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Population Waves and Fertility Fluctuations: Social Security Implications

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1. Introduction

Long range population projections are extremely sensitive to the underlying fertility assumptions. Experience has shown that fertility rates are particularly difficult to forecast. Human fertility responds in a dynamic if complex manner to changes in the environment as historical studies have shown. Different models have been proposed to account for these fertility fluctuations. The purpose of the present paper is to discuss some of the methods by which fertility variations can be incorporated into population forecasts.[1] These fertility variations can in some situations lead to population waves which have important consequences for the financing of pay as you go social security systems. This will be illustrated in the context of a specific example based on Canadian data.

Malthus argued that there are positive checks on population growth through environmental constraints. Subsequent models have assumed that a human population will arrive at a long-run equilibrium growth rate e.g., Liebenstein (1963), Solow (1956), Lee (1973). Lee (1974) was one of the first authors to analyze the impact of incorporating an age structure into the model when the population is subject to environmental constraints. This gave rise to a response lag in the adjustment process and thus to control cycles or limit cycles as the population exhibited long-run regular oscillations about its equilibrium growth path. Additional analysis on the dynamics of this type of limit cycle behaviour has been carried out by Frauenthal (1975), Swick (1981), Smith (1981(a)) and Frauenthal and Swick (1983).

Socio-economic arguments for the existence and persistence of long-run natural cycles in human fertility have been given by Easterlin (1961, 1968, 1978). Easterlin's original observation was that women born in large cohorts tend to produce less children than women born in small cohorts. Since then Easterlin's analysis has been refined and tested empirically particularly in the context of the recent fertility rates in the United States. Although the evidence on the existence of Easterlin cycles is conflicting, (cf. Smith (1981(b)) this idea provides the inspiration for some of the recent demographic models which lead to limit cycle oscillations.

Most traditional demographic projections assume either that fertility rates will remain constant or tend towards some ultimate level. This is often the case when demographic forecasts are made in connection with national social security plans. In a continuous time framework this leads to the Lotka model where the population grows at an exponential rate corresponding to the dominant real root of the characteristic equation. Keyfitz (1977) gives a good discussion of this type of population dynamics known as stable population theory. The dominant complex root gives rise to periodic fluctuations of one generation in length which become progressively dampened as the population evolves under fixed rates of mortality and fertility. (For the purposes of this paper emigration and immigration are assumed to be zero). These one-generational cycles in population dynamics are often known as the echo effect and are captured within the framework of conventional population projection techniques. From the perspective of national social security projections, conventional procedures pick up these effects. Coale (1972) gives a detailed analysis of the evolution of a population towards its stable state under a fixed regime of mortality and fertility.

In addition to the Easterlin effect and the echo effect there is another method of generating population waves. Such waves will occur if there are regular fluctuations in the birth rate due to periodic patterns in climate, harvests, epidemics or economic cycles. Studies in historical demography have shown evidence of such cycles as for example Goubert's (1965) analysis of baptismal records in pre-industrial France. More recently Wrigley and Schofield (1981) have found evidence of fertility oscillations in England which are related to those in real-wage fluctuations but with a time-lag of 40 years.[2] Periodic fluctuations in the fertility rate can be incorporated directly into population projections by assuming a secular periodic pattern in fertility rates. A mathematical analysis of some of the consequences of this type of assumption has been given by Coale (1972).

In a practical setting it may well be that the influence of both of the major cycle generating mechanisms is present and it may be very difficult to separate out the relative contribution of each. Smith (1981(b)) has attempted to decompose recent United States fertility patterns into period and cohort effects and finds that period effects seem to be more significant. Congden (1980) incorporates both period and cohort effects into a forecast of the birth rate in the Greater London area until the year 1991. Congden's analysis uses a variant of the Easterlin hypothesis to incorporate cohort effects. The inclusion of either or both these effects in long-run population projections can radically influence the relative age-compositon of a population. The resulting population dynamics can differ significantly from those arising from stable population theory. This in turn can have important consequences for the long-run financing and stability of a pay as you go national social security system since the ratio of retired individuals to those of working age evolves in a very different fashion under both assumptions. One of the chief aims of this paper is to illustrate some of the major changes which occur when fertility fluctuations of the types just discussed are incorporated in long-run social security projections.

