}p_a^{(\tau)}}\, {}_{r+x-a}p_a^{(j)}. \tag{6.6}$$

For any $r' > r_1$, divide (6.6) with $r = r' + t$, by Equation (6.5) with $u = r'$, to obtain

$$\frac{_{r'+t}p_x^{(j)}}{_{r'}p_x^{(\tau)}} = \frac{_{r'+t+x-a}p_a^{(j)}}{_{r'+x-a}p_a^{(\tau)}},$$

and we conclude that

$$sel(j) \leq r_1. \tag{6.7}$$

From (6.4) we see immediately that

$$sel(\tau) \leq sel(j). \tag{6.8}$$

Using (6.3), we can integrate to obtain

$$\frac{{}_rp_x^{(\tau)}}{{}_{r_0}p_x^{(\tau)}}\,\mu_x^{(j)}(r) = \frac{{}_{r+x-a}p_a^{(\tau)}}{{}_{r_0+x-a}p_a^{(\tau)}}\,\mu_a^{(j)}(r + x - a),\ r \geq r_0.$$

Invoking (6.4) again, we cancel out the hazard rate terms to conclude that

$$sel(\tau) \leq r_0 = sel(j). \tag{6.9}$$

Inequalities (6.7)–(6.9) complete the proof. □

Define the *select period of the entire model* to be the maximum of $sel(j)$ over all j. Then we have the following, as a consequence of parts (a) and (c).

Corollary to Theorem 6.1

For any $r \geq 0$, the select period of the model is less than, or equal to, r, if and only if $sel'(j) \leq r$ for all j. □

We have now generalized the Section 3 result, in which we derived the statement of this corollary for $r=0$.

In this paper we have followed the traditional viewpoint that the effects of selection wear off after some fixed select period applicable to all attained ages. Conceivably, this period could vary by attained age. It is possible to formulate a very general definition of select period, which will handle all such cases, and then derive the same result as in Theorem 6.1. We will not pursue this further in this work.

Note from part (c) that $sel(j)$ can be larger that $sel'(j)$, illustrating the assertion made above that (6.4) does not imply (6.2).

We saw already, from Example 3.1, that equality need not hold in part (a). A curious fact is that when this does not hold, then $sel(\tau)$ must be less than the "second highest" of the values of $sel'(j)$. The precise statement is as follows.

Proposition 6.2.

Suppose that

(i) $sel'(j) \leq r_0,\ j = 1, 2, \ldots, m - 1,$

and

(ii) $sel'(m) > r_0$.

Then either $sel(\tau)=sel(m)$, or $sel(\tau)\leq r_0$.

Proof

If $sel(\tau)>r_0$, then for $r>sel(\tau)$,

$$\mu_x^{(\tau)}(r) = \mu_a^{(\tau)}(r + x - a).$$

Write the above terms as summations over all j. Since $r>r_0$, we can cancel out all terms with $j=m$ to arrive at

$$\mu_x^{(j)}(r) = \mu_a^{(j)}(r + x - a),$$

and we conclude that $sel(m)\leq sel(\tau)$. From Theorem (6.1)(a), equality holds. □

We conclude the paper by showing that it is not equivalent to define $sel(j)$ as the infimum of all r satisfying

$$_tp_{[x]+r}^{(j)} = {}_tp_{[a]+r+x-a}^{(j)}, \quad x \geq a,\ t \geq 0, \tag{6.10}$$

as one may be tempted to do, by analogy with the single-decrement case. We cannot employ the same trick as in proving Proposition (5.1), since the denominators involve τ rather than j. The following example shows this conclusively.

Example 6.1

Consider the double-decrement model, with $a=0$, in which for all x and t,

$$f_x(t,1) = \frac{1}{2}\,e^{-t},\ f_x(t,2) = \frac{1}{2}\,\phi(x)\,e^{-\tau\phi(x)}$$

where ϕ is a non-negative function such that $\phi(0)=1$, but ϕ is not identically 1. From this we integrate to get

$$_tp_x^{(1)} = \frac{1}{2}\,e^{-t}, \qquad _tp_x^{(2)} = \frac{1}{2}\,e^{-t\phi(x)},$$

$$_tp_x^{(\tau)} = \frac{1}{2}\,[e^{-t} + e^{-t\phi(x)}],$$

$$ {}_tp^{(1)}_{[0]+x+r} = \frac{{}_{x+r+t}p_0^{(1)}}{{}_{x+r}p_0^{(\tau)}} = \frac{e^{-(x+r+t)}}{e^{-(x+r)} + e^{-(x+r)}} = \frac{1}{2}e^{-t}, $$

and

$$ {}_tp^{(1)}_{[x]+r} = \frac{{}_{r+t}p_x^{(1)}}{{}_rp_x^{(\tau)}} = \frac{e^{-(r+t)}}{e^{-r} + e^{-r\phi(x)}}. $$

Take $j=1$. The above calculations show that (6.10) holds for $r=0$. However if $r>0$ and x is such that $\phi(x)$ is not equal to 1, the equation in (6.10) fails. Therefore, $sel(1)=\infty$.

