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Applying the Cost of Capital Approach to Extrapolating an Implied Volatility Surface

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Introduction

- AEGON Context: European based life insurer that needs to develop market consistent financial statements

- Basic idea: use observed market prices for hedgeable risk
  use cost of capital to price non-hedgeable risk

- Practical Problem: “Holes” in observed market data

- Can we apply the cost of capital concepts developed for insurance liabilities to fill the “holes”?

- Key ideas
  1. Assume Law of Large Numbers Applies where appropriate
  2. Start with simple Best Estimate (Black Scholes)
  3. Consider risk of current period loss (Contagion Event)
  4. Consider potential future losses (Parameter Risk)
  5. Revise Best Estimate assumptions if appropriate

Option Pricing – Current Period Loss

- Starting Point: Assume Black Scholes delta hedging world is best estimate model
- Risk Neutral process for stock price

$$dS = (r-q)Sdt + \sigma S dz$$

$$\frac{\partial V}{\partial t} + (r-q)S\frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} - rV = 0$$

- Concept of “implied volatility σ^{imp}” used to describe market condition

$$\text{Observed Price} = V(t, S, \sigma^{imp})$$

- Data goes out about 15 years for S&P 500

S&P 500 Implied Vols at June 30, 2009 for a number of different maturities

Option Pricing – Current Period Loss

- Starting Point: Black Scholes delta hedging
- Key issue is our ability to value the gain/loss in a given period. If S->JS then unhedged loss UHL is

$$UHL = V(t, JS) - V(t, S) - (J-1)S\frac{\partial V}{\partial S}$$

- Under Black Sholes assumptions: $J = \exp[\mu\Delta t + \sigma z\sqrt{\Delta t}]$

$$E[UHL] = \tfrac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2}\Delta t + \ldots$$