The layout of the paper is as follows. In the next section there is a brief account of a model developed by Lee (1974) which shows how limit cycles can arise in a population with an environmental constraint. It will be shown that the population evolution can be summarized by a non-linear difference equation which has quite interesting dynamical behaviour. In particular, control cycles of different lengths can occur depending on the magnitude of the adjustment parameters. For some parameter values the period of the control cycle corresponds to two generations as suggested by the Easterlin hypothesis.

In Section 3 we examine Frauenthal and Swicks' (1983) generalization of the classic Lotka renewal equation in mathematical demography to incorporate an Easterlin effect. In this model cohort fertility characteristics are determined by the size of the cohort at birth relative to the corresponding equilibrium cohort size at birth. For some plausible parameter values this model gives rise to limit cycles of two generations in length corresponding to an Easterlin effect.

A brief analysis of the incorporation of secular periodic fluctuations in the fertility rates is given in Section 4.

Section 5 incorporates the various types of fertility fluctuations into the projection of an actual national population and analyzes their impact. From the viewpoint of a social security system the evolution of the dependency ratio over time is of particular importance. The dependency ratio corresponds to the ratio of the retired lives to those of working age. Initially it is assumed that the population evolves according to a fixed regime of fertility rates and survival rates. Subsequently the impact of different types of fertility fluctuations are analyzed in terms of evolution of the dependency ratio over time.

2. Introduction to Limit Cycle Behaviour

To illustrate the origin of long period oscillations related to cohort characteristics, we discuss an insightful model developed by Lee (1974).

It is assumed that one time unit is equal to a single generation. The number of births at time t is represented by B_t and depends on the size of the adult population at time t, A_t as well as the birth rate. The adjustment process operates through the dependence of the birth rate on A_t. The birth rate, b, decreases as the cohort size increases. The proportion of births surviving to adulthood is s where s is assumed independent of time. Thus the population evolves according to the following two equations.

$$B_t = b(A_t)A_t \tag{1}$$

$$A_t = sB_{t-1} \tag{2}$$

Again following Lee we assume a simple linear functional form for b.

$$b(A_t) = a - cA_t \tag{3}$$

where a and c are constants.

It follows that the equilibrium birth cohort size will be:

$$\bar{B} = \frac{as - 1}{cs^2} \tag{4}$$

From equation (4) the value of c required to produce a given equilibrium $\bar{B}$ can be calculated if a and s are known.

The product (as) corresponds to the maximum net reproduction rate which occurs when the population is very small. The dynamical behaviour of the system depends on the size of (as) as indicated.

Range of as	Behaviour of population
$1 < as < 2$	Population converges directly to equilibrium
$2 < as < 3$	Population converges in an oscillating fashion to its equilibrium value
$3 < as$	Equilibrium is either unstable or population evolves in limit cycles

In order to illustrate numerically some aspects of this behaviour assume the following parameter values.

$$\bar{B} = 20$$

$$\bar{A} = 16$$

$$s = 0.8$$

The different dynamical behaviour of the system can be investigated by taking values of a as follows:

1.5, 2.5, 3.5, 4.0

Under these four assumptions the parameters are selected so that the same equilibrium number of births is obtained in all cases; $\bar{B} = 20$. In addition at time t=1 the number of births in each case is assumed to be 2.5. Table 1 gives details of the evolution of the birth sequences under the four assumptions. Under the first two the birth sequence converges directly to its equilibrium value whereas when a = 3.5 the

