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# A PRACTICAL ALGORITHM FOR APPROXIMATING THE PROBABILITY OF RUIN

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ABSTRACT

For a compound Poisson process, the sequence of record jumps generated by the maximal loss random variable is approximated by an i.i.d. sequence of mixed exponential or gamma random variables. This is then used to approximate the probability of ultimate ruin. This approximation is compared to the Cramér-Lundberg, Beekman-Bowers, and de Vylder approximations. It is shown to be sufficiently accurate for practical purposes.

## 1. INTRODUCTION

The traditional compound Poisson process is often used to model an insurance company's aggregate claims process $S(t)$. This model can briefly be described as follows: the aggregate claims occurring in $(0,t]$ are defined as

$$S(t) = \sum_{k=1}^{N(t)} X_k, \qquad t > 0,$$

where $N(t)$ is the number of claims in $(0,t]$ and $X_k > 0$ is the size of the $k$-th claim, $k = 1, 2, \ldots$ . It is assumed that the $X_k$'s are independent and identically distributed (i.i.d.) random variables with known cumulative distribution function (cdf) $P(x)$. The $N(t)$ process is assumed to be Poisson process with rate $\lambda$. The expected number of claims in $(0,t]$ is $\lambda t$, and $p_1$ is the expected claim size. The premiums are paid continuously at rate $c = (1+\theta)\lambda p_1$, where $\theta$ is the relative security loading, making $ct$ the total premium income in $(0,t]$. The insurance company's surplus at time $t$, given an initial reserve of $u \geq 0$, is $U(t)$, where

$$U(t) = u + ct - S(t).$$

The company is said to be ruined if $U(t) < 0$ for any $t > 0$. The probability of ultimate ruin is given by

$$\Psi(u) = Pr[U(t) < 0 | U(0) = u].$$

Following Bühlmann [8], the terms record points, record highs and record jumps are defined as follows:

**Definition 1.** *Let* $S_0=0$ *and* $S_n=\Sigma_{i=1}^{n} Y_i$, *where the* $Y_i$*'s are i.i.d. random variables capable of taking both positive and negative values. The first* record point *is* $(0,S_0)$. *The point* $(n,S_n)$ *is called a* record point *if* $S_i<S_n$ *for all* $i=0, 1, \ldots, n-1$. *In this case* $S_n$ *is called a* record high *and is designated by* $M_k$ *if* $k$ *further record points lie to the left of* $S_n$. *For* $k=1, 2, \ldots$, *the differences* $L_k=M_k-M_{k-1}$ *are known as* record jumps.

Another way of looking at the evolution of the risk reserve process is to use the maximal aggregate loss random variable $L$, given by

$$L = \sup_{t\geq 0}\{S(t) - ct\}.$$

It is well known (see Bowers et al. [5]) that,

$$L = \sum_{k=1}^{N} L_k, \tag{1}$$

where the $L_k$'s are positive i.i.d. record jumps, and $N$ is a geometric random variable with

$$Pr[N = n] = \frac{\theta}{1+\theta}\left(\frac{1}{1+\theta}\right)^{n+1}, \qquad n = 0, 1, \ldots .$$

The $L_k$'s are absolutely continuous random variables with probability density function (pdf) $h(y)$ satisfying

$$h(y) = \frac{1 - P(y)}{p_1}, \qquad y > 0. \tag{2}$$

This approach leads to the following expression for $\Psi(u)$:

$$\begin{aligned}\Psi(u) &= Pr[L > u] \\ &= 1 - Pr[L \leq u] \\ &= 1 - \frac{\theta}{1+\theta}\sum_{n=0}^{\infty}\left(\frac{1}{1+\theta}\right)^{n} H_n^*(u)\end{aligned} \tag{3}$$

where $H_n^*(u)$ is the $n$-fold convolution of $H_1^*(u)$ with itself, that is,

$$H_1^*(u) = H(u) = \int_0^u h(x)dx$$

$$H_0^*(u) = \begin{cases} 1 & \text{if } u \geq 0 \\ 0 & \text{if } u < 0 \end{cases}$$

and

$$H_n^*(u) = \int_0^u H_{n-1}^*(u - x)dH(x).$$

The numerical evaluation of $\Psi(u)$ is one of the fundamental and more challenging problems of actuarial risk theory. Even though Equation (3) appears to offer a direct approach for evaluating $\Psi(u)$, in practice, direct evaluation of $\Psi(u)$ is extremely difficult except in a few special cases. An example of such a special case is a claim size distribution that is exponential, or a mixture of exponentials or shifted exponentials; see Dufresne and Gerber [12], [13]. For a few other distributions, explicit expressions for $\Psi(u)$ have been obtained in series form; see, for example, Beekman [2], Gerber, Goovaerts and Kaas [18], Shiu [29], and Willmot [33].

If the claim size distribution is completely known, $\Psi(u)$ can be calculated accurately using numerical multiple integration to obtain $H_n^*(u)$ and then summing as in Equation (3). Unfortunately, the degree of accuracy in such calculations may be spurious since, in the "real world,"the assumption of any particular claim size distribution may be very uncertain indeed. In addition, Pentikäinen [25, section 7] cautioned that if the initial data are inaccurate, then it is meaningless to demand essentially greater accuracy from the calculation technique. This suggests that calculations may have to be based on the sample moments generated by the claims' data rather than on some "fitted" distribution. It is therefore desirable to be able to obtain ruin probabilities using only the sample moments of the claim size distribution.

In view of the difficulties associated with evaluating $\Psi(u)$, several simple "practical" approximations have been developed. One of the oldest approximations is

$$\Psi(u) \approx e^{-Ru},$$

where $R$ (the adjustment coefficient) is the smallest positive solution to the equation

$$1 + (1 + \theta)p_1 r = M_X(r),$$

where $M_X(r) = E[e^{rX}]$. Although this approximation is generally inaccurate, it serves as an upper bound on $\Psi(u)$. This approximation has been improved to yield the so-called Cramér-Lundberg approximation

$$\Psi(u) \approx \frac{\theta p_1}{M_X'(R) - (1 + \theta)p_1} e^{-Ru}; \tag{4}$$

see Bühlmann [8, p. 151]. In practice, this approximation may be difficult to apply because it requires knowledge of $M_X(r)$, which in turn requires complete knowledge of $P(x)$. Brockett and Cox [7] provided a method for using the moments of the $X_k$'s to estimate the bounds on $R$.

Another approximation is the Beekman-Bowers approximation developed by Beekman [1] and Bowers [4]. Here $\Psi(u)$ is approximated with the aid of a gamma distribution, that is,

$$\Psi(u) \approx \frac{1}{1 + \theta}[1 - G(u)], \tag{5}$$

where

$$G(u) = \int_0^u \frac{b^{-a}y^{a-1}e^{-y/b}}{\Gamma(a)}dy$$

is a gamma cdf with the parameters $a$ and $b$ given by

$$b = \frac{2p_3}{3p_2} + \frac{p_2}{2\theta}\left(1 - \frac{\theta}{p_1}\right)$$

and

$$a = \frac{1}{b}\frac{p_2}{2\theta}\left(1 + \frac{\theta}{p_1}\right),$$

where $p_j = E[X_1^j]$. Grandell and Segerdahl [21] used the gamma claim size distribution to compare the Beekman-Bowers approximation and the Cramér-Lundberg approximation. They showed that the Beekman-Bowers approximation is not as accurate as the Cramér-Lundberg approximation. Seal [28, p. 62] cautioned against the use of the Beekman-Bowers approximation because it did not appear to give accurate results except when claims were exponentially distributed. Another potential problem with the Beekman-Bowers approximation is that it requires the evaluation of the incomplete gamma function, a function that may not be readily available.

De Vylder [10] provided another practical approximation. He essentially approximated the entire risk reserve process $U(t)$ by a simpler process $\tilde{U}(t)$ defined as

$$\tilde{U}(t) = u + (1 + \tilde{\theta})\tilde{\lambda}\tilde{p}_1 - \tilde{S}(t),$$

where $\tilde{\lambda}$ is the new Poisson parameter, $\tilde{S}(t)$ is the aggregate claims process generated by i.i.d. exponential claim sizes, each with mean $\tilde{p}_1$, and $\tilde{\theta}$ is the new loading. This approximation is completed by equating the first three moments

$$E[U^k(t)] = E[\tilde{U}_k(t)], \qquad k = 1, 2, 3.$$

This yields the following estimates,

$$\tilde{p}_1 = \frac{p_3}{3p_2}$$

$$\tilde{\theta} = \frac{2p_1p_3}{3p_2^2}\theta$$

$$\tilde{\lambda} = \frac{9p_2^3}{2p_3^2}\lambda.$$

These parameters completely specify $\tilde{U}(t)$. The resulting approximation is

$$\Psi(u) \approx \frac{1}{1 + \tilde{\theta}} \exp\left( -\frac{u\tilde{\theta}}{\tilde{p}_1(1 + \tilde{\theta})} \right). \tag{6}$$

Bohman [3] provided a "rule of thumb" for approximating the probability of ruin for both finite and infinite time. For infinite time, this rule is essentially the same as de Vylder's.

Other authors have developed approximations. A recursive algorithm for approximating $\Psi(u)$, which eliminated the propagation of errors, was provided by Goovaerts and de Vylder [19]. Using a different recursive approach, Panjer [24] and Dickson [11] discretized the distribution $H(x)$ and then used Panjer's [23] recursive formula to approximate $\Psi(u)$. Unfortunately, both types of recursive approximations suffer from the same problem: being recursive, they cannot provide $\Psi(u)$ directly without obtaining *all* the intermediate approximating values. Meyers and Beekman [22] used a different approximation entirely: they made a piecewise linear approximation to $H(x)$, then used numerical methods to invert the resulting characteristic function.

Recently there has been an interest in using nonparametric techniques to approximate the ruin function $\Psi(u)$. These estimates are based on the actual claims data generated. The interested reader should review Frees [16] and Croux and Veraverbeke [9].

In the sequel, a simple and practical algorithm for approximating $\Psi(u)$ is developed. This approximation uses only the first four sample moments of the claim size distribution, and hence it does not require complete information on $P(x)$. Ruin probabilities generated by this approximation are compared to some exact results and to the Cramér-Lundberg, Beekman-Bowers and de Vylder approximations.

## 2. THE MODEL

The essential idea behind de Vylder's approximation was to replace the claim size random variable $X_k$ by an exponential random variable $\tilde{X}_k$. This can also be viewed as replacing the random variable $L_k$ by an exponential random variable $\tilde{L}_k$. There are two interesting aspects of de Vylder's approximation: (a) of the three parameters he estimated $(\tilde{\lambda}, \tilde{\theta}, \tilde{p}_1)$, it turned out that $\tilde{\lambda}$ was not needed, and (b) $\tilde{\theta} \neq \theta$. Note that since $\Psi(0) = 1/(1+\theta)$, then for small values of $u$, de Vylder's approximation may lose some of its accuracy. It therefore seems reasonable to ignore $\lambda$ altogether and to keep the loading as $\theta$ to ensure an adequate approximation for small values of $u$. The next step is to find three parameters that can contribute to the accuracy of the approximation. At this point the following definition is needed:

**Definition 2.** *A mixed exponential pdf, $g(x)$, will be called an "n-th order" mixed exponential distribution if, for some positive integer n, it can be written in the form*

$$g(x) = \sum_{i=1}^{n} a_i b_i e^{-b_i x}$$

*with $b_i > 0$, $a_i \neq 0$, and $\Sigma_{i=1}^{n} a_i = 1$.*

It is well known that mixed exponential claim size pdf's yield mixed exponential distributions for the corresponding record jump random variables. They also yield explicit and easily obtainable ruin probabilities; see Bowers et al. [5, chapter 12.6]. Gamma distributions with a shape parameter of 2 also yield explicit ruin probabilities.

With this information, the simplest approach is to extend de Vylder's approximation by using a second-order mixed exponential distribution or, if this is not possible, a gamma distribution with shape parameter of 2. One can accomplish this by using one of two approaches: (1) replace the pdf of the claim size random variable, $X_1$, with a second-order mixed exponential

or a gamma distribution with shape parameter 2, or (2) replace the pdf of the record jump random variable, $L_1$, with a second-order mixed exponential or a gamma distribution.