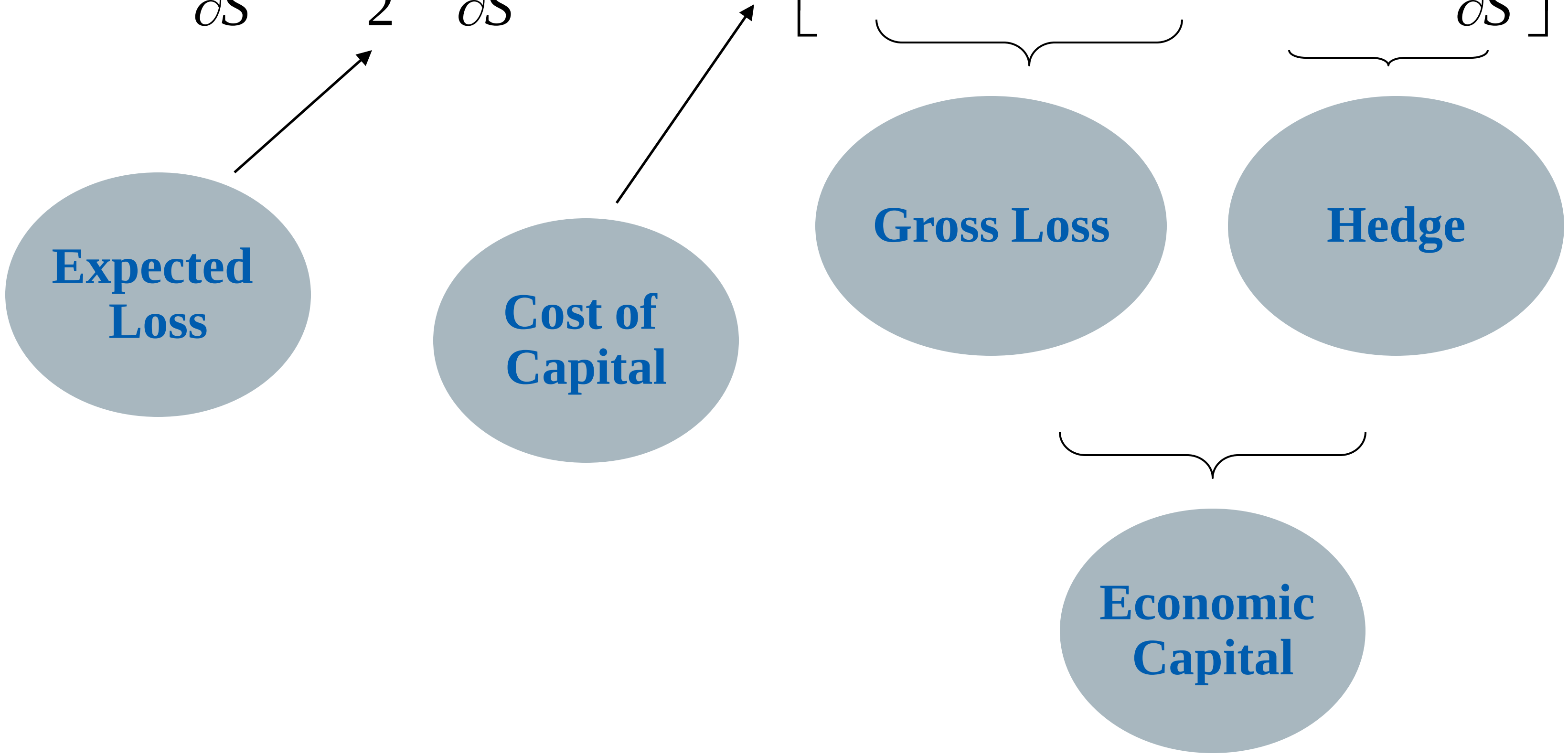
$$VAR[UHL] = o(\Delta t^2)$$

- Must hold capital to cover possible
  - Mis estimation of the mean (parameter risk)
  - Unexpected large up or down movement (contagion risk)

Option Pricing – Current Period Loss

- Choose an appropriate J and cost of capital π then

$$\frac{\partial V}{\partial t} + (r-q)S\frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} - rV = -\pi\left[V(t,JS) - V(t,S) - (J-1)S\frac{\partial V}{\partial S}\right]$$



Option Pricing – Current Period Loss

- Choose a reasonable J and cost of capital π

$$\frac{\partial V}{\partial t}+(r-q)S\frac{\partial V}{\partial S}+\frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2}-rV=-\pi\left[V(t,JS)-V(t,S)-(J-1)S\frac{\partial V}{\partial S}\right]$$

- Equivalent to new “contagion loaded” process

$$dS=[r-q-\pi(J-1)]Sdt+\sigma Sdz+(J-1)Sdq$$

- Formally a simple version of Merton’s 1973 jump diffusion model, interpretation is new
- Reasonably compact (infinite series) closed form solution available (See Haug’s “Option Pricing Formulas” 1997).

Option Pricing – Contagion Issues

- Cost of Capital must cover frictional cost plus target return to shareholder $\pi = \tau r + \beta M + \alpha$

- Quantity $UHL = V(t, JS) - V(t, S) - (J - 1)S \dfrac{\partial V}{\partial S}$
  - is negative if option is concave rather than convex
  - Same as mortality/longevity issue

- For vanilla puts and calls might want to use $J = .6$ for puts but $J = 1.4$ for calls

- Numerical examples assume we are dealing with puts

Large Maturity Approximation

- Over a long time (e.g. 15+ years) the jump process can be approximated by a modified Black Scholes model