Table 1 Population Evolution under Four Different Birth Rate Parameters

Time in Generations	Numbers of Births a=1.5	a=2.5	a=3.5	a=4.0
1	2.50	2.50	2.50	2.50
2	2.94	4.69	6.44	7.31
3	3.44	8.28	14.30	17.52
4	4.01	13.13	21.63	22.30
5	4.65	17.64	18.45	16.66
6	5.36	19.72	21.02	22.78
7	6.15	20.00	19.09	15.81
8	7.00	20.00	20.66	23.10
9	7.91	20.00	19.44	15.23
10	8.87	20.00	20.42	23.22
11	9.85	20.00	19.65	14.99
12	10.85	20.00	20.27	23.25
13	11.85	20.00	19.78	14.94
14	12.81	20.00	20.17	23.26
15	13.73	20.00	19.86	14.93
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
40	19.96	20.00	20.00	23.26
41	19.97	20.00	20.00	14.92
42	19.98	20.00	20.00	23.26
43	19.98	20.00	20.00	14.92
44	19.98	20.00	20.00	23.26
45	19.99	20.00	20.00	14.92
46	19.99	20.00	20.00	23.26
47	19.99	20.00	20.00	14.92
48	19.99	20.00	20.00	23.26
49	19.99	20.00	20.00	14.92
50	20.00	20.00	20.00	23.26
51	20.00	20.00	20.00	14.92
52	20.00	20.00	20.00	23.26
53	20.00	20.00	20.00	14.92
54	20.00	20.00	20.00	23.26
55	20.00	20.00	20.00	14.92

population oscillates around the equilibrium level prior to convergence. In the final column the birth sequence corresponding to a = 4 ultimately develops into a regular periodic motion with limit cycles of period equal to two generations. As the values of a are increased the limit cycle behaviour becomes more complex and the period length increases. Eventually a stage is reached where the oscillations become unstable.

3. Cohort Fertility Variations

In the last section the population model leading to limit cycle behaviour was a very simple one. Recent papers by Frauenthal and Swick (1983) and Swick (1981) have shown how cohort fertility models can be incorporated into a continuous time model of population dynamics. For some parameter values limit cycle behaviour is obtained. In this section we trace these recent developments and comment on some of the main results. The point of departure is Lotka's renewal equation.

Consider a closed human population. Let B(t) denote the number of female births from time t to (t+dt). Let l(s,t) denote the probability that a woman born at time (t-s) will survive to time s and m(s,t) the probability that a woman born at time (t-s) will give birth to a daughter when she is aged between s and s + ds. Assume that G(t) is the number of female births to women present in the population at time t=0. Then the evolution of the female population is given by:

$$B(t) = G(t) + \int_0^t B(t-s)\, l(s,t)\, m(s,t)\, ds \qquad (5)$$

$$t > 0$$

For a constant regime of fertility and mortality

$$l(s,t) = l(s)$$

$$m(s,t) = m(s)$$

In this case the solution of equation (5) gives rise to Stable Population Theory, Keyfitz (1977, Chapter 5).

To incorporate an Easterlin effect, Frauenthal and Swick assumed that the current birth rate is a function of the size of the cohort at birth so that:

$$m(s,t) = m(s) \quad M\{B(t-s)\} \tag{6}$$

This assumption gives rise to significant changes when it is incorporated in the Lotka renewal equation.

The net fertility function $\phi(s)$ is defined by;

$$\phi(s) = m(s)\ l(s)$$

It is convenient to assume that this "equilibrium" or standard fertility function has been rescaled so that

$$R_o = \int_o^\infty \phi(s)\ ds = 1$$

If α and β are the ages within which births occur then the limits of integration ~~in (6)~~ are α and β.

With these assumptions the modified renewal equation for $t > \beta$ becomes

$$B(t) = \int_\alpha^\beta \phi(s)\ B(t-s)\ M[B(t-s)]ds \tag{7}$$

To reflect the Easterlin hypothesis it is assumed that as B increases M(B) decreases so that an adjustment mechanism is built into the population dynamics. Notice also that the population has an equilibrium birth level equal to E if $M(E) = 1$.