After comparing the numerical approximations generated by these two approaches, it turns out that approximations based on the former approach are not as accurate as those based on the latter. If $X_1$ happens in fact to be a second-order mixed exponential or a gamma random variable with shape parameter 2, then both approaches yield the exact answers. Note the $L_1$ has one very convenient property that would make it easier to approximate than $X_1$: the pdf of $L_1$ always exists and is always non-increasing (Equation (2)). This is true even though $X_1$ may be discrete! In the sequel the second approach is used.

To be specific, $L$ is approximated by $\hat{L}$ as follows:

$$\hat{L} = \sum_{k=1}^{N} \hat{L}_k \tag{7}$$

where the $\hat{L}_k$'s are i.i.d. random variables and $N$ is the same geometric distribution in Equation (1). Even though $h(x)$ is a non-increasing function of $x$, for the sake of simplicity, it is not assumed that the pdf of $\hat{L}_k$ is non-increasing. The random variable $\hat{L}_1$ is assumed to have a pdf $\hat{h}(x)$, given by

$$\hat{h}(x) = \begin{cases} \dfrac{p}{\beta_1} e^{-x/\beta_1} + \dfrac{q}{\beta_2} e^{-x/\beta_2}, & \text{if } \nu > 1/2 \\ \beta^{-2} x e^{-x/\beta} & \text{otherwise} \end{cases} \tag{8}$$

where

$$\nu = \frac{Var[L_1]}{(E[L_1])^2},$$

and $p$ and $q$ are mixing constants satisfying

$$p + q = 1, \qquad 0 < p < \infty.$$

Since $p$ may exceed 1, it cannot be viewed as a "mixing probability."

The parameters $\beta_1$ and $\beta_2$ are positive and are determined by matching the first two moments of $L_1$ and $\hat{L}_1$ as follows: let $\mu_j = E[L_1^j]$ and $p_j = E[X_1^j]$

and $\hat{\mu}_j = E[\hat{L}_1^j]$. From Bowers et al. Equation (12.5.4), the moment generating function (mgf) of $L_1$ is

$$M_{L_1}(r) = \frac{1}{p_1 r}[M_X(r) - 1]$$

where $M_X(r)$ is the mgf of $X_1$. Expanding both sides of this equation as a power series in $r$ yields the results

$$\mu_j = \frac{p_{j+1}}{(j+1)p_1}, \qquad j = 1, 2, \ldots, \tag{9}$$

while

$$\hat{\mu}_j = j!(p\beta_1^j + q\beta_2^j), \qquad j = 1, 2, \ldots .$$

Equating the first two moments, that is, $\mu_j = \hat{\mu}_j$, for $j = 1, 2$ gives

$$p\beta_1 + q\beta_2 = \mu_1$$
$$p\beta_1^2 + q\beta_2^2 = \frac{\mu_2}{2}$$

This pair of simultaneous equations yields two pairs of solutions:

$$\beta_1 = \mu_1\left[1 + \sqrt{\frac{q(\nu - 1)}{2p}}\right], \quad \beta_2 = \mu_1\left[1 - \frac{p}{q}\sqrt{\frac{q(\nu - 1)}{2p}}\right] \tag{10}$$

and

$$\beta_1 = \mu_1\left[1 - \sqrt{\frac{q(\nu - 1)}{2p}}\right], \quad \beta_2 = \mu_1\left[1 + \frac{p}{q}\sqrt{\frac{q(\nu - 1)}{2p}}\right] \tag{11}$$

where

$$\nu = (\mu_2 - \mu_1^2)/\mu_1^2. \tag{12}$$

### 3. CONDITIONS ON $p$

At this point it is instructive to investigate the constraints needed to ensure that $\hat{h}(x)$ is a pdf. There are two cases to consider: (i) $p > 1$ and (ii) $0 < p \leq 1$.

*3.1* p > 1

Since $p>1$, then $\sqrt{q(\nu-1)/2p}$ exists if and only if $\nu\leq 1$. Now for $\hat{h}(x)$ to be a pdf, the following two conditions [Equations (13) and (14)] must hold:

$$0 < \beta_2 < \beta_1 < \infty, \tag{13}$$

and for a non-negative intercept, that is, $\hat{h}(0)\geq 0$,

$$\frac{p}{\beta_1} + \frac{q}{\beta_2} \geq 0. \tag{14}$$

Since $p>1$ and $q<0$, then $p/q<-1$. This implies that $\beta_1>\beta_2$ in Equation (11). In Equation (10) the opposite result holds, that is, $\beta_1<\beta_2$, which violates Equation (13). As a result, Equation (10) cannot be used to define $\beta_1$ and $\beta_2$ when $p>1$. It is easily seen that $\beta_2>0$ in Equation (11) if and only if $p>\xi_1>1$, where

$$\xi_1 = \frac{2}{\nu+1}. \tag{15}$$

Substituting $\beta_1$ and $\beta_2$, from Equation (11), into Inequality (14) yields the inequality

$$\frac{1}{2(1-\nu)} > f(p) > 1, \qquad 0 < \nu < 1, \tag{16}$$

where

$$f(p) = \frac{\left(p-\frac{1}{2}\right)^2}{p(p-1)}, \qquad p > 1.$$

Let $\xi_2>0$ be the unique root such that

$$f(\xi_2) = \frac{1}{2(1-\nu)},$$

that is,

$$\xi_2 = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2\nu-1}}\right), \qquad 1/2 < \nu < 1. \tag{17}$$

Since $f(p)$ is a decreasing function of $p$, then for $p>1$, it follows that Inequalities (14) and (16) hold if $p \geq \xi_2$. It can easily be proved that $\xi_2 > \xi_1$ in the region $\frac{1}{2} < \nu < 1$. Hence, to ensure that Equation (11) yields a pdf when $p>1$ and $\frac{1}{2} < \nu < 1$, it is necessary and sufficient that $p > \xi_2$.

Since Equation (10) does not yield a pdf, it will not be used in the sequel; that is, only Equation (11) will be used to define $\beta_1$ and $\beta_2$ for $p>0$. The case $0 < \nu \leq \frac{1}{2}$ is discussed in Section 4.2.

### 3.2 $0 < p \leq 1$

Here the conditions $\beta_1 > 0$ and $\beta_2 > 0$ automatically ensure that $\hat{h}(x)$ is a pdf. From Equation (11), $\beta_2 > 0$ while $\beta_1 > 0$ if and only if $p > 1 - \xi_1$. So, to ensure that $\hat{h}(x)$ is a pdf, $p$ must satisfy $1 - \xi_1 < p \leq 1$.

## 4. ESTIMATING $p$

To ensure that the approximating pdf $\hat{h}(x)$ yields exact results for second-order mixed exponentials, the third moment of $L_1$ and of $\hat{L}_1$ must be equated; that is, solve

$$p\beta_1^3 + q\beta_2^3 = \frac{\mu_3}{6}.$$

This yields the equation

$$\frac{\left(p - \frac{1}{2}\right)}{(1-p)} \sqrt{\frac{(1-p)(\nu-1)}{2p}} = h_3, \tag{18}$$

where

$$h_3 = \frac{1}{(\nu-1)}\left[\frac{\mu_3}{6\mu_1^3} - 1 - \frac{3}{2}(\nu-1)\right]. \tag{19}$$

Unfortunately Equation (18) will not always have a solution in $p$ that satisfies the conditions of Section 3. Four cases must be considered: $\{\nu > 1\}$, $\{\nu = 1\}$, $\{\frac{1}{2} < \nu < 1\}$, and $\{\nu \leq \frac{1}{2}\}$. As will be seen, a mixed exponential can be fitted only if $\nu > \frac{1}{2}$; I cannot give an intuitive explanation for this.

*4.1* $\nu > 1$

When $\nu > 1$, that is, $0 < p < 1$, the solution to Equation (18) is given by

$$\hat{p} = \frac{1}{2}\left[1 + sign(h_3)\sqrt{\frac{1}{1+g}}\right], \tag{20}$$

where

$$g = \frac{2h_3^2}{|(\nu - 1)|} \tag{21}$$

and

$$sign(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

From Section 3.2, if $\hat{p} < 1 - \xi_1$, then $\hat{h}(x)$ will not be a pdf. In such cases, I have found, from trial and error, that setting $p = 0.8(\nu - 1)/(\nu + 1) + 0.2$ provides a good approximation. This can be summarized as follows:

**Result 1.** *If* $\nu > 1$, *then*

$$p = \begin{cases} \hat{p} & \text{if } \hat{p} > (\nu - 1)/(\nu + 1) \\ 0.2 + 0.8(\nu - 1)/(\nu + 1) & \text{otherwise.} \end{cases} \tag{22}$$

*where* $\hat{p}$ *is defined in Equation (20).*

*4.2* $\nu = 1$

Here, it must be the case that $\beta_1 = \beta_2 = \mu_1$ regardless of the value of $p$, so simply set $p = 1$ and the exponential distribution results, that is,

$$\hat{h}(x) = \frac{1}{\mu_1} e^{-x/\mu_1}. \tag{23}$$

*4.3* $\frac{1}{2} < \nu < 1$

In Section 3.1, it was proved that $p$ must be greater than or equal to $\xi_2$ to ensure that $\hat{h}(x)$ is a pdf. But to solve Equation (18), one must have $h_3 < 0$ and $g > 1$. If these hold, then the solution to Equation (18) is

$$p^* = \frac{1}{2}\left[1 + \sqrt{\frac{g}{g-1}}\right]. \tag{24}$$

If either $h_3 \geq 0$ or $g \leq 1$, then Equation (18) has no solution. In addition, if the solution $p^*$ is such that $p^* < \xi_2$, then it cannot be used. In these situations, I have found, from trial and error, that setting $p = 6\xi_2$ yields acceptable approximations.

**Result 2.** *If* $\frac{1}{2} < \nu < 1$,

$$p = \begin{cases} p^* & \text{if } h_3 < 0, g > 1 \text{ and } p^* \geq \xi_2; \\ 6\xi_2 & \text{otherwise.} \end{cases} \tag{25}$$

*4.4* $0 < \nu \leq \frac{1}{2}$

So far the case $0 < \nu \leq \frac{1}{2}$ has not been considered. It has already been proved that Equation (11) cannot be used to fit a second-order mixed exponential pdf when $\nu \leq \frac{1}{2}$. One simple pdf that yields explicit ruin probabilities is the gamma with shape parameter 2. It was for this reason that $\hat{h}(x)$ was defined as it was in Equation (8):

$$\hat{h}(x) = \beta^{-2} x e^{-x/\beta}, \qquad \text{if } \nu \leq \tfrac{1}{2}, \tag{26}$$

with

$$\beta = \frac{\mu_1}{2}.$$

## 5. THE PROBABILITY OF RUIN

When $L_1$ has a simple exponential pdf (that is, $p = 1$ and $\beta_1 = \mu_1$), it is well known that the probability of ruin is

$$\Psi(u) = \frac{1}{(1+\theta)} \exp\left(-\frac{\theta u}{(1+\theta)\mu_1}\right). \tag{27}$$

When $L_1$ has a mixed exponential distribution with parameters $p$, $\beta_1$ and $\beta_2$, one can use the results of Bowers et al. [5, pp. 362–63] to show that the corresponding probability of ruin is

$$\Psi(u) = C_1 e^{-r_1 u} + C_2 e^{-r_2 u} \tag{28}$$

where $r_1$, $r_2$, $C_1$ and $C_2$ are constants. In particular, $r_1$ and $r_2$ are the roots of the quadratic,

$$r^2 - \left[\frac{(\theta + q)}{(1 + \theta)\beta_1} + \frac{(\theta + p)}{(1 + \theta)\beta_2}\right] r + \frac{\theta}{(1 + \theta)\beta_1\beta_2} = 0, \tag{29}$$

with $r_1 \geq r_2$. The constants $C_1$ and $C_2$ are given by

$$C_2 = \frac{r_2[r_1(p\beta_2 + q\beta_1) - 1]}{(1 + \theta)(r_1 - r_2)} \tag{30}$$

$$C_2 = \frac{1}{1 + \theta} - C_1. \tag{31}$$

When $L_1$ has the gamma pdf given by Equation (26), $\Psi(u)$ is given by

$$\Psi(u) = C_3 e^{-r_3 u} + C_4 e^{-r_4 u}, \tag{32}$$

where $r_3$, $r_4$, $C_3$, and $C_4$ are constants. In particular, the constants $r_3$ and $r_4$ are the roots of the quadratic

$$(1 + \theta)(1 - \beta r)^2 = 1, \tag{33}$$

with $r_3 \geq r_4$. The constants $C_3$ and $C_4$ are given by

$$C_3 = \frac{1 - \sqrt{1 + \theta}}{2(1 + \theta)} \tag{34}$$

$$C_4 = \frac{1}{1 + \theta} - C_3. \tag{35}$$

## 6. THE ALGORITHM

The method of approximating $\Psi(u)$ described in this paper can be summarized in the following algorithm:

Step 0: Get the sample moments $p_1$, $p_2$, $p_3$, $p_4$ and the values of $u$ and $\theta$.
Step 1: Compute $\mu_k$, $k = 1, 2, 3$ using Equation (9).