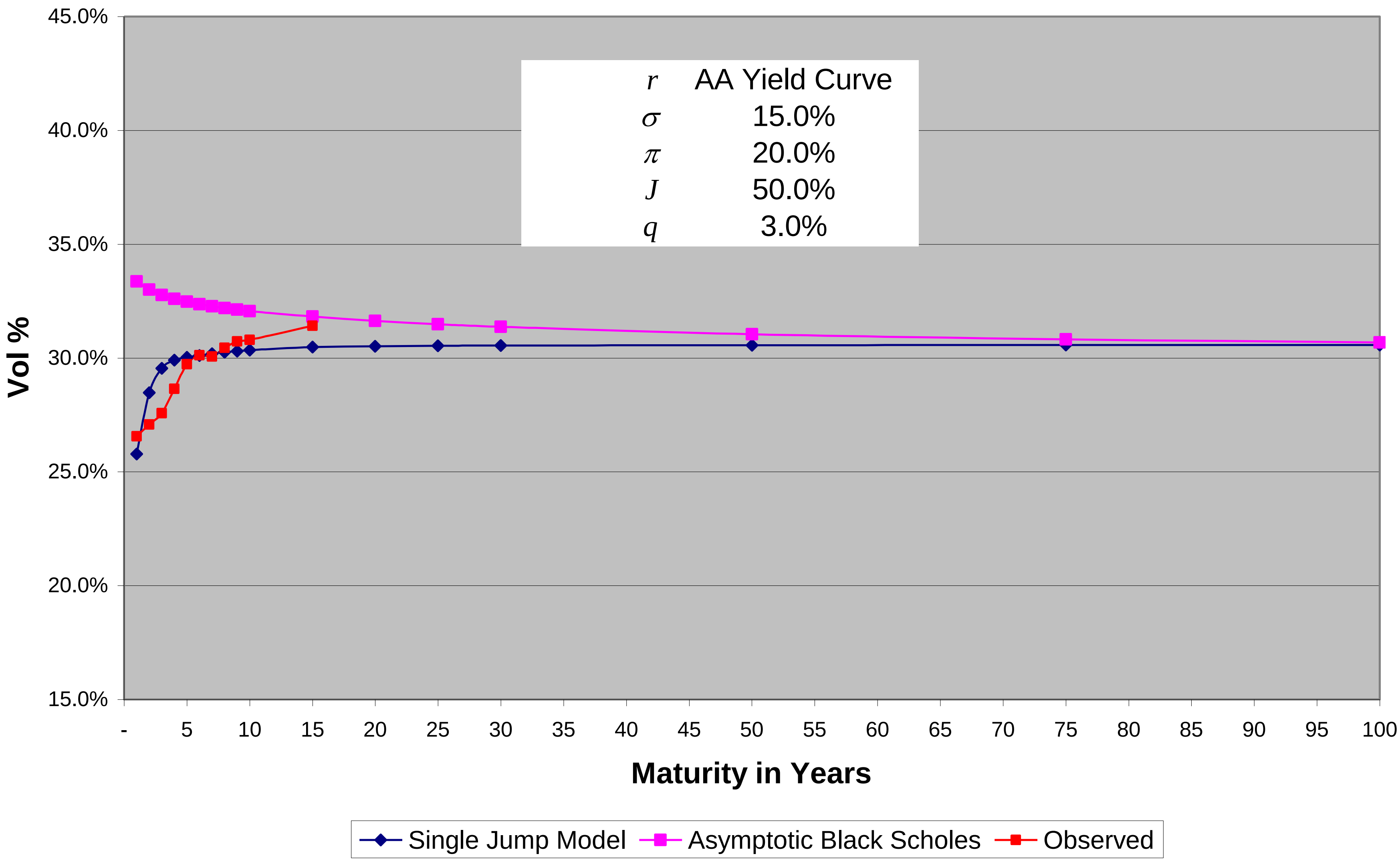
$$dS = [r - q - \pi(J-1)]Sdt + \sigma Sdz + (J-1)Sdq,$$

"converges" to

$$dS = [r - q - \pi(J - 1 - \ln(J) - \ln(J)^2/2)]Sdt + \sqrt{\sigma^2 + \pi\ln(J)^2}\,Sdz.$$

- Allows standard Black Scholes formula to be used instead of series solution
- “Asymptotic Black Scholes Approximation”

At the Money Implied Volatility



Step 3 Parameter Risk

- Back to Black Scholes for a moment…
- Assume new information arrives that causes us to change our best estimate volatility assumption from σ^2 to a new value $\hat{\sigma}^2 = \sigma^2 + \Delta\sigma^2$
- Need capital to cover the loss $\hat{V} - V$
- New system of valuation equations

$$\frac{\partial V}{\partial t} + (r-q)S\frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} - rV = -\pi\left[\hat{V}(t,S) - V(t,S)\right],$$

$$\frac{\partial \hat{V}}{\partial t} + (r-q)S\frac{\partial \hat{V}}{\partial S} + \frac{\hat{\sigma}^2 S^2}{2}\frac{\partial^2 \hat{V}}{\partial S^2} - r\hat{V} = -\pi\left[\hat{V}^{(2)}(t,S) - \hat{V}(t,S)\right],$$

$$\frac{\partial \hat{V}^{(2)}}{\partial t} + \ldots$$

Parameter Risk

- In theory, must specify volatility assumptions for entire hierarchy of volatility assumptions

$$\sigma^2, \sigma^2 + \Delta\sigma^2, \sigma^2 + \Delta{\sigma_2}^2, \ldots$$