Different functional forms for M have been examined in the literature. Perhaps the simplest is:

$$B\ M(B) = B[1 + \gamma(\frac{E}{B} - 1)] + g(B-E) \tag{8}$$

where $g(0) = 0 = g'(0)$ and $\gamma = -EM'(E)$

If g is ignored we obtain

$$M(B) = 1 + \gamma\left(\frac{E}{B} - 1\right)$$

From this equation note that if the number of births B lies below E the equilibrium level that the term $(\frac{E}{B} - 1)$ is positive. If γ is positive this means that M(B), which is the net reproduction rate for the cohort, will exceed the equilibrium value (of 1). In the same way if B exceeds E the net reproduction rate for the cohort will be below 1 for γ positive. If we ignore the issue of overlapping generations[3] this mechanism can give rise to regular cycles in the population when the initial birth level is different from the equilibrium level E.

Numerical and analytical investigations of the solutions of (7) indicate that the parameter γ plays a key role in determining the solution behaviour. These results are contained in papers by Swick, (1981), Frauenthal and Swick (1983) and Smith (1981(a)) and may be summarized as follows.

1. $\gamma < 0$. In general this leads to population extinction as initial disturbances are magnified through time.
2. $0 < \gamma < 1$. In general the birth trajectories converge towards the equilibrium level with oscillations that are damped.
3. $1 < \gamma < \gamma_0$. In this region cycles of two generations in length are generated. The birth level oscillates around E with period equal to 2μ where

$$\mu = \int_{\alpha}^{\beta} \emptyset(s)\, ds \qquad (9)$$

4. $\gamma > \gamma_0$. In this region the population is unstable.

The parameter γ_0 is closely related to the dominant root of the characteristic equation of the corresponding renewal equation based on the revised birth rate dynamics.

4. Periodic Fertility Fluctuations

In this section we consider situations where the fertility rates are subject to a regular periodic cycle. A detailed analysis has been given by Coale (1972).

One method of modelling a periodic cycle is to assume that

$$\emptyset(s,t) = \emptyset(s, t+T) \qquad (10)$$

where $\emptyset$ follows a cycle of T years long. Assuming constant mortality and that fertility fluctuations rise and fall in the same proportions at each age then,

$$\emptyset(s,t) = \emptyset(a)\cdot F(t) \qquad (11)$$

where F is a periodic function of period T. Under these circumstances the birth trajectory is

$$B(t) = e^{rt}H(t) \qquad (12)$$

where H is a periodic function of cycle length T and r is the growth rate of the population.

Assume for simplicity that fertility fluctuations occur with a single frequency with a small amplitude. Then the periodic fluctuations in births depend both on the frequency and amplitude of the fertility fluctuations. In general the amplitude of the birth cycles will differ from those of the fertility cycles. The length of the fertility cycle relative to the length of a generation has an important influence on the result.

When the fertility rate fluctuations have a large amplitude the resultant birth fluctuations about the exponential trend are no longer of the same basic form as the fertility fluctuations. Coale (1972) gives an analysis of the resulting behaviour in this case.

5. Application to Long-Range Population Projections

The purpose of this section is to discuss the incorporation of the various types of mechanisms for inducing birth rate fluctuations into long-range population projections. In particular this leads to relative shifts in the age structure which can have profound social and economic consequences in the context of a pay as you go national social security system. For the purposes of this analysis the Canadian population is used. The methodology is applicable to other national population projections. Initially we present some demographic details of the Canadian population. For a more complete summary of the Canadian population and its probable evolution under conventional assumptions, see Brown (1982).

During the last 60 years Canadian fertility rates have shown an initial increase followed by a decline and then a substantial rise followed by a dramatic drop during the last two decades. This pattern is confirmed in Table 2 which displays the net reproduction rates and the total fertility rates for Canada for the period 1925-1980. When the age specific fertility rates are examined the age specific fertility rates for age groups 15-19, 20-24, 25-29 and 30-34 illustrate the greatest cyclical tendency over time. Fertility rates for women over age 35 have basically declined over this time period. Thus as far as pure periodic changes are concerned the pattern is more complicated than that decribed in the previous section where it was assumed that the changes affected all ages in the same proportions. The shape of the net fertility function changed over this time period with a reduction of both the mean and the variation of the net maternity function (see Keyfitz and Flieger (1971) page 63).