Step 2: Compute $\nu = (\mu_2 - \mu_1^2)/\mu_1^2$.
Step 3: Check to see if $\{\nu > 1\}$, $\{\nu = 1\}$, $\{1/2 < \nu < 1\}$, or $\{\nu \leq 1/2\}$.
Step 4: (i) If $\{\nu > 1\}$ then:
Compute $h_3$ and $g$ using Equations (19) and (21), respectively;
Compute $\hat{p}$ using Equation (20);
Compute $p$ using Equation (22);
Compute $\beta_1$ and $\beta_2$ using Equation (11);
Compute $r_1$, $r_2$, $C_1$, and $C_2$ using Equations (29), (30), and (31);
Compute $\Psi(u)$ using Equation (28).
(ii) If $\{\nu = 1\}$, then compute $\Psi(u)$ using Equation (27).
(iii) If $\{1/2 < \nu < 1\}$, then:
Compute $h_3$ and $g$ using Equations (19) and (21), respectively;
Compute $\xi_2$ using Equation (17);
Compute $p$ using Equation (25);
Compute $\beta_1$ and $\beta_2$ using Equation (11);
Compute $r_3$, $r_4$, $C_3$, and $C_4$ using Equations (29), (30), and (31);
Compute $\Psi(u)$ using Equation (28).
(iv) If $\{\nu \leq 1/2\}$, then:
Compute $\beta = \mu_1/2$;
Compute $r_3$, $r_4$, $C_3$ and $C_4$ using Equations (33), (34) and (35);
Compute $\Psi(u)$ using Equation (32).

## 7. COMPARING APPROXIMATIONS

In this section, the relative errors produced by the algorithm introduced in this paper are compared to those produced by de Vylder's approximation, Beekman-Bowers' approximation, Cramér-Lunderg's approximation, and an approximation to the Cramér-Lundberg approximation based on using the first four sample moments. The term $\epsilon$ denotes the relative error, that is,

$$\epsilon = \frac{approx - exact}{exact}.$$

If $|\epsilon| < 5 \times 10^{-d}$, the approximation is said to be accurate to $d$ significant digits. The relative errors are checked for an accuracy of at least three significant digits by multiplying them by 200: if $200\epsilon < 1$, then there is at

least three-digit accuracy. Note that the true degree of accuracy depends not only on the approximation technique, but also on the accuracy of the data and the appropriateness of the compound Poisson model.

In the Appendix, seven tables (based on published results) are used to display the relative errors. These tables are constructed by using already published claim size distributions and ruin probabilities, specifically:

1. Table 1 uses the results provided by Willmot [33] for the gamma claim size distribution with cdf

$$P(x) = \int_0^x \frac{\alpha(\alpha x)^{\alpha-1}e^{-\alpha x}}{\Gamma(\alpha)}dx$$

and a loading of $\theta = 0.25$.

2. Table 2 uses Grandell's [20, table 1, p. 20) gamma distribution with the same $P(x)$ as in Table 1 except that $\alpha = 0.01$.
3. Table 3 uses Bohman's [3] second-order mixed exponential distribution with

$$P(x) = 1 - ae^{-\alpha_1 x} - (1 - a)e^{-\alpha_2 x}$$

where $\alpha_1 = a/(1-a)$, $\alpha_2 = 1/\alpha_1$, and

$$a = \frac{1}{2}\left[1 + \sqrt{\frac{s^2 - 1}{7 + s^2}}\right].$$

4. Table 4 uses Bowers et al.'s second-order mixed exponential distribution, example (12.10), adjusted so that $p_1 = 1$,

$$P(x) = 1 - \frac{1}{2}e^{-5x/7} - \frac{1}{2}e^{-3x/5}.$$

5. Table 5 uses Dufresne and Gerber's [13] second-order mixed exponential distribution adjusted so that $p_1 = 1$,

$$P(x) = 1 - 4e^{-7x/4} - 3e^{-7x/3}.$$

6. Table 6 uses Wikstad's [32] third-order mixed exponential,

$$P(x) = 1 - 0.0039793e^{-0.014631x} - 0.1078392e^{-0.190206x} - 0.8881815e^{-5.514588x}.$$

Here $p_1 = 1$, $p_2 = 43.1982$, $p_3 = 7717.23$, and $p_4 = 86920.5576$.

7. Table 7 uses the lognormal distribution of Thorin and Wikstad [30] with $p_1 = 1$, $p_2 = 25.53372$, $p_3 = 16647.24$, and $p_4 = e^{19.44}$.
8. Tables 8–14 in the Appendix are based on the data given in Reckin, Schwark and Snyder [27] and several different retention levels. In particular, the data are as follows:

DATA: $x$ IN $1,000

| $x$ | $Pr[X_1 = x]$ | $x$ | $Pr[X_1 = x]$ |
|---|---|---|---|
| 0 | 0 | 63 | 0.0036 |
| 2 | 0.3075 | 73 | 0.0041 |
| 4 | 0.2066 | 83 | 0.0019 |
| 6 | 0.2240 | 93 | 0.0013 |
| 11 | 0.0859 | 100 | 0.0158 |
| 16 | 0.0362 | 130 | 0.0005 |
| 22 | 0.0277 | 155 | 0.0018 |
| 26 | 0.0220 | 226 | 0.0034 |
| 33 | 0.0194 | 355 | 0.0006 |
| 44 | 0.0096 | 550 | 0.0003 |
| 50 | 0.0276 | 1000 | 0.0002 |

In each of Tables 8–14, the unit of currency is adjusted to ensure that the claim size random variable has a mean of one unit, and the reserve $u$ is expressed in these units. For example, in Table 8, one unit is $12,008.60, so $u = 10$ implies a reserve of $120,086. In Tables 13 and 14, on the other hand, one unit represents $7,721.40, so $u = 10$ implies a reserve of $77,214.

To apply these ruin approximations, the raw sample moments were adjusted to ensure the sample mean is always one unit. This is accomplished by dividing $p_k$ by $p_1^k$.

RAW SAMPLE MOMENTS FOR DIFFERENT RETENTION LIMITS

| Retention | $p_1$ | $p_2$ | $p_3$ | $p_4$ |
|---|---|---|---|---|
| 1,000 | 12.0086 | 979.4402 | 350,204.5370 | 249,225,368.088 |
| 200 | 11.5622 | 619.4168 | 70,201.9136 | 10,574,336.275 |
| 100 | 10.9982 | 455.7218 | 33,200.4386 | 2,872,570.150 |
| 50 | 9.6085 | 253.0017 | 10,160.7403 | 461,384.131 |
| 25 | 7.7214 | 115.7300 | 2,373.1866 | 54,256.132 |

ADJUSTED MOMENTS
FOR DIFFERENT RETENTION LIMITS

| Retention | $p_1$ | $p_2$ | $p_3$ | $p_4$ |
|---|---|---|---|---|
| 1,000 | 1 | 6.792 | 202.230 | 11,984.578 |
| 200 | 1 | 4.633 | 45.418 | 591.686 |
| 100 | 1 | 3.768 | 24.956 | 196.329 |
| 50 | 1 | 2.740 | 11.454 | 54.130 |
| 25 | 1 | 1.941 | 5.155 | 15.264 |

The exact results ($\Psi(u)$) are obtained by using the technique suggested by Ramsay [26] to improve Goovaerts' and de Vylder's [19] stable algorithm for obtaining ruin probabilities.

The following notation is used in the relative error in the tables:

- *MY* = approximations are based on my algorithm of Section 6.
- *DV* = de Vylder's approximation Equation (6).
- *BB* = Beekman-Bowers approximation Equation (5).
- *CL* = Cramér-Lundberg approximation Equation (4).
- *CL*4 = Cramér-Lundberg approximation with $M_X(r)$ approximated using only the first four moments, that is, as a fourth-degree polynomial in $r$.

Note in Table 1, $\nu \leq \frac{1}{2}$ when $\alpha \geq 7$; this means that the approximation *MY* uses a gamma distribution with shape parameter 2 for $L_1$; see Equation (8). From Tables 1 to 6, my algorithm is, in general, clearly better than the others except the full Cramér-Lundberg approximation (*CL*); it yielded roughly the same degree of accuracy as *CL*. As expected, *MY* produces exact results when $P(x)$ is a second-order mixed exponential or a gamma distribution with shape parameter 2; see Tables 1 ($\alpha = 1, 2$), 3, 4, and 5. Note that the Cramér-Lundberg approximation was remarkably accurate for even relatively small values of $u$.

For the lognormal case (Table 7) these approximations do not fare well. This is not surprising because they all implicitly assumed that the claims distribution had an exponentially decreasing pdf, which leads to an exponentially decreasing ruin function. However, the lognormal distribution has a very heavy right tail, which decreases at a subexponential rate. To be precise, Thorin and Wikstad [30] showed that for the lognormal pdf with parameters

$$\mu_X = E[lnX] \quad \text{and} \quad \sigma_X^2 = Var[lnX]$$

the probability of ruin is asymptotically given by

$$\Psi(u) \sim \frac{\sigma_X^3[\tau(u)]^2}{\theta\sqrt{2\pi}} e^{-\frac{1}{2\sigma_X^2}\tau(u)^2},$$

where

$$\tau(u) = ln(u/p_2).$$

This does not represent an exponential decrease in $\Psi(u)$. For a detailed treatment of the ruin problem for heavy tailed distributions, see von Bahr [31], Embrechts and Veraverbeke [14] or Embrechts and Villasenor [15]. The only approximations available for these types of distributions are the asymptotic approximations for large $u$.

From the results of Tables 8–14, these approximations appear to be adequate in practice, when applied to real data. They all, with the exception of the CL4, yield two or three significant digits, especially as the retention limit increases. Owing to the many assumptions involved in "real life" situations, I think actuaries should find such accuracy acceptable.

## 8. INVERSE RUIN

A very important practical problem that needs to be addressed is that of the calculation of the inverse ruin function. The inverse ruin function, $\Psi^{-1}(p)$, gives the initial reserve $u_p$ such that $\Psi(u_p)=p$. In fact, Brender [6] and Reckin, Schwark and Snyder [27] pointed out that, in practice, actuaries are apt to be much more interested in finding the reserve level (contingency surplus) $u_p$ for a given probability of ruin, $p$, than in finding $p$ for a given level of reserve $u$. Unfortunately, this important problem is rarely discussed in the literature.

Neither the series expansion method nor the discretization will immediately yield $\Psi^{-1}(p)$. However, they can be used to numerically obtain $u_p$ from the equation

$$\Psi(u_p) = p \tag{36}$$

by using such standard numerical techniques as the Newton-Raphson method or the secant method (if $\Psi'(u)$ is difficult to obtain). These methods require the choice of a starting value $\hat{u}_0$, which is close to the unknown solution $u_p$, and repeated evaluations of $\Psi(u)$, a very slow process at best. Frees [16], [17] suggested the use of quantile and stochastic approximation techniques

to estimate $u_p$. These techniques essentially require a computer simulation of the risk process; this may limit their usefulness.

Using my algorithm, one can also use Equation (36), but with $\Psi(u)$ given by either Equation (27), (28) or (32). It is now a simple matter to obtain use of $p$ either directly or by the Newton-Raphson method.

