Applying the Cost of Capital Approach to Extrapolating an Implied Volatility Surface

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Introduction

- AEGON Context: European based life insurer that needs to develop market consistent financial statements

- Basic idea: use observed market prices for hedgeable risk use cost of capital to price non-hedgeable risk

- Practical Problem: “Holes” in observed market data

- Can we apply the cost of capital concepts developed for insurance liabilities to fill the “holes”?

- Key ideas
  1. Assume Law of Large Numbers Applies where appropriate
  2. Start with simple Best Estimate (Black Scholes)
  3. Consider risk of current period loss (Contagion Event)
  4. Consider potential future losses (Parameter Risk)
  5. Revise Best Estimate assumptions if appropriate
Starting Point: Assume Black Scholes delta hedging world is best estimate model

Risk Neutral process for stock price

\[ dS = (r - q)Sdt + \sigma Sdz \]

\[ \frac{\partial V}{\partial t} + (r - q)S \frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} - rV = 0 \]

Concept of “implied volatility $\sigma^{imp}$” used to describe market condition

Observed Price = $V(t, S, \sigma^{imp})$

Data goes out about 15 years for S&P 500
S&P 500 Implied Vols at June 30, 2009
for a number of different maturities
Starting Point: Black Scholes delta hedging

Key issue is our ability to value the gain/loss in a given period. If $S \rightarrow JS$ then unhedged loss $UHL$ is

$$UHL = V(t, JS) - V(t, S) - (J - 1)S \frac{\partial V}{\partial S}$$

Under Black Sholes assumptions: $J = \exp[\mu \Delta t + \sigma \Delta t^{1/2}]$

$$E[UHL] = \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial S^2} \Delta t + ...$$

$$\text{VAR}[UHL] = o(\Delta t^2)$$

Must hold capital to cover possible
- Mis estimation of the mean (parameter risk)
- Unexpected large up or down movement (contagion risk)
Option Pricing – Current Period Loss

- Choose an appropriate $J$ and cost of capital $\pi$ then

$$\frac{\partial V}{\partial t} + (r - q)S \frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} - rV = -\pi \left[ V(t, JS) - V(t, S) - (J - 1)S \frac{\partial V}{\partial S} \right]$$

- Expected Loss
- Cost of Capital
- Gross Loss
- Hedge
- Economic Capital
Choose a reasonable $J$ and cost of capital $\pi$

$$\frac{\partial V}{\partial t} + (r - q)S \frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} - rV = -\pi \left[ V(t, JS) - V(t, S) - (J - 1)S \frac{\partial V}{\partial S} \right]$$

Equivalent to new “contagion loaded” process

$$dS = [r - q - \pi(J - 1)]Sdt + \sigma Sdz + (J - 1)Sdq$$

Formally a simple version of Merton’s 1973 jump diffusion model, interpretation is new

Reasonably compact (infinite series) closed form solution available (See Haug’s “Option Pricing Formulas” 1997).
Option Pricing – Contagion Issues

- Cost of Capital must cover frictional cost plus target return to shareholder  \( \pi = \tau r + \beta M + \alpha \)

- Quantity  \( UHL = V(t, JS) - V(t, S) - (J - 1)S \frac{\partial V}{\partial S} \)
  - is negative if option is concave rather than convex
  - Same as mortality/longevity issue

- For vanilla puts and calls might want to use \( J = .6 \) for puts but \( J = 1.4 \) for calls

- Numerical examples assume we are dealing with puts
Large Maturity Approximation

- Over a long time (e.g. 15+ years) the jump process can be approximated by a modified Black Scholes model

\[ dS = [r - q - \pi(J - 1)]Sdt + \sigma Sdz + (J - 1)Sdq, \]

"converges" to

\[ dS = [r - q - \pi(J - 1 - \ln(J) - \ln(J)^2 / 2)]Sdt + \sqrt{\sigma^2 + \pi \ln(J)^2} Sdz. \]

- Allows standard Black Scholes formula to be used instead of series solution
- “Asymptotic Black Scholes Approximation”
Step 3 Parameter Risk