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DISCUSSION OF PRECEDING PAPER

ELIAS S.W. SHIU:

Dr. Promislow is to be thanked for writing this paper, extending the classical theory of multiple decrements to incorporate select and ultimate models. I am particularly appreciative of the insightful definition for the symbol

$$ {}_tp_x^{(j)}. $$

For the past 15 years I had been telling my students that such a symbol made no sense.

I agree with Dr. Promislow that "withdrawal rates. . . almost certainly depend more on duration than on attained age." Indeed, for some policies, withdrawals are permitted only at policy anniversary. Such forces of withdrawal are infinite at positive integers and zero elsewhere. Consequently, the corresponding probability density functions $f(t,j)$ do not exist. A paper attempting to alleviate this difficulty is [8].

In the classical multiple-decrement theory, the various forces of decrement are assumed to act independently of each other. This may not be realistic. Actuaries have discussed this difficulty for more than a century. For example, in 1874 Makeham [5, p. 322] wrote: "It will be observed that these solutions all proceed upon the assumption that the extermination of small pox does not affect the mortality arising from other causes. This must be proved before any reliance can be placed upon the conclusions arrived at." Another obvious example of this dependency problem is policy surrender. A policyholder knowing himself to be terminally ill is very unlikely to surrender his life insurance policy; that is, as soon as a policyholder finds out that he is terminally ill, his force of withdrawal becomes zero or very close to zero. I now quote Hilary Seal [7, p. 698]: "The policyholder who let his policy lapse by failure to pay a due premium was presumably not near death or disability so far as these hazards can be foreseen. Lapse is thus likely to be probabilistically dependent on the other. . . causes of decrement. . . . This difficulty had been mentioned in the controversies of the 1760s and 1870s, but to this day actuaries have not attempted to resolve it." It would be interesting to see a solution to this problem.

I would like to take this opportunity to pose another problem. Exercise 7.45 on page 229 of *Actuarial Mathematics* [1] is to derive a continuous version of the celebrated Hattendorff Theorem. Because of its difficulty, it is a starred problem. Indeed, in the earlier editions of the *ACTEX Study*

Manual for the Course 150 Examination, no solution to this exercise was given. An elegant solution can be found on pages 6 and 7 of Hickman's paper [3]. In fact, the solution was given in the context of multiple decrements. The idea is to decompose the discounted premium into two parts—the first part contributes to the discounted value of the reserve and the second part is the discounted cost of insurance based upon the instantaneous net amount at risk. The theorem then follows from a straightforward integration by parts. (Actually, the more general formula stated in exercise 16.13 on page 479 of *Actuarial Mathematics* [1] also follows from this method.) My question is whether one can adapt this elegant proof to the discrete case; that is, decompose the premium and then do a summation by parts. (Two recent papers on the Hattendorff Theorem are [6] and [9].)

To conclude this discussion, I would like to mention that the theory of multiple decrements is related to the theory of competing risks in the statistical literature. A concise survey on competing risks is the article by Gail [2]. Chapter 7 of the book by Kalbfleisch and Prentice [4] is on competing risks.

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(AUTHOR'S REVIEW OF DISCUSSION)

S. DAVID PROMISLOW:

I thank Dr. Shiu for his informative remarks. He clearly indicates that there is much scope for future research in the field of multiple decrement theory.

Particularly intriguing is his challenge to build dependence into the model. As Dr. Shiu indicates, this is a project of great practical significance, as well as one which could present many interesting mathematical and statistical problems.

There is also practical interest in finding methods to handle decrements for which density functions and forces of decrement do not exist. Dr. Shiu mentions withdrawal as one example. Another common type of decrement with this same feature is retirement.

Regarding Dr. Shiu's final problem, I do see one source of difficulty in adapting Hickman's proof of the Hattendorf Theorem to the discrete case. In the continuous case, it is shown that the second moment (and hence the variance) of L, the loss random variable, is the same as the second moment of another random variable R, the discounted net amount at risk, that is, the present value of the difference between the benefit paid at the time of failure and the reserve at that time. This is not true in the discrete case. Looking at statement (b) of Theorem 7.1, on page 215 of *Actuarial Mathematics* by Bowers et al. (Shiu's reference [1]), we see that this would be true if the p_{x+h} were missing from Formula (7.10.5) on the preceding page, since we would be left in that formula with the density function for K, the curtate-future-lifetime of (x). However, the presence of this extra term indicates that in the discrete case, the second moment of R will be greater than that of L. It seems to me an interesting problem to try to develop some good intuitive, verbal explanation for this behavior of the random variables L and R.