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## APPENDIX

### TABLE 1

COMPARISON OF RELATIVE ERRORS PRODUCED BY RAMSAY'S ALGORITHM WITH THOSE PRODUCED BY WILLMOT'S GAMMA DISTRIBUTION [33]

| | | | 200ε | | | | |
|---|---|---|---|---|---|---|---|
| α | u | Ψ(u) | *MY* | *DV* | *BB* | *CL* | *CL4* |
| 0.50 | 0.10 | 0.786173 | 0.32 | −3.49 | 0.35 | −4.66 | 0.59 |
| 0.50 | 0.25 | 0.767738 | 0.42 | −2.67 | 0.69 | −3.84 | 1.40 |
| 0.50 | 0.50 | 0.739747 | 0.34 | −1.77 | 0.98 | −2.93 | 2.27 |
| 0.50 | 1.00 | 0.689448 | 0.06 | −0.74 | 1.13 | −1.87 | 3.23 |
| 0.50 | 1.50 | 0.644071 | −0.11 | −0.17 | 1.06 | −1.26 | 3.73 |
| 0.50 | 2.00 | 0.602368 | −0.18 | 0.18 | 0.89 | −0.88 | 3.99 |
| 0.50 | 2.50 | 0.563728 | −0.19 | 0.40 | 0.69 | −0.63 | 4.12 |
| 0.50 | 3.00 | 0.527777 | −0.17 | 0.53 | 0.48 | −0.46 | 4.16 |
| 0.50 | 5.00 | 0.406239 | −0.05 | 0.70 | −0.26 | −0.14 | 3.98 |
| 0.50 | 10.00 | 0.211856 | 0.05 | 0.48 | −1.13 | −0.01 | 2.85 |
| 1.00 | 0.10 | 0.784159 | −0.00 | −0.00 | −0.00 | 0.00 | 3.66 |
| 1.00 | 0.25 | 0.760984 | −0.00 | −0.00 | −0.00 | 0.00 | 3.62 |
| 1.00 | 0.50 | 0.723870 | 0.00 | 0.00 | 0.00 | 0.00 | 3.55 |
| 1.00 | 1.00 | 0.654985 | 0.00 | 0.00 | 0.00 | 0.00 | 3.42 |
| 1.00 | 1.50 | 0.592655 | −0.00 | −0.00 | −0.00 | 0.00 | 3.29 |
| 1.00 | 2.00 | 0.536256 | −0.00 | −0.00 | −0.00 | 0.00 | 3.15 |
| 1.00 | 2.50 | 0.485225 | −0.00 | −0.00 | −0.00 | 0.00 | 3.02 |
| 1.00 | 3.00 | 0.439049 | −0.00 | −0.00 | −0.00 | 0.00 | 2.88 |
| 1.00 | 5.00 | 0.294304 | 0.00 | 0.00 | 0.00 | 0.00 | 2.35 |
| 1.00 | 10.00 | 0.108268 | −0.00 | −0.00 | −0.00 | 0.00 | 1.03 |
| 2.00 | 0.10 | 0.783443 | −0.00 | 3.25 | −0.43 | 4.21 | 6.56 |
| 2.00 | 0.25 | 0.757171 | −0.00 | 1.87 | −1.01 | 2.81 | 5.11 |
| 2.00 | 0.50 | 0.711975 | −0.00 | 0.54 | −1.40 | 1.44 | 3.67 |
| 2.00 | 1.00 | 0.624303 | 0.00 | −0.46 | −1.16 | 0.38 | 2.48 |
| 2.00 | 1.50 | 0.545309 | 0.00 | −0.67 | −0.56 | 0.10 | 2.08 |
| 2.00 | 2.00 | 0.475824 | 0.00 | −0.68 | −0.01 | 0.03 | 1.89 |
| 2.00 | 2.50 | 0.415080 | 0.00 | −0.64 | 0.44 | 0.01 | 1.76 |
| 2.00 | 3.00 | 0.362064 | 0.00 | −0.58 | 0.77 | 0.00 | 1.64 |
| 2.00 | 5.00 | 0.209585 | −0.00 | −0.34 | 1.37 | −0.00 | 1.18 |
| 2.00 | 10.00 | 0.053430 | 0.00 | 0.29 | 0.27 | −0.00 | 0.04 |

TABLE 1—*Continued*

| | | | 200ε | | | | |
|---|---|---|---|---|---|---|---|
| α | u | Ψ(u) | *MY* | *DV* | *BB* | *CL* | *CL4* |
| 2.75 | 0.10 | 0.783367 | 0.39 | 4.52 | −0.60 | 5.81 | 7.72 |
| 2.75 | 0.25 | 0.756359 | 0.46 | 2.42 | −1.49 | 3.66 | 5.52 |
| 2.75 | 0.50 | 0.708246 | 0.21 | 0.41 | −2.10 | 1.60 | 3.39 |
| 2.75 | 1.00 | 0.612585 | −0.11 | −0.84 | −1.55 | 0.25 | 1.92 |
| 2.75 | 1.50 | 0.526867 | −0.15 | −0.97 | −0.53 | 0.03 | 1.59 |
| 2.75 | 2.00 | 0.452702 | −0.13 | −0.90 | 0.32 | 0.00 | 1.47 |
| 2.75 | 2.50 | 0.388930 | −0.12 | −0.81 | 0.95 | −0.00 | 1.36 |
| 2.75 | 3.00 | 0.334139 | −0.10 | −0.72 | 1.39 | 0.00 | 1.26 |
| 2.75 | 5.00 | 0.182035 | −0.05 | −0.35 | 2.01 | 0.00 | 0.85 |
| 2.75 | 10.00 | 0.039878 | 0.08 | 0.58 | −0.36 | 0.00 | −0.17 |
| 10.00 | 0.10 | 0.783343 | 2.11 | 7.89 | −0.96 | 9.85 | 10.83 |
| 10.00 | 0.25 | 0.755721 | 1.46 | 3.40 | −2.92 | 5.27 | 6.21 |
| 10.00 | 0.50 | 0.702026 | −1.22 | −1.11 | −4.62 | 0.62 | 1.51 |
| 10.00 | 1.00 | 0.581211 | −1.42 | −1.80 | −1.91 | −0.25 | 0.57 |
| 10.00 | 1.50 | 0.478377 | −0.86 | −1.32 | 0.64 | 0.05 | 0.80 |
| 10.00 | 2.00 | 0.394452 | −0.67 | −1.20 | 2.04 | −0.01 | 0.67 |
| 10.00 | 2.50 | 0.325135 | −0.41 | −1.01 | 2.98 | 0.00 | 0.62 |
| 10.00 | 3.00 | 0.268013 | −0.16 | −0.83 | 3.53 | −0.00 | 0.55 |
| 10.00 | 5.00 | 0.123742 | 0.85 | −0.11 | 3.25 | −0.00 | 0.29 |
| 10.00 | 10.00 | 0.017923 | 3.40 | 1.70 | −5.29 | −0.00 | −0.37 |

Note: *MY* = approximations based on the algorithm of Section 6
*DV* = de Vylder's approximation, Equation (6)
*BB* = Beekman-Bowers approximation, Equation (5)
*CL* = Cramér-Lundberg approximation, Equation (4)

TABLE 2

COMPARISON OF RELATIVE ERRORS PRODUCED BY RAMSAY'S ALGORITHM WITH THOSE PRODUCED BY GRANDELL'S GAMMA DISTRIBUTION [20]

| | | | $200\epsilon$ | | | | |
|---|---|---|---|---|---|---|---|
| $u$ | $\theta$ | $\Psi(u)$ | *MY* | *DV* | *BB* | *CL* | *CL4* |
| 300 | 0.10 | 0.521140 | −0.01 | 0.54 | −0.25 | −0.05 | 0.88 |
| 600 | 0.10 | 0.308670 | 0.03 | 0.39 | −1.64 | −0.00 | 0.73 |
| 900 | 0.10 | 0.182870 | 0.02 | 0.19 | −1.81 | −0.00 | 0.52 |
| 1200 | 0.10 | 0.108340 | 0.01 | −0.00 | −1.34 | −0.00 | 0.31 |
| 1500 | 0.10 | 0.064180 | 0.02 | −0.18 | −0.45 | 0.01 | 0.12 |
| 1800 | 0.10 | 0.038030 | −0.02 | −0.41 | 0.67 | −0.02 | −0.12 |
| 2100 | 0.10 | 0.022530 | −0.02 | −0.60 | 2.04 | −0.02 | −0.32 |
| 2400 | 0.10 | 0.013350 | −0.07 | −0.83 | 3.55 | −0.05 | −0.56 |
| 2700 | 0.10 | 0.007910 | −0.10 | −1.04 | 5.20 | −0.07 | −0.78 |
| 3000 | 0.10 | 0.004680 | 0.16 | −0.97 | 7.27 | 0.19 | −0.72 |

Note: *MY* = approximations based on the algorithm of Section 6
*DV* = de Vylder's approximation, Equation (6)
*BB* = Beekman-Bowers approximation, Equation (5)
*CL* = Cramér-Lundberg approximation, Equation (4)
*CL4* = Cramér-Lundberg approximation with $M(r)$ approximated using only the first four moments, that is, as a fourth-degree polynomial in $r$.

TABLE 3

COMPARISON OF RELATIVE ERRORS PRODUCED BY RAMSAY'S ALGORITHM WITH THOSE PRODUCED BY BOHMAN'S SECOND-ORDER MIXED EXPONENTIAL DISTRIBUTION [3]

| | | | $200\epsilon$ | | | | |
|---|---|---|---|---|---|---|---|
| $u$ | $\theta$ | $\Psi(u)$ | *MY* | *DV* | *BB* | *CL* | *CL4* |
| 2 | 0.20 | 0.703701 | 0.00 | 0.33 | 3.68 | −0.05 | 3.60 |
| 2 | 0.40 | 0.537148 | 0.00 | 0.96 | 4.01 | −0.09 | 14.23 |
| 2 | 0.60 | 0.431993 | 0.00 | 1.59 | 3.73 | −0.11 | 27.79 |
| 2 | 0.80 | 0.360166 | −0.00 | 2.16 | 3.35 | −0.13 | 42.24 |
| 2 | 1.00 | 0.308250 | 0.00 | 2.66 | 2.98 | −0.14 | 56.76 |
| 4 | 0.20 | 0.618945 | 0.00 | 0.36 | 1.65 | −0.00 | 3.47 |
| 4 | 0.40 | 0.433784 | 0.00 | 0.92 | 0.21 | −0.00 | 13.02 |
| 4 | 0.60 | 0.328304 | 0.00 | 1.44 | −1.18 | −0.00 | 24.31 |
| 4 | 0.80 | 0.261581 | −0.00 | 1.87 | −2.30 | 0.00 | 35.47 |
| 4 | 1.00 | 0.216148 | 0.00 | 2.22 | −3.16 | 0.00 | 45.85 |
| 6 | 0.20 | 0.544539 | 0.00 | 0.33 | 0.28 | 0.00 | 3.29 |
| 6 | 0.40 | 0.350461 | 0.00 | 0.80 | −1.96 | 0.00 | 11.72 |
| 6 | 0.60 | 0.249640 | 0.00 | 1.18 | −3.67 | −0.00 | 20.77 |
| 6 | 0.80 | 0.190100 | −0.00 | 1.45 | −4.88 | 0.00 | 28.74 |
| 6 | 1.00 | 0.151671 | 0.00 | 1.63 | −5.72 | 0.00 | 35.25 |
| 8 | 0.20 | 0.479078 | 0.00 | 0.31 | −0.68 | 0.00 | 3.12 |
| 8 | 0.40 | 0.283144 | 0.00 | 0.68 | −3.22 | 0.00 | 10.44 |
| 8 | 0.60 | 0.189824 | 0.00 | 0.91 | −4.86 | −0.00 | 17.28 |
| 8 | 0.80 | 0.138153 | −0.00 | 1.02 | −5.84 | 0.00 | 22.20 |
| 8 | 1.00 | 0.106428 | 0.00 | 1.05 | −6.42 | 0.00 | 25.09 |
| 10 | 0.20 | 0.421487 | 0.00 | 0.28 | −1.37 | 0.00 | 2.94 |
| 10 | 0.40 | 0.228757 | 0.00 | 0.56 | −3.91 | 0.00 | 9.16 |
| 10 | 0.60 | 0.144341 | 0.00 | 0.65 | −5.22 | −0.00 | 13.85 |
| 10 | 0.80 | 0.100402 | −0.00 | 0.60 | −5.79 | 0.00 | 15.86 |
| 10 | 1.00 | 0.074680 | 0.00 | 0.47 | −5.96 | 0.00 | 15.38 |

Note: *MY* = approximations based on the algorithm of Section 6
*DV* = de Vylder's approximation, Equation (6)
*BB* = Beekman-Bowers approximation, Equation (5)
*CL* = Cramér-Lundberg approximation, Equation (4)
*CL4* = Cramér-Lundberg approximation with $M(r)$ approximated using only the first four moments, that is, as a fourth-degree polynomial in $r$.