- Back to Black Scholes for a moment...
- Assume new information arrives that causes us to change our best estimate volatility assumption from \( \sigma^2 \) to a new value \( \hat{\sigma}^2 = \sigma^2 + \Delta \sigma^2 \)
- Need capital to cover the loss \( \hat{V} - V \)
- New system of valuation equations

\[
\frac{\partial V}{\partial t} + (r - q)S \frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} - rV = -\pi \left[ \hat{V}(t, S) - V(t, S) \right],
\]

\[
\frac{\partial \hat{V}}{\partial t} + (r - q)S \frac{\partial \hat{V}}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 \hat{V}}{\partial S^2} - r\hat{V} = -\pi \left[ \hat{V}^{(2)}(t, S) - \hat{V}(t, S) \right],
\]

\[
\frac{\partial \hat{V}^{(2)}}{\partial t} + ...
\]
Parameter Risk

- In theory, must specify volatility assumptions for entire hierarchy of volatility assumptions

\[ \sigma^2, \sigma^2 + \Delta \sigma^2, \sigma^2 + \Delta \sigma^2, \ldots \]

- Example: geometric hierarchy

\[ \Delta \sigma_n^2 - \Delta \sigma_{n-1}^2 = \alpha^{n-1} \Delta \sigma^2 \]

- Formal solution is a stochastic volatility model where volatility jumps from one level to the next with transition intensity equal to cost of capital (Brute Force)

\[ \sigma^2 \xrightarrow{\pi} \sigma^2 + \Delta \sigma^2 \xrightarrow{\pi} \sigma^2 + \Delta \sigma^2 \xrightarrow{\pi} \ldots \]

- Closed form solutions for some special cases e.g. \( \alpha = 1 \) or \( \alpha = 0 \).
Implied Volatility - Brute Force Parameter Risk

Constantly Increasing Shock Hierarchy

\( r = 5.0\% \)
\( \sigma = 15.0\% \)
\( \Delta \sigma = 15.0\% \)
\( \pi = 20.0\% \)
\( q = 3.0\% \)

Strike as % of current price
Parameter Risk: At the Money Implied Volatility

Constantly Increasing Shock Hierarchy

\[ r = 5.0\% \]
\[ \sigma = 15.0\% \]
\[ \Delta \sigma = 15.0\% \]
\[ \pi = 20.0\% \]
\[ q = 3.0\% \]
Parameter Risk

- Good News! Parameter Risk is actually fairly easy to do in practice
- Can replace shock hierarchy with a deterministic model (mean of the hierarchy) $\sigma^2 + \beta \Delta \sigma^2$
- Final valuation model

$$\frac{\partial V}{\partial t} + (r - q)S \frac{\partial V}{\partial S} + \frac{(\sigma^2 + \beta \Delta \sigma^2)S^2}{2} \frac{\partial^2 V}{\partial S^2} - rV = -\pi[1 - (1 - \alpha)\beta] \frac{\partial V}{\partial \beta}$$

- Has convenient closed form solutions
Put the pieces together

- Put parameter and contagion risk together

\[
\frac{\partial V}{\partial t} + (r - q)S \frac{\partial V}{\partial S} + \pi [1 - (1 - \alpha) \beta] \frac{\partial V}{\partial \beta} + \frac{(\sigma^2 + \beta \Delta \sigma^2)S^2}{2} \frac{\partial^2 V}{\partial S^2} - rV
\]

\[
= -\pi \left[ V(t, JS) - V(t, S) - (J - 1)S \frac{\partial V}{\partial S} \right].
\]

- If we want to fit June 30, 2009 S&P 500 market data can use
  - \( J = 50\% \), \( \pi = 20\% \), \( q = 3.0\% \)
  - \( \Delta \sigma^2 = 10\% \), \( \alpha = 50\% \)

- Reasonable fit for first 15 years
Put the pieces together

Implied Volatility - Cost of Capital Model

- $r$ (AA Yield Curve): 15.0%
- $\sigma$ (volatility): 15.0%
- $\Delta \sigma$ (change in volatility): 10.0%
- $a$ (alpha): 50.0%
- $\pi$ (beta): 20.0%
- $J$ (jump): 50.0%
- $q$ (discount rate): 3.0%

Strike as % of current price vs. Vol %

Legend:
- 1
- 3
- 5
- 10
- 15
Final Step: Extrapolation

- Fit not perfect but appears to capture major risk issues

- As of June 30, 2009 we are still in financial crisis mode

- Conclusion: must respect market data for first 15 years but can use more “reasonable” parameters after that time

- Example: Assume $\pi$ goes to 10% after 15 years
At the Money Implied Volatility Extrapolation Assumptions

Vol %

Maturity in Years

Single Jump Model  Asymptotic Black Scholes  Observed  Jump Model Fwd Vol
Asymptotic Black Scholes Implied Volatility

Maturity in Years

Vol %

Spot Volatility  Fwd Volatility