Table 2 Summary of Fertility Statistics
Canada 1925-1980

Year	Net Reproduction Rate (Period)	Total Fertility Rate
1925	1.28	3.13
1930	1.38	3.28
1935	1.19	2.76
1940	1.22	2.76
1945	1.35	3.02
1950	1.58	3.46
1955	1.78	3.83
1960	1.82	3.90
1965	1.48	3.15
1970	1.10	2.33
1975	.88	1.85
1980	.81	1.70

In terms of cohort fertility there is also evidence of a long-term cycle. Table 3 gives the net reproduction rate for Canada by year of birth for the same period. For comparison purposes the corresponding cohort net reproduction rates for the United States are also shown. The cyclical effect emerges much more clearly in the U.S. data but further analysis of the data with respect to the impact of immigration and population growth would be required before any conclusions could be made. The figures seem to indicate that there is a long period cycle of length about 50 years in both data sets. Whether or not the social and economic conditions that gave rise to this cycle will be repeated in the future is a different question. Some observers have suggested that many of the changes which are taking place in women's status in society point to a permanent reduction in the fertility rate. This rationale permeates most projections which are made in connection with social security financing. Although this is probably the most likely scenario there is considerable benefit in analyzing other scenarios based on different fertility rate assumptions.

As far as the Canadian data are concerned it is very difficult to isolate the relative contribution of "period" and cohort effects in the empirical statistics. Even if this were possible there is no guarantee that future fertility rates would evolve in a corresponding fashion. Our procedure will be to project the population forward under different fertility assumptions and comment on the results.

The starting point is the 1981 Canadian population by age and sex. Current mortality levels are assumed to prevail in the future. There

Table 3 Cohort Net Reproduction Rates
Canada and the United States

Cohort Year of Birth	Net Reproduction Rate	
	Canada	United States
1910	1.25	.96
1915	1.23	1.05
1920	1.33	1.18
1925	1.46	1.34
1930	1.51	1.45
1935	1.52	1.48
1940	1.36	1.31
1945	1.14	1.06
1950	.98	.89
1955	.89	.70

Table 4 1981 Age-Sex Composition of Canadian Population

Age Range	Percentages in each age-range Male	Female
0-4	7.5	7.1
5-9	7.6	7.0
10-14	8.2	7.6
15-19	9.8	9.2
20-24	9.6	9.4
25-29	9.0	8.9
30-34	8.5	8.3
34-39	6.8	6.6
40-44	5.6	5.4
45-49	5.3	5.1
50-54	5.2	5.1
55-59	4.7	5.0
60-64	3.8	4.2
65-69	3.2	3.7
70-74	2.3	2.9
75-79	1.5	2.1
80-84	0.8	1.3
85-89	0.4	0.7
90 +	0.2	0.4
Total	100.0	100.0
Total Numbers of each sex	12,068,285	12,274,890

is thus no allowance for mortality improvements. The age-sex composition of the 1981 populations is summarized in Table 4. Note the bulge in the 15-35 age group corresponding to the baby boom in the 1950's and 1960's. It has been assumed that the equilibrium fertility rate is such that the population doubles in approximately 270 years or eleven generations. This corresponds to a very slow growth assumption. The equilibrium age fertility schedule assumed is 126% of the 1981 age-fertility schedule and is given in Table 5.

Table 5 Equilibrium Age-Specific Fertility Rates Used in Population Projections

Age Range	Rates per 1000 Women
15-19	34.8
20-24	126.1
25-29	163.0
30-34	87.3
35-39	24.4
40-44	3.9
45-49	0.3

Table 6 summarizes the results of the projection of the base population under these assumptions. It is convenient to represent this projection by using the ratios of the population aged 65 and over to the populations aged between 20 and 65. This ratio is denoted by RA. It is a very useful index of the current cost of pay as you go national social security plans. Contributions from the population in the working ages are used under this system to pay the pensions of the retired population. It is assumed that age 65 is the retirement age.