TABLE 4

COMPARISON OF RELATIVE ERRORS PRODUCED BY RAMSAY'S ALGORITHM WITH THOSE PRODUCED BY BOWERS ET AL.'S SECOND-ORDER MIXED EXPONENTIAL DISTRIBUTION [5]

| | | | 200ε | | | | |
|---|---|---|---|---|---|---|---|
| $u$ | $\theta$ | $\Psi(u)$ | *MY* | *DV* | *BB* | *CL* | *CL4* |
| 10 | 0.20 | 0.199211 | 0.00 | 0.26 | −0.94 | 0.00 | 1.58 |
| 10 | 0.40 | 0.063403 | 0.00 | −0.22 | −0.96 | −0.00 | 1.49 |
| 10 | 0.60 | 0.026936 | 0.00 | −1.70 | −0.48 | −0.00 | −5.18 |
| 10 | 0.80 | 0.013840 | −0.00 | −3.88 | −0.11 | 0.00 | −17.69 |
| 10 | 1.00 | 0.008111 | 0.00 | −6.43 | −0.01 | 0.00 | −33.70 |
| 20 | 0.20 | 0.048606 | 0.00 | −0.26 | −0.13 | 0.00 | −0.01 |
| 20 | 0.40 | 0.005862 | 0.00 | −2.95 | 3.25 | −0.00 | −10.06 |
| 20 | 0.60 | 0.001233 | 0.00 | −7.95 | 6.90 | −0.00 | −34.08 |
| 20 | 0.80 | 0.000373 | −0.00 | −14.27 | 9.50 | 0.00 | −65.01 |
| 20 | 1.00 | 0.000145 | 0.00 | −21.11 | 10.92 | 0.00 | −95.68 |
| 30 | 0.20 | 0.011859 | 0.00 | −0.79 | 1.80 | 0.00 | −1.58 |
| 30 | 0.40 | 0.000542 | 0.00 | −5.64 | 9.84 | −0.00 | −20.95 |
| 30 | 0.60 | 0.000056 | 0.00 | −14.00 | 17.80 | −0.00 | −58.69 |
| 30 | 0.80 | 0.000010 | −0.00 | −24.12 | 23.62 | 0.00 | −100.05 |
| 30 | 1.00 | 0.000003 | 0.00 | −34.67 | 27.13 | 0.00 | −134.56 |
| 40 | 0.20 | 0.002894 | 0.00 | −1.31 | 4.30 | 0.00 | −3.15 |
| 40 | 0.40 | 0.000050 | 0.00 | −8.30 | 17.73 | −0.00 | −31.22 |
| 40 | 0.60 | 0.000003 | 0.00 | −19.86 | 30.86 | −0.00 | −79.65 |
| 40 | 0.80 | 0.000000 | −0.00 | −33.44 | 40.79 | 0.00 | −125.99 |
| 40 | 1.00 | 0.000000 | 0.00 | −47.20 | 47.71 | 0.00 | −158.95 |
| 50 | 0.20 | 0.000706 | 0.00 | −1.83 | 7.14 | 0.00 | −4.70 |
| 50 | 0.40 | 0.000005 | 0.00 | −10.92 | 26.59 | −0.00 | −40.90 |
| 50 | 0.60 | 0.000000 | 0.00 | −25.54 | 46.04 | −0.00 | −97.50 |
| 50 | 0.80 | 0.000000 | −0.00 | −42.27 | 66.16 | 0.00 | −145.20 |
| 50 | 1.00 | 0.000000 | 0.00 | −58.78 | 120.40 | 0.00 | −174.25 |

Note: *MY* = approximations based on the algorithm of Section 6
*DV* = de Vylder's approximation, Equation (6)
*BB* = Beekman-Bowers approximation, Equation (5)
*CL* = Cramér-Lundberg approximation, Equation (4)
*CL4* = Cramér-Lundberg approximation with $M(r)$ approximated using only the first four moments, that is, as a fourth-degree polynomial in $r$.

TABLE 5

COMPARISON OF RELATIVE ERRORS PRODUCED BY RAMSAY'S ALGORITHM WITH THOSE PRODUCED BY DUFRESNE AND GERBER'S SECOND-ORDER MIXED EXPONENTIAL DISTRIBUTION [13]

| | | | $200\epsilon$ | | | | |
|---|---|---|---|---|---|---|---|
| $u$ | $\theta$ | $\Psi(u)$ | *MY* | *DV* | *BB* | *CL* | *CL4* |
| 5 | 0.20 | 0.276212 | 0.00 | −0.27 | 1.04 | 0.00 | 0.87 |
| 5 | 0.40 | 0.104813 | 0.00 | −0.18 | 1.56 | 0.00 | 1.75 |
| 5 | 0.60 | 0.048897 | −0.00 | 0.64 | 1.10 | 0.00 | −0.17 |
| 5 | 0.80 | 0.026314 | 0.00 | 2.11 | 0.32 | 0.00 | −5.45 |
| 5 | 1.00 | 0.015704 | 0.00 | 4.06 | −0.46 | −0.00 | −13.52 |
| 10 | 0.20 | 0.089684 | 0.00 | 0.05 | 0.72 | 0.00 | 0.29 |
| 10 | 0.40 | 0.014773 | 0.00 | 1.60 | −2.27 | 0.00 | −2.96 |
| 10 | 0.60 | 0.003607 | −0.00 | 5.06 | −6.87 | 0.00 | −13.44 |
| 10 | 0.80 | 0.001155 | 0.00 | 10.14 | −11.39 | 0.00 | −30.09 |
| 10 | 1.00 | 0.000449 | 0.00 | 16.48 | −15.19 | −0.00 | −50.21 |
| 15 | 0.20 | 0.029120 | 0.00 | 0.36 | −0.77 | 0.00 | −0.28 |
| 15 | 0.40 | 0.002082 | 0.00 | 3.39 | −8.40 | 0.00 | −7.55 |
| 15 | 0.60 | 0.000266 | −0.00 | 9.58 | −17.82 | 0.00 | −25.82 |
| 15 | 0.80 | 0.000051 | 0.00 | 18.50 | −26.36 | 0.00 | −51.61 |
| 15 | 1.00 | 0.000013 | 0.00 | 29.66 | −33.29 | −0.00 | −79.68 |
| 20 | 0.20 | 0.009455 | 0.00 | 0.68 | −2.79 | 0.00 | −0.85 |
| 20 | 0.40 | 0.000294 | 0.00 | 5.19 | −15.41 | 0.00 | −12.04 |
| 20 | 0.60 | 0.000020 | −0.00 | 14.19 | −29.54 | 0.00 | −37.39 |
| 20 | 0.80 | 0.000002 | 0.00 | 27.19 | −41.74 | 0.00 | −70.41 |
| 20 | 1.00 | 0.000000 | 0.00 | 43.64 | −51.27 | −0.01 | −103.36 |
| 25 | 0.20 | 0.003070 | 0.00 | 1.00 | −5.13 | 0.00 | −1.43 |
| 25 | 0.40 | 0.000041 | 0.00 | 7.01 | −22.75 | 0.00 | −16.42 |
| 25 | 0.60 | 0.000001 | −0.00 | 18.91 | −41.22 | 0.00 | −48.18 |
| 25 | 0.80 | 0.000000 | 0.00 | 36.23 | −56.16 | 0.00 | −86.82 |
| 25 | 1.00 | 0.000000 | 0.00 | 58.48 | −64.57 | −0.01 | −122.37 |

Note: *MY* = approximations based on the algorithm of Section 6
*DV* = de Vylder's approximation, Equation (6)
*BB* = Beekman-Bowers approximation, Equation (5)
*CL* = Cramér-Lundberg approximation, Equation (4)
*CL4* = Cramér-Lundberg approximation with $M(r)$ approximated using only the first four moments, that is, as a fourth-degree polynomial in $r$.

TABLE 6

COMPARISON OF RELATIVE ERRORS PRODUCED BY RAMSAY'S ALGORITHM WITH THOSE PRODUCED BY WIKSTAD'S THIRD-ORDER MIXED EXPONENTIAL DISTRIBUTION [32]

| | | | $200\epsilon$ | | | | |
|---|---|---|---|---|---|---|---|
| $u$ | $\theta$ | $\Psi(u)$ | *MY* | *DV* | *BB* | *CL* | *CL4* |
| 10 | 0.10 | 0.799318 | −2.25 | −10.85 | 9.57 | −13.34 | 2.99 |
| 10 | 0.20 | 0.661063 | −4.25 | −16.29 | 11.14 | −22.84 | 19.29 |
| 10 | 0.30 | 0.560997 | −5.98 | −19.16 | 10.94 | −29.55 | 39.17 |
| 10 | 0.40 | 0.485721 | −7.47 | −20.69 | 10.46 | −34.38 | 60.37 |
| 10 | 0.50 | 0.427320 | −8.76 | −21.47 | 10.05 | −37.93 | 82.07 |
| 10 | 0.60 | 0.380853 | −9.88 | −21.81 | 9.79 | −40.60 | 103.87 |
| 10 | 0.70 | 0.343100 | −10.85 | −21.88 | 9.65 | −42.65 | 125.58 |
| 10 | 0.80 | 0.311884 | −11.69 | −21.78 | 9.62 | −44.26 | 147.10 |
| 10 | 0.90 | 0.285682 | −12.44 | −21.59 | 9.65 | −45.55 | 168.37 |
| 10 | 1.00 | 0.263404 | −13.10 | −21.33 | 9.74 | −46.58 | 189.36 |
| 100 | 0.10 | 0.539334 | 0.46 | 2.21 | 2.31 | −0.02 | 14.22 |
| 100 | 0.20 | 0.345541 | 1.18 | 5.40 | −3.38 | −0.02 | 31.42 |
| 100 | 0.30 | 0.246064 | 1.81 | 8.00 | −7.27 | −0.01 | 43.15 |
| 100 | 0.40 | 0.187778 | 2.29 | 9.95 | −9.80 | −0.01 | 49.40 |
| 100 | 0.50 | 0.150318 | 2.66 | 11.37 | −11.52 | −0.01 | 51.29 |
| 100 | 0.60 | 0.124564 | 2.94 | 12.41 | −12.74 | −0.01 | 49.97 |
| 100 | 0.70 | 0.105935 | 3.16 | 13.18 | −13.65 | −0.00 | 46.34 |
| 100 | 0.80 | 0.091916 | 3.33 | 13.76 | −14.35 | −0.01 | 41.10 |
| 100 | 0.90 | 0.081031 | 3.47 | 14.20 | −14.90 | −0.01 | 34.77 |
| 100 | 1.00 | 0.072359 | 3.58 | 14.54 | −15.35 | −0.01 | 27.73 |
| 1000 | 0.10 | 0.021017 | −0.41 | −2.05 | 8.67 | −0.02 | −15.76 |
| 1000 | 0.20 | 0.001767 | −2.82 | −13.42 | 67.18 | −0.02 | −81.95 |
| 1000 | 0.30 | 0.000321 | −6.48 | −29.49 | 145.80 | −0.01 | −141.99 |
| 1000 | 0.40 | 0.000092 | −10.46 | −45.72 | 231.30 | −0.01 | −175.40 |
| 1000 | 0.50 | 0.000036 | −14.31 | −60.27 | 317.23 | −0.01 | −190.33 |
| 1000 | 0.60 | 0.000017 | −17.84 | −72.72 | 400.27 | −0.01 | −196.33 |
| 1000 | 0.70 | 0.000009 | −21.02 | −83.21 | 478.81 | −0.00 | −198.62 |
| 1000 | 0.80 | 0.000006 | −23.85 | −92.00 | 552.22 | −0.01 | −199.48 |
| 1000 | 0.90 | 0.000004 | −26.36 | −99.40 | 620.36 | −0.01 | −199.80 |
| 1000 | 1.00 | 0.000003 | −28.59 | −105.67 | 683.41 | −0.01 | −199.92 |

Note: *MY* = approximations based on the algorithm of Section 6
*DV* = de Vylder's approximation, Equation (6)
*BB* = Beekman-Bowers approximation, Equation (5)
*CL* = Cramér-Lundberg approximation, Equation (4)
*CL4* = Cramér-Lundberg approximation with $M(r)$ approximated using only the first four moments, that is, as a fourth-degree polynomial in $r$.

TABLE 7

COMPARISON OF RELATIVE ERRORS PRODUCED BY RAMSAY'S ALGORITHM WITH THOSE PRODUCED BY THORIN AND WIKSTAD'S LOGNORMAL DISTRIBUTION [30]

| | | | $200\epsilon$ | | | |
|---|---|---|---|---|---|---|
| $u$ | $\theta$ | $\Psi(u)$ | *MY* | *DV* | *BB* | *CL4* |
| 100 | 0.05 | 0.550740 | 33.94 | −41.23 | −29.56 | −56.64 |
| 100 | 0.10 | 0.343950 | 50.63 | −38.96 | −35.86 | −1.75 |
| 100 | 0.15 | 0.235730 | 52.89 | −28.46 | −30.63 | 68.52 |
| 100 | 0.20 | 0.173090 | 45.18 | −16.18 | −22.17 | 148.01 |
| 100 | 0.25 | 0.133840 | 31.59 | −4.11 | −13.13 | 233.58 |
| 100 | 0.30 | 0.107650 | 15.12 | 7.06 | −4.47 | 323.10 |
| 100 | 1.00 | 0.025350 | −134.15 | 83.46 | 56.72 | 1613.23 |
| 1000 | 0.05 | 0.041990 | −99.09 | 110.18 | 83.28 | 362.19 |
| 1000 | 0.10 | 0.010990 | −169.86 | 171.07 | 171.59 | 1022.26 |
| 1000 | 0.15 | 0.005740 | −173.25 | 159.33 | 190.26 | 1432.64 |
| 1000 | 0.20 | 0.003840 | −170.97 | 137.40 | 189.13 | 1655.22 |
| 1000 | 0.25 | 0.002880 | −169.26 | 118.33 | 183.96 | 1774.20 |
| 1000 | 0.30 | 0.002300 | −168.04 | 103.64 | 179.03 | 1836.77 |

Note: *MY* = approximations based on the algorithm of Section 6
*DV* = de Vylder's approximation, Equation (6)
*BB* = Beekman-Bowers approximation, Equation (5)
*CL4* = Cramér-Lundberg approximation with $M(r)$ approximated using only the first four moments, that is, as a fourth-degree polynomial in $r$.

TABLE 8

COMPARISON OF RELATIVE ERRORS PRODUCED BY RAMSAY'S ALGORITHM WITH THOSE PRODUCED BY RECKIN, SCHWARK AND SNYDER [27] FOR INFINITE RETENTION PER CLAIM

| | | | $200\epsilon$ | | | |
|---|---|---|---|---|---|---|
| $\theta$ | $u$ | $\Psi(u)$ | *MY* | *DV* | *BB* | *CL4* |
| 0.100 | 0 | 0.90909091 | −0.00 | −29.76 | −0.00 | −29.31 |
| 0.100 | 10 | 0.62660774 | −2.61 | −3.34 | 2.31 | −2.55 |
| 0.100 | 20 | 0.47721561 | −1.36 | 5.61 | 2.88 | 6.72 |
| 0.100 | 30 | 0.37251562 | 1.65 | 9.72 | 2.45 | 11.15 |
| 0.100 | 40 | 0.29589384 | 2.89 | 10.23 | 0.60 | 11.95 |
| 0.100 | 50 | 0.23717805 | 2.71 | 8.83 | −1.74 | 10.84 |
| 0.100 | 100 | 0.08003352 | −1.46 | −1.95 | −7.01 | 1.35 |
| 0.200 | 0 | 0.83333333 | 0.00 | −48.53 | 0.00 | −39.55 |
| 0.200 | 10 | 0.43160197 | −5.67 | 1.67 | 0.71 | 13.21 |
| 0.200 | 20 | 0.27336595 | −0.67 | 19.56 | 6.14 | 31.67 |
| 0.200 | 30 | 0.18372007 | 6.45 | 25.28 | 7.57 | 37.23 |
| 0.200 | 40 | 0.12908357 | 7.65 | 21.10 | 3.95 | 32.37 |
| 0.200 | 50 | 0.09267680 | 5.04 | 12.36 | −1.46 | 22.74 |
| 0.200 | 100 | 0.01731687 | −2.72 | −22.80 | −5.55 | −15.95 |
| 0.300 | 0 | 0.76923077 | 0.00 | −61.46 | 0.00 | −41.54 |
| 0.300 | 10 | 0.31810314 | −8.47 | 9.23 | 2.08 | 36.14 |
| 0.300 | 20 | 0.17737952 | 1.75 | 34.35 | 13.09 | 60.98 |
| 0.300 | 30 | 0.10907122 | 12.33 | 38.03 | 14.98 | 61.56 |
| 0.300 | 40 | 0.07215670 | 11.38 | 24.72 | 7.00 | 43.66 |
| 0.300 | 50 | 0.04921654 | 4.71 | 5.77 | −3.53 | 20.15 |
| 0.300 | 100 | 0.00627498 | 2.48 | −46.62 | −2.39 | −46.50 |
| 0.400 | 0 | 0.71428571 | −0.00 | −70.91 | −0.00 | −39.88 |
| 0.400 | 10 | 0.24645221 | −10.79 | 17.37 | 5.08 | 61.49 |
| 0.400 | 20 | 0.12558042 | 5.18 | 47.84 | 20.90 | 89.16 |
| 0.400 | 30 | 0.07328565 | 17.92 | 46.73 | 21.41 | 79.19 |
| 0.400 | 40 | 0.04717622 | 13.39 | 22.68 | 7.52 | 44.38 |
| 0.400 | 50 | 0.03141052 | 2.49 | −5.69 | −8.36 | 6.81 |
| 0.400 | 100 | 0.00299275 | 14.32 | −65.01 | 3.14 | −76.72 |
| 0.500 | 0 | 0.66666667 | −0.00 | −78.11 | −0.00 | −36.31 |
| 0.500 | 10 | 0.19829729 | −12.63 | 25.30 | 8.78 | 87.36 |
| 0.500 | 20 | 0.09465148 | 9.02 | 59.51 | 28.37 | 114.34 |
| 0.500 | 30 | 0.05357595 | 22.67 | 52.07 | 26.19 | 89.97 |
| 0.500 | 40 | 0.03411333 | 14.05 | 17.66 | 6.13 | 37.79 |
| 0.500 | 50 | 0.02243742 | −0.49 | −18.06 | −14.31 | −11.23 |
| 0.500 | 100 | 0.00168102 | 30.87 | −78.00 | 10.89 | −102.21 |

Note: *MY* = approximations based on the algorithm of Section 6
*DV* = de Vylder's approximation, Equation (6)
*BB* = Beekman-Bowers approximation, Equation (5)
*CL4* = Cramér-Lundberg approximation with $M(r)$ approximated using only the first four moments, that is, as a fourth-degree polynomial in $r$.

TABLE 9

COMPARISON OF RELATIVE ERRORS PRODUCED BY RAMSAY'S ALGORITHM WITH THOSE PRODUCED BY RECKIN, SCHWARK AND SNYDER [27] FOR $200,000 RETENTION LIMIT PER CLAIM

| | | | 200ε | | | |
|---|---|---|---|---|---|---|
| θ | u | Ψ(u) | MY | DV | BB | CL4 |
| 0.100 | 0 | 0.90909091 | −0.00 | −7.19 | −0.00 | −6.26 |
| 0.100 | 10 | 0.59867598 | 0.11 | 0.56 | 0.57 | 1.41 |
| 0.100 | 20 | 0.41144235 | −0.43 | −0.09 | −2.03 | 0.62 |
| 0.100 | 30 | 0.28184123 | −0.33 | −0.08 | −2.67 | 0.50 |
| 0.100 | 40 | 0.19304759 | −0.20 | −0.06 | −2.60 | 0.39 |
| 0.100 | 50 | 0.13223135 | −0.08 | −0.04 | −2.10 | 0.28 |
| 0.100 | 100 | 0.01993787 | 0.53 | 0.05 | 3.78 | −0.27 |
| 0.200 | 0 | 0.83333333 | 0.00 | −12.80 | 0.00 | −8.28 |
| 0.200 | 10 | 0.39592748 | −0.29 | 0.94 | −2.03 | 4.59 |
| 0.200 | 20 | 0.20325157 | −1.01 | −0.38 | −4.97 | 2.06 |
| 0.200 | 30 | 0.10360884 | −0.33 | −0.28 | −3.49 | 0.98 |
| 0.200 | 40 | 0.05278711 | 0.47 | −0.08 | −0.37 | 0.00 |
| 0.200 | 50 | 0.02689796 | 1.23 | 0.09 | 3.74 | −0.99 |
| 0.200 | 100 | 0.00092393 | 5.14 | 0.97 | 33.63 | −5.89 |
| 0.300 | 0 | 0.76923077 | 0.00 | −17.30 | 0.00 | −7.25 |
| 0.300 | 10 | 0.28141936 | −0.91 | 1.03 | −4.45 | 8.24 |
| 0.300 | 20 | 0.11426119 | −1.13 | −0.69 | −5.78 | 2.71 |
| 0.300 | 30 | 0.04591584 | 0.70 | −0.34 | −0.51 | −0.63 |
| 0.300 | 40 | 0.01842072 | 2.88 | 0.34 | 7.59 | −3.58 |
| 0.300 | 50 | 0.00739335 | 5.00 | 0.93 | 17.42 | −6.58 |
| 0.300 | 100 | 0.00007698 | 16.00 | 3.98 | 88.00 | −20.84 |
| 0.400 | 0 | 0.71428571 | −0.00 | −20.99 | −0.00 | −4.25 |
| 0.400 | 10 | 0.21109349 | −1.60 | 0.85 | −6.41 | 11.42 |
| 0.400 | 20 | 0.07059032 | −0.61 | −0.83 | −4.91 | 1.80 |
| 0.400 | 30 | 0.02331375 | 2.89 | −0.04 | 5.11 | −4.97 |
| 0.400 | 40 | 0.00767145 | 7.22 | 1.51 | 19.59 | −10.82 |
| 0.400 | 50 | 0.00252685 | 11.43 | 2.86 | 36.88 | −16.68 |
| 0.400 | 100 | 0.00000979 | 33.95 | 9.88 | 168.29 | −43.26 |
| 0.500 | 0 | 0.66666667 | −0.00 | −24.07 | −0.00 | 0.03 |
| 0.500 | 10 | 0.16501146 | −2.31 | 0.48 | −7.96 | 13.75 |
| 0.500 | 20 | 0.04681457 | 0.58 | −0.70 | −2.81 | −0.73 |
| 0.500 | 30 | 0.01310942 | 6.20 | 0.74 | 12.56 | −11.79 |
| 0.500 | 40 | 0.00364629 | 13.40 | 3.55 | 34.62 | −21.03 |
| 0.500 | 50 | 0.00101612 | 20.44 | 6.02 | 61.09 | −30.14 |
| 0.500 | 100 | 0.00000171 | 59.59 | 19.05 | 279.20 | −69.03 |

Note: *MY* = approximations based on the algorithm of Section 6
*DV* = de Vylder's approximation, Equation (6)
*BB* = Beekman-Bowers approximation, Equation (5)
*CL4* = Cramér-Lundberg approximation with $M(r)$ approximated using only the first four moments, that is, as a fourth-degree polynomial in $r$.