Note from Table 6 that this ratio increases steadily for the next fifty years. This effect is due to the aging of those individuals born in the baby boom of the 1950's and 1960's. In Table 6 1981 is counted as time t=0 in the projection. Note that the highest value of the RA ratio occurs in the early 2030's. After this the numbers of individuals born during the baby boom will fall off sharply.[4] Since the projection is based on an unchanging equilibrium fertility rate the population tends towards a stable state and the RA ratio tends towards an ultimate level of 27.6%. Note however that there are variations in this ratio which become progressively damped as the population evolves towards its stable state. These cyclical variations are about one generation in length. It should be noted that the ratio used here is obtained from two broad age groups within the population so that there is a smoothing effect taking place when this ratio is computed.

Another ratio that is sometimes used in these circumstances is the ratio of young people aged 20 and under to the population aged 21 to 65. This ratio denoted in this paper as YA is helpful in analyzing the relative share of society's resources to be used for educating and care of the young. In view of the particular structure of the Canadian population this ratio will decline over the next three decades and under the assumed constant fertility and mortality regime oscillate very slowly towards its equilibrium value.

The sum of these two ratios denoted by TD may be of interest in that it is a measure of total dependency. This ratio does not fluctuate as much as the RA ratio. Again it reaches its maximum in the

Table 6 Evolution of Various Ratios for Population Projection under Fixed Mortality and Fertility.

Base Population Canada 1981

Time	RA%	YA%	TD%
0	15.7	57.0	72.7
5	16.2	51.8	68.0
10	17.6	52.3	69.9
15	18.7	53.3	72.0
20	19.2	53.6	72.8
25	18.9	49.7	68.6
30	19.6	47.0	66.6
35	22.1	47.3	69.4
40	25.2	49.3	74.5
45	28.5	51.1	79.7
50	31.0	51.5	82.6
55	30.1	50.2	80.3
60	28.0	49.2	77.2
65	25.9	49.1	75.1
70	26.8	50.2	77.0
75	28.2	50.7	78.9
80	28.5	50.4	78.9
85	27.8	49.8	77.6
90	27.1	49.7	76.7
95	27.0	49.9	76.9
100	27.6	50.3	77.9
110	27.8	50.0	77.9
120	27.3	49.9	77.2
130	27.7	50.2	77.9
140	27.6	50.0	77.6
150	27.5	50.0	77.5
175	27.6	50.0	77.6
200	27.6	50.0	77.6

2030's. However the ultimate value at which it stabilizes is 77.6% which is not too far below its maximum of 82.6%.

The conclusion from Table 6 is that the population reaches a fairly stable state within three generations under constant fertility and mortality assumptions. After this the proportions of the population in the various age groups of interest are relatively constant. Under these circumstances, contributions to a pay as you go national social security plan would be also relatively stable.

The situation is quite different when periodic fluctuations are incorporated into the fertility forecasts. In Table 7 the equilibrium fertility rate is the same as that of Table 6. However in Table 7 it has been assumed that fertility rates will fluctuate with a period of 26 years. It has been assumed that the fertility rate changes can be modelled over time using a sine curve and that the maximum departure from the equilibrium rate is 21% of the equilibrium rate in any one direction. This figure was arrived at by assuming the 1981 fertility level was at the lowest point of the cycle. There is, of course, nothing sacrosanct about this assumption concerning the frequency and amplitude of the fertility fluctuations. Fairly long periods are required to generate a substantial degree of periodicity to the various ratios because of smoothing across age groups.

Inspection of Table 7 indicates that the periodic fluctuations in the fertility rates are transmitted to the various ratios. For convenience we have shown the incidence of the peaks and troughs of the

Table 7 Evolution of Various Ratios for Population Projection under Fixed Mortality and Oscillating Fertility with a Period of 26 Years.