TABLE 10

COMPARISON OF RELATIVE ERRORS PRODUCED BY RAMSAY'S ALGORITHM WITH THOSE PRODUCED BY RECKIN, SCHWARK AND SNYDER [27] FOR $100,000 RETENTION LIMIT PER CLAIM

| | | | 200ϵ | | | |
|---|---|---|---|---|---|---|
| $\theta$ | $u$ | $\Psi(u)$ | *MY* | *DV* | *BB* | *CL4* |
| 0.100 | 0 | 0.90909091 | −0.00 | −3.08 | −0.00 | −2.15 |
| 0.100 | 10 | 0.55733138 | −0.51 | −0.28 | −0.57 | 0.48 |
| 0.100 | 20 | 0.34655047 | −0.44 | −0.28 | −1.31 | 0.28 |
| 0.100 | 30 | 0.21534573 | −0.25 | −0.16 | −1.28 | 0.21 |
| 0.100 | 40 | 0.13381172 | −0.05 | −0.03 | −0.91 | 0.15 |
| 0.100 | 50 | 0.08314805 | 0.15 | 0.10 | −0.33 | 0.09 |
| 0.100 | 100 | 0.00770265 | 1.15 | 0.76 | 4.21 | −0.21 |
| 0.200 | 0 | 0.83333333 | 0.00 | −5.58 | 0.00 | −1.62 |
| 0.200 | 10 | 0.34417616 | −1.42 | −0.82 | −2.53 | 1.78 |
| 0.200 | 20 | 0.14531790 | −0.57 | −0.39 | −2.19 | 0.76 |
| 0.200 | 30 | 0.06124251 | 0.66 | 0.41 | 0.06 | 0.12 |
| 0.200 | 40 | 0.02580755 | 1.91 | 1.24 | 3.12 | −0.50 |
| 0.200 | 50 | 0.01087526 | 3.17 | 2.06 | 6.71 | −1.12 |
| 0.200 | 100 | 0.00014449 | 9.61 | 6.28 | 29.51 | −4.16 |
| 0.300 | 0 | 0.76923077 | 0.00 | −7.64 | 0.00 | 0.97 |
| 0.300 | 10 | 0.22908635 | −2.07 | −1.18 | −3.93 | 3.30 |
| 0.300 | 20 | 0.06999375 | 0.53 | 0.31 | −0.83 | 0.46 |
| 0.300 | 30 | 0.02130859 | 3.90 | 2.54 | 5.49 | −1.62 |
| 0.300 | 40 | 0.00648575 | 7.36 | 4.83 | 13.27 | −3.64 |
| 0.300 | 50 | 0.00197407 | 10.89 | 7.15 | 22.10 | −5.64 |
| 0.300 | 100 | 0.00000515 | 29.59 | 19.30 | 78.98 | −15.22 |
| 0.400 | 0 | 0.71428571 | −0.00 | −9.37 | −0.00 | 4.97 |
| 0.400 | 10 | 0.16135606 | −2.22 | −1.19 | −4.52 | 4.54 |
| 0.400 | 20 | 0.03741618 | 3.08 | 1.99 | 2.60 | −1.16 |
| 0.400 | 30 | 0.00862561 | 9.75 | 6.43 | 14.54 | −5.56 |
| 0.400 | 40 | 0.00198773 | 16.72 | 11.05 | 28.94 | −9.80 |
| 0.400 | 50 | 0.00045805 | 23.93 | 15.78 | 45.39 | −13.94 |
| 0.400 | 100 | 0.00000030 | 64.03 | 41.32 | 158.60 | −33.13 |
| 0.500 | 0 | 0.66666667 | −0.00 | −10.85 | −0.00 | 9.96 |
| 0.500 | 10 | 0.11874899 | −1.81 | −0.80 | −4.34 | 5.27 |
| 0.500 | 20 | 0.02168046 | 7.07 | 4.68 | 7.77 | −4.20 |
| 0.500 | 30 | 0.00392458 | 18.20 | 12.11 | 26.75 | −11.63 |
| 0.500 | 40 | 0.00071000 | 30.08 | 19.94 | 49.80 | −18.67 |
| 0.500 | 50 | 0.00012844 | 42.61 | 28.07 | 76.63 | −25.44 |
| 0.500 | 100 | 0.00000002 | 117.35 | 74.40 | 281.47 | −55.19 |

Note: *MY* = approximations based on the algorithm of Section 6
*DV* = de Vylder's approximation, Equation (6)
*BB* = Beekman-Bowers approximation, Equation (5)
*CL4* = Cramér-Lundberg approximation with $M(r)$ approximated using only the first four moments, that is, as a fourth-degree polynomial in $r$.

TABLE 11

COMPARISON OF RELATIVE ERRORS PRODUCED BY RAMSAY'S ALGORITHM WITH THOSE PRODUCED BY RECKIN, SCHWARK AND SNYDER [27] FOR $50,000 RETENTION LIMIT PER CLAIM

| | | | 200ε | | | |
|---|---|---|---|---|---|---|
| θ | u | Ψ(u) | MY | DV | BB | CL4 |
| 0.100 | 0 | 0.90909091 | −0.00 | −0.31 | −0.00 | 0.50 |
| 0.100 | 10 | 0.46897113 | −0.45 | −0.42 | −0.49 | 0.15 |
| 0.100 | 20 | 0.24162044 | −0.29 | −0.28 | −0.39 | 0.05 |
| 0.100 | 30 | 0.12448098 | −0.13 | −0.12 | −0.21 | −0.03 |
| 0.100 | 40 | 0.06413175 | 0.04 | 0.03 | 0.02 | −0.12 |
| 0.100 | 50 | 0.03304020 | 0.21 | 0.18 | 0.26 | −0.21 |
| 0.100 | 100 | 0.00119883 | 1.10 | 0.99 | 1.73 | −0.58 |
| 0.200 | 0 | 0.83333333 | 0.00 | −0.56 | 0.00 | 2.77 |
| 0.200 | 10 | 0.24817311 | −0.97 | −0.89 | −1.10 | 0.68 |
| 0.200 | 20 | 0.07338009 | 0.23 | 0.21 | 0.14 | 0.04 |
| 0.200 | 30 | 0.02169332 | 1.48 | 1.35 | 1.60 | −0.57 |
| 0.200 | 40 | 0.00641294 | 2.74 | 2.50 | 3.17 | −1.17 |
| 0.200 | 50 | 0.00189562 | 4.02 | 3.68 | 4.83 | −1.75 |
| 0.200 | 100 | 0.00000426 | 11.63 | 10.73 | 14.79 | −3.64 |
| 0.300 | 0 | 0.76923077 | 0.00 | −0.77 | 0.00 | 6.50 |
| 0.300 | 10 | 0.14377452 | −0.98 | −0.88 | −1.14 | 1.10 |
| 0.300 | 20 | 0.02643832 | 2.47 | 2.27 | 2.51 | −0.95 |
| 0.300 | 30 | 0.00485951 | 6.08 | 5.57 | 6.61 | −2.90 |
| 0.300 | 40 | 0.00089305 | 9.78 | 8.96 | 10.97 | −4.79 |
| 0.300 | 50 | 0.00016407 | 13.62 | 12.48 | 15.58 | −6.59 |
| 0.300 | 100 | 0.00000003 | 39.73 | 36.71 | 48.68 | −10.79 |
| 0.400 | 0 | 0.71428571 | −0.00 | −0.96 | −0.00 | 11.30 |
| 0.400 | 10 | 0.08932177 | −0.26 | −0.20 | −0.43 | 1.01 |
| 0.400 | 20 | 0.01086549 | 6.72 | 6.18 | 6.91 | −3.43 |
| 0.400 | 30 | 0.00132058 | 14.12 | 12.94 | 15.12 | −7.60 |
| 0.400 | 40 | 0.00016044 | 21.88 | 20.02 | 23.92 | −11.61 |
| 0.400 | 50 | 0.00001948 | 30.10 | 27.50 | 33.39 | −15.39 |
| 0.400 | 100 | 0.00000000 | 83.46 | 76.19 | 210.31 | −28.68 |
| 0.500 | 0 | 0.66666667 | −0.00 | −1.11 | −0.00 | 16.86 |
| 0.500 | 10 | 0.05864604 | 1.23 | 1.20 | 1.06 | 0.23 |
| 0.500 | 20 | 0.00495464 | 13.00 | 11.95 | 13.34 | −7.49 |
| 0.500 | 30 | 0.00041798 | 25.79 | 23.58 | 27.20 | −14.65 |
| 0.500 | 40 | 0.00003524 | 39.52 | 36.02 | 42.38 | −21.42 |
| 0.500 | 50 | 0.00000297 | 54.43 | 49.50 | 59.11 | −27.70 |
| 0.500 | 100 | 0.00000000 | 0.00 | 0.00 | 0.00 | 0.00 |

Note: *MY* = approximations based on the algorithm of Section 6
*DV* = de Vylder's approximation, Equation (6)
*BB* = Beekman-Bowers approximation, Equation (5)
*CL4* = Cramér-Lundberg approximation with $M(r)$ approximated using only the first four moments, that is, as a fourth-degree polynomial in $r$.

TABLE 12

COMPARISON OF RELATIVE ERRORS PRODUCED BY RAMSAY'S ALGORITHM WITH THOSE PRODUCED BY RECKIN, SCHWARK AND SNYDER [27] FOR $50,000 RETENTION LIMIT PER CLAIM

| | | | Actual Values for Each Approximation | | | |
|---|---|---|---|---|---|---|
| $\theta$ | $u$ | $\Psi(u)$ | *MY* | *DV* | *BB* | *CL4* |
| 0.100 | 0 | 0.90909091 | 0.90909091 | 0.90770319 | 0.90909091 | 0.91135471 |
| 0.100 | 10 | 0.46897113 | 0.46791436 | 0.46799353 | 0.46783175 | 0.46931840 |
| 0.100 | 20 | 0.24162044 | 0.24126629 | 0.24128806 | 0.24115085 | 0.24168390 |
| 0.100 | 30 | 0.12448098 | 0.12440192 | 0.12440328 | 0.12435264 | 0.12445945 |
| 0.100 | 40 | 0.06413175 | 0.06414422 | 0.06413983 | 0.06413658 | 0.06409262 |
| 0.100 | 50 | 0.03304020 | 0.03307409 | 0.03306921 | 0.03308328 | 0.03300564 |
| 0.100 | 100 | 0.00119883 | 0.00120542 | 0.00120477 | 0.00120923 | 0.00119533 |
| 0.200 | 0 | 0.83333333 | 0.83333333 | 0.83100416 | 0.83333333 | 0.84488916 |
| 0.200 | 10 | 0.24817311 | 0.24696830 | 0.24706769 | 0.24681150 | 0.24901594 |
| 0.200 | 20 | 0.07338009 | 0.07346483 | 0.07345624 | 0.07343105 | 0.07339299 |
| 0.200 | 30 | 0.02169332 | 0.02185337 | 0.02183944 | 0.02186709 | 0.02163127 |
| 0.200 | 40 | 0.00641294 | 0.00650066 | 0.00649313 | 0.00651468 | 0.00637543 |
| 0.200 | 50 | 0.00189562 | 0.00193373 | 0.00193049 | 0.00194137 | 0.00187904 |
| 0.200 | 100 | 0.00000426 | 0.00000450 | 0.00000448 | 0.00000457 | 0.00000418 |
| 0.300 | 0 | 0.76923077 | 0.76923077 | 0.76625704 | 0.76923077 | 0.79424839 |
| 0.300 | 10 | 0.14377452 | 0.14307183 | 0.14313967 | 0.14295360 | 0.14456295 |
| 0.300 | 20 | 0.02643832 | 0.02676537 | 0.02673902 | 0.02677007 | 0.02631223 |
| 0.300 | 30 | 0.00485951 | 0.00500719 | 0.00499495 | 0.00502016 | 0.00478915 |
| 0.300 | 40 | 0.00089305 | 0.00093673 | 0.00093308 | 0.00094205 | 0.00087168 |
| 0.300 | 50 | 0.00016407 | 0.00017524 | 0.00017430 | 0.00017685 | 0.00015866 |
| 0.300 | 100 | 0.00000003 | 0.00000004 | 0.00000004 | 0.00000004 | 0.00000003 |
| 0.400 | 0 | 0.71428571 | 0.71428571 | 0.71087008 | 0.71428571 | 0.75464039 |
| 0.400 | 10 | 0.08932177 | 0.08920615 | 0.08923304 | 0.08913008 | 0.08977191 |
| 0.400 | 20 | 0.01086549 | 0.01123041 | 0.01120111 | 0.01124113 | 0.01067925 |
| 0.400 | 30 | 0.00132058 | 0.00141384 | 0.00140604 | 0.00142038 | 0.00127040 |
| 0.400 | 40 | 0.00016044 | 0.00017799 | 0.00017649 | 0.00017963 | 0.00015113 |
| 0.400 | 50 | 0.00001948 | 0.00002241 | 0.00002215 | 0.00002273 | 0.00001798 |
| 0.400 | 100 | 0.00000000 | 0.00000000 | 0.00000000 | 0.00000000 | 0.00000000 |
| 0.500 | 0 | 0.66666667 | 0.66666667 | 0.66295038 | 0.66666667 | 0.72285973 |
| 0.500 | 10 | 0.05864604 | 0.05900650 | 0.05899879 | 0.05895757 | 0.05871434 |
| 0.500 | 20 | 0.00495464 | 0.00527674 | 0.00525055 | 0.00528501 | 0.00476908 |
| 0.500 | 30 | 0.00041798 | 0.00047189 | 0.00046727 | 0.00047483 | 0.00038737 |
| 0.500 | 40 | 0.00003524 | 0.00004220 | 0.00004158 | 0.00004270 | 0.00003146 |
| 0.500 | 50 | 0.00000297 | 0.00000377 | 0.00000370 | 0.00000384 | 0.00000256 |
| 0.500 | 100 | 0.00000000 | 0.00000000 | 0.00000000 | 0.00000000 | 0.00000000 |