Base Population Canada 1981

Time*	RA%	YA%	TD%
0	15.7	57.0	72.7
10	17.6	50.5	68.1
20	19.2	55.7	74.9
30	19.9	46.7	66.6
40	24.7	47.1	71.8
50	31.8	57.3	89.1
51	32.0	36.9	88.9
68	24.5	48.5	73.0
82	30.8	52.5	83.3
96	24.0	50.6	74.6
109	30.9	51.5	82.4
123	23.8	51.7	75.5
135	31.0	51.8	82.8
149	23.7	51.6	75.3
161	31.0	51.8	82.8
175	23.7	51.6	75.3
187	31.0	51.8	82.8

* After fifty years have elapsed the times shown correspond to the peaks and troughs of the RA ratio. Thus a peak was obtained after 51 years and the next trough after 68 years. Eventually the peaks occur every 26 years and the troughs have the same periodicity.

retired to active ratio. The other ratios show similar behaviour but achieve their maxima and minima at different times. They illustrate the same periodic cycle with a period of 26 years. Although the fertility rate underlying the projections deviates by up to 21% on either side of its equilibrium value, the amplitude of the deviations for the various ratios is less than this. For the RA ratio the maximum deviation is about 13.5% on either side of its average value. For the YA ratio the maximum deviation is 10.6% on either side of its mean value and the corresponding figure for the TD ratio is about 9%. It is interesting to note that the relative swings in the magnitude of these ratios is much less than the corresponding swings in the fertility rates.

When the length of the period is 52 years or two generations the various ratios eventually oscillate with periods of 52 years at different phases.[5] In this case the amplitudes of the ratios about their means are as follows:

Ratio	% Deviation about Mean
RA	9.3
YA	10.9
TD	8.6

We now turn to the incorporation of a fertility rate variation related to cohort size according to the models outlined in Section 3. The method used involves a slight modification of the ones discussed earlier. An equilibrium growth rate for the population is computed from the base case projection which incorporates constant mortality and constant (equilibrium) fertility assumptions. The number of births each year assuming the population grows at the equilibrium growth rate r is:

$$B_o e^{rt}$$

For a cohort of women aged x in this stable reference population the stable reference cohort is:

$$B_o \; e^{r(t-x)}$$

The size of the actual cohort at birth when we include the Easterlin effect is:

$$B(t-x)$$

The fertility rates at time t for the cohort born at (t-x) are obtained by multiplying the equilibrium fertility rates by:

$$1 + \gamma \left\{ \frac{B_o e^{r(t-x)}}{B(t-x)} - 1 \right\}$$

If γ is positive and if the actual cohort size lies below the stable reference cohort (or equilibrium cohort size) then the actual fertility rates will be greater than the equilibrium rates and conversely.

In order to start the projection we need actual cohort birth sizes from 1931. The actual birth sequence was adjusted because of the rapid population growth during these years and the resulting sequence represented approximately by a simple mathematical function.[6] The projection was carried out for different values of γ.

For $0 < \gamma < 1$ the population tends towards a stable state fairly quickly after the impact of the baby boom fades away. For example when $\gamma = \frac{1}{2}$ the oscillations quickly diminish in amplitude after a couple of cycles. For $\gamma = 1\frac{1}{2}$ the population evolution is more dramatic. The relative composition of the age structure goes through some fairly turbulent swings but the amplitude of the periodic movements in the ratios reduces significantly over time and the population tends towards a stable state. We attribute this result to the impact of overlapping generations. Smith (1981b) notes that while a particular individual is born into one single cohort his offspring will belong to different cohorts and so this tends to restore the population to equilibrium.