Note: *MY* = approximations based on the algorithm of Section 6
*DV* = de Vylder's approximation, Equation (6)
*BB* = Beekman-Bowers approximation, Equation (5)
*CL4* = Cramér-Lundberg approximation with $M(r)$ approximated using only the first four moments, that is, as a fourth-degree polynomial in $r$.

TABLE 13

COMPARISON OF RELATIVE ERRORS PRODUCED BY RAMSAY'S ALGORITHM WITH THOSE PRODUCED BY RECKIN, SCHWARK AND SNYDER [27] FOR $25,000 RETENTION LIMIT PER CLAIM

| | | | 200¢ | | | |
|---|---|---|---|---|---|---|
| $\theta$ | $u$ | $\Psi(u)$ | *MY* | *DV* | *BB* | *CL4* |
| 0.100 | 0 | 0.90909091 | −0.00 | 1.61 | −0.00 | 2.28 |
| 0.100 | 10 | 0.35659709 | 0.01 | −0.07 | 0.47 | 0.31 |
| 0.100 | 20 | 0.13844221 | 0.33 | 0.32 | 0.79 | 0.42 |
| 0.100 | 30 | 0.05374837 | 0.65 | 0.71 | 0.63 | 0.52 |
| 0.100 | 40 | 0.02086566 | 0.98 | 1.11 | 0.24 | 0.64 |
| 0.100 | 50 | 0.00809872 | 1.34 | 1.55 | −0.25 | 0.79 |
| 0.100 | 100 | 0.00007061 | 5.30 | 5.88 | −1.67 | 3.65 |
| 0.200 | 0 | 0.83333333 | 0.00 | 2.97 | 0.00 | 5.77 |
| 0.200 | 10 | 0.14800809 | 0.13 | 0.04 | 1.01 | 0.68 |
| 0.200 | 20 | 0.02567839 | 1.44 | 1.82 | 0.69 | 0.36 |
| 0.200 | 30 | 0.00445287 | 2.85 | 3.72 | −0.66 | 0.13 |
| 0.200 | 40 | 0.00077113 | 4.55 | 5.91 | −2.25 | 0.17 |
| 0.200 | 50 | 0.00013318 | 6.82 | 8.69 | −3.60 | 0.76 |
| 0.200 | 100 | 0.00000002 | 67.83 | 73.46 | 34.07 | 49.63 |
| 0.300 | 0 | 0.76923077 | 0.00 | 4.14 | 0.00 | 10.34 |
| 0.300 | 10 | 0.06900137 | 0.81 | 0.97 | 1.48 | 0.63 |
| 0.300 | 20 | 0.00595419 | 4.15 | 5.67 | 0.56 | −1.07 |
| 0.300 | 30 | 0.00051277 | 7.95 | 10.90 | −1.69 | −2.36 |
| 0.300 | 40 | 0.00004393 | 12.93 | 17.39 | −3.67 | −2.63 |
| 0.300 | 50 | 0.00000372 | 20.50 | 26.62 | −3.85 | −0.65 |
| 0.300 | 100 | 0.00000000 | −242.63 | −245.30 | −866.03 | −234.02 |
| 0.400 | 0 | 0.71428571 | −0.00 | 5.15 | −0.00 | 15.72 |
| 0.400 | 10 | 0.03524392 | 2.23 | 2.98 | 2.25 | −0.23 |
| 0.400 | 20 | 0.00164481 | 8.75 | 12.33 | 1.40 | −4.41 |
| 0.400 | 30 | 0.00007641 | 16.50 | 23.15 | −0.81 | −7.61 |
| 0.400 | 40 | 0.00000350 | 27.40 | 37.52 | −1.47 | −8.33 |
| 0.400 | 50 | 0.00000016 | 46.26 | 60.65 | 3.73 | −3.14 |
| 0.400 | 100 | 0.00000000 | 0.00 | 0.00 | 0.00 | 0.00 |
| 0.500 | 0 | 0.66666667 | −0.00 | 6.04 | −0.00 | 21.72 |
| 0.500 | 10 | 0.01937137 | 4.42 | 6.12 | 3.56 | 2.05 |
| 0.500 | 20 | 0.00052288 | 15.34 | 21.98 | 3.72 | −9.76 |
| 0.500 | 30 | 0.00001399 | 28.78 | 41.10 | 2.79 | −15.59 |
| 0.500 | 40 | 0.00000037 | 49.14 | 68.40 | 5.76 | −16.79 |
| 0.500 | 50 | 0.00000001 | 87.61 | 116.77 | 26.09 | −7.04 |
| 0.500 | 100 | 0.00000000 | 0.00 | 0.00 | 0.00 | 0.00 |

Note: *MY* = approximations based on the algorithm of Section 6
*DV* = de Vylder's approximation, Equation (6)
*BB* = Beekman-Bowers approximation, Equation (5)
*CL4* = Cramér-Lundberg approximation with $M(r)$ approximated using only the first four moments, that is, as a fourth-degree polynomial in $r$.

TABLE 14

COMPARISON OF RELATIVE ERRORS PRODUCED BY RAMSAY'S ALGORITHM WITH THOSE PRODUCED BY RECKIN, SCHWARK AND SNYDER [27] FOR $25,000 RETENTION LIMIT PER CLAIM

| | | | Actual Values for Each Approximation | | | |
|---|---|---|---|---|---|---|
| $\theta$ | $u$ | $\Psi(u)$ | *MY* | *DV* | *BB* | *CL4* |
| 0.100 | 0 | 0.90909091 | 0.90909091 | 0.91641348 | 0.90909091 | 0.91946518 |
| 0.100 | 10 | 0.35659709 | 0.35662350 | 0.35647346 | 0.35743897 | 0.35715265 |
| 0.100 | 20 | 0.13844221 | 0.13867216 | 0.13866375 | 0.13899180 | 0.13873066 |
| 0.100 | 30 | 0.05374837 | 0.05392232 | 0.05393848 | 0.05391760 | 0.05388787 |
| 0.100 | 40 | 0.02086566 | 0.02096756 | 0.02098140 | 0.02089093 | 0.02093195 |
| 0.100 | 50 | 0.00809872 | 0.00815318 | 0.00816150 | 0.00808861 | 0.00813071 |
| 0.100 | 100 | 0.00007061 | 0.00007248 | 0.00007269 | 0.00007002 | 0.00007190 |
| 0.200 | 0 | 0.83333333 | 0.83333333 | 0.84572248 | 0.83333333 | 0.85739352 |
| 0.200 | 10 | 0.14800809 | 0.14810453 | 0.14803453 | 0.14875533 | 0.14851176 |
| 0.200 | 20 | 0.02567839 | 0.02586288 | 0.02591184 | 0.02576675 | 0.02572418 |
| 0.200 | 30 | 0.00445287 | 0.00451633 | 0.00453559 | 0.00443820 | 0.00445576 |
| 0.200 | 40 | 0.00077113 | 0.00078867 | 0.00079391 | 0.00076245 | 0.00077180 |
| 0.200 | 50 | 0.00013318 | 0.00013772 | 0.00013896 | 0.00013078 | 0.00013369 |
| 0.200 | 100 | 0.00000002 | 0.00000002 | 0.00000002 | 0.00000002 | 0.00000002 |
| 0.300 | 0 | 0.76923077 | 0.76923077 | 0.78515648 | 0.76923077 | 0.80898485 |
| 0.300 | 10 | 0.06900137 | 0.06928074 | 0.06933663 | 0.06951130 | 0.06921723 |
| 0.300 | 20 | 0.00595419 | 0.00607764 | 0.00612307 | 0.00597081 | 0.00592227 |
| 0.300 | 30 | 0.00051277 | 0.00053316 | 0.00054072 | 0.00050845 | 0.00050671 |
| 0.300 | 40 | 0.00004393 | 0.00004677 | 0.00004775 | 0.00004313 | 0.00004335 |
| 0.300 | 50 | 0.00000372 | 0.00000410 | 0.00000422 | 0.00000365 | 0.00000371 |
| 0.300 | 100 | 0.00000000 | 0.00000000 | 0.00000000 | 0.00000000 | 0.00000000 |
| 0.400 | 0 | 0.71428571 | 0.71428571 | 0.73268557 | 0.71428571 | 0.77043009 |
| 0.400 | 10 | 0.03524392 | 0.03563602 | 0.03576917 | 0.03564011 | 0.03520325 |
| 0.400 | 20 | 0.00164481 | 0.00171680 | 0.00174622 | 0.00165629 | 0.00160854 |
| 0.400 | 30 | 0.00007641 | 0.00008271 | 0.00008525 | 0.00007610 | 0.00007350 |
| 0.400 | 40 | 0.00000350 | 0.00000398 | 0.00000416 | 0.00000348 | 0.00000336 |
| 0.400 | 50 | 0.00000016 | 0.00000019 | 0.00000020 | 0.00000016 | 0.00000015 |
| 0.400 | 100 | 0.00000000 | 0.00000000 | 0.00000000 | 0.00000000 | 0.00000000 |
| 0.500 | 0 | 0.66666667 | 0.66666667 | 0.68678845 | 0.66666667 | 0.73905973 |
| 0.500 | 10 | 0.01937137 | 0.01979900 | 0.01996407 | 0.01971591 | 0.01917258 |
| 0.500 | 20 | 0.00052288 | 0.00056297 | 0.00058033 | 0.00053261 | 0.00049737 |
| 0.500 | 30 | 0.00001399 | 0.00001601 | 0.00001687 | 0.00001419 | 0.00001290 |
| 0.500 | 40 | 0.00000037 | 0.00000046 | 0.00000049 | 0.00000038 | 0.00000033 |
| 0.500 | 50 | 0.00000001 | 0.00000001 | 0.00000001 | 0.00000001 | 0.00000001 |
| 0.500 | 100 | 0.00000000 | 0.00000000 | 0.00000000 | 0.00000000 | 0.00000000 |

Note: *MY* = approximations based on the algorithm of Section 6
*DV* = de Vylder's approximation, Equation (6)
*BB* = Beekman-Bowers approximation, Equation (5)
*CL4* = Cramér-Lundberg approximation with $M(r)$ approximated using only the first four moments, that is, as a fourth-degree polynomial in $r$.