However as γ becomes larger the operation of this smoothing procedure proceeds more slowly. For example when $\gamma = 2$ there are significant periodic fluctuations in the population after 300 years although there is a small reduction in the amplitude of these waves over time. Table 8 gives the peaks and troughs of the RA ratio for $\gamma = 2$. Note that the cycles are two generations in length. This means that for a sufficiently high adjustment factor a population will oscillate for a very long time with periods of about two generations if it starts with the type of initial conditions we have used here. These initial conditions could also be viewed as resulting from a long period oscillation in the period fertility rates. One inference from these

Table 8 Evolution of Various Ratios for Population Projection under Fixed Mortality and Easterlin-type Fertility $\gamma = 2$

Base Population Canada 1981

Time*	RA	YA	TD
0	15.7	57.0	72.7
10	17.6	36.3	53.9
20	19.2	28.8	48.0
30	22.7	44.6	67.2
40	32.5	70.9	103.4
50	39.7	78.8	118.5
82	16.6	42.5	59.1
110	35.6	57.0	92.7
137	20.7	47.0	67.7
163	32.4	54.8	87.2
190	22.9	48.5	71.4
216	30.4	53.4	83.8
242	24.4	45.4	73.8
269	29.2	52.5	81.7
297	25.4	50.0	75.3

* The first peak of the RA ratio is attained at t = 50. Thereafter the t values correspond to successive peaks and troughs in the RA ratio. The length of the period is about 54 years or two generations.

projections is that a combination of period and cohort effects can in certain circumstances lead to a persistence of long-run cyclical behaviour in human populations. The resulting oscillations have a long period of the order of two generations.

In terms of long-range economic and social planning this type of growth is much more difficult to handle and plan for than the constant growth rates predicted by stable population theory. The population is always in a process of adjustment towards its equilibrium value or away from it. Indeed it is just equal to the equilibrium value twice every two generations. Since the implications of this type of behaviour differ radically from those of conventional assumptions, it may be instructive to include it as a possible scenario.

If the population were to evolve as in Table 8 then we would expect social and economic forces to bring about pressure for change. In the context of a national social security plan, one possible response of this nature would be to maintain the year-by-year level of the RA ratio at its equilibrium value of 27.6% by adjusting the retirement age appropriately. If this were done, under the demographic assumptions of Table 8 then the retirement age would vary from a maximum of 69 years to a minimum of 58 years. The retirement age would also oscillate with the same periodic frequency of two generations and the amplitude of the oscillations would diminish over time.

Footnotes

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1 The mathematical models used in this paper to generate fertility oscillations have already been described in the literature. However, it is useful to briefly review them here before proceeding to the actual appications.

2 Wrigley and Schofield's (1981) main conclusion on this point is summarized in Chapter 10 of their book on page 438.

> Although the nature and functioning of the social and economic institutions producing the long lag between a change in the trend of real wages and a corresponding change in fertility will not be fully understood without much further work, the fact that there were wide slow oscillations in fertility which broadly mirrored the real-wage fluctuations but with a time lag of about 40 years is of the greatest interest itself. Fertility responded to real-wage changes and vice-versa as if the system was operating homeostatically but with very slow feedback between the two components resulting in wide, leisurely swings about a notional equilibrium point.

3 The consequences of overlapping generations have been mentioned by Smith (1981a). We discuss this point later in Section 5.

4 The projections of Table 6 covering the period to 2071 (90 years) reveal a pattern broadly similar to that given by Robert Brown (1982) (Table 2, page 377). The differences arise mainly from,

(i) A different partition of the age interval. In Brown's paper the intervals are 0-17, 18-64 and 65+, whereas the present paper uses 0-20, 21-64, 65+.

(ii) Different fertility assumptions. The present paper assumes a constant fertility rate in Table 6 of slightly over 2.1, whereas in Table 2 of Brown's paper the constant fertility rate is 1.8.

(iii) For the purposes of Table 6 immigration was ignored. In Table 2 of Brown's paper immigration was assumed to be 100,000 per annum.

5 For space reasons the detailed figures for this projection are not shown.

Footnotes continued

6 The function was selected so that the adjusted birth sequence had 1931 births equal to 1981 births. The number of births increased by equal increments until 1956 when it reached 160% of the 1931 level. From 1956 until 1981 the number of births decreased in level steps. It should be pointed out that there are a variety of ways of using the historical birth sequence to obtain input data on births for this purpose. Our procedure is destined to capture the broad trend of births in Canada in a manner that is simple and suitable for inclusion in our model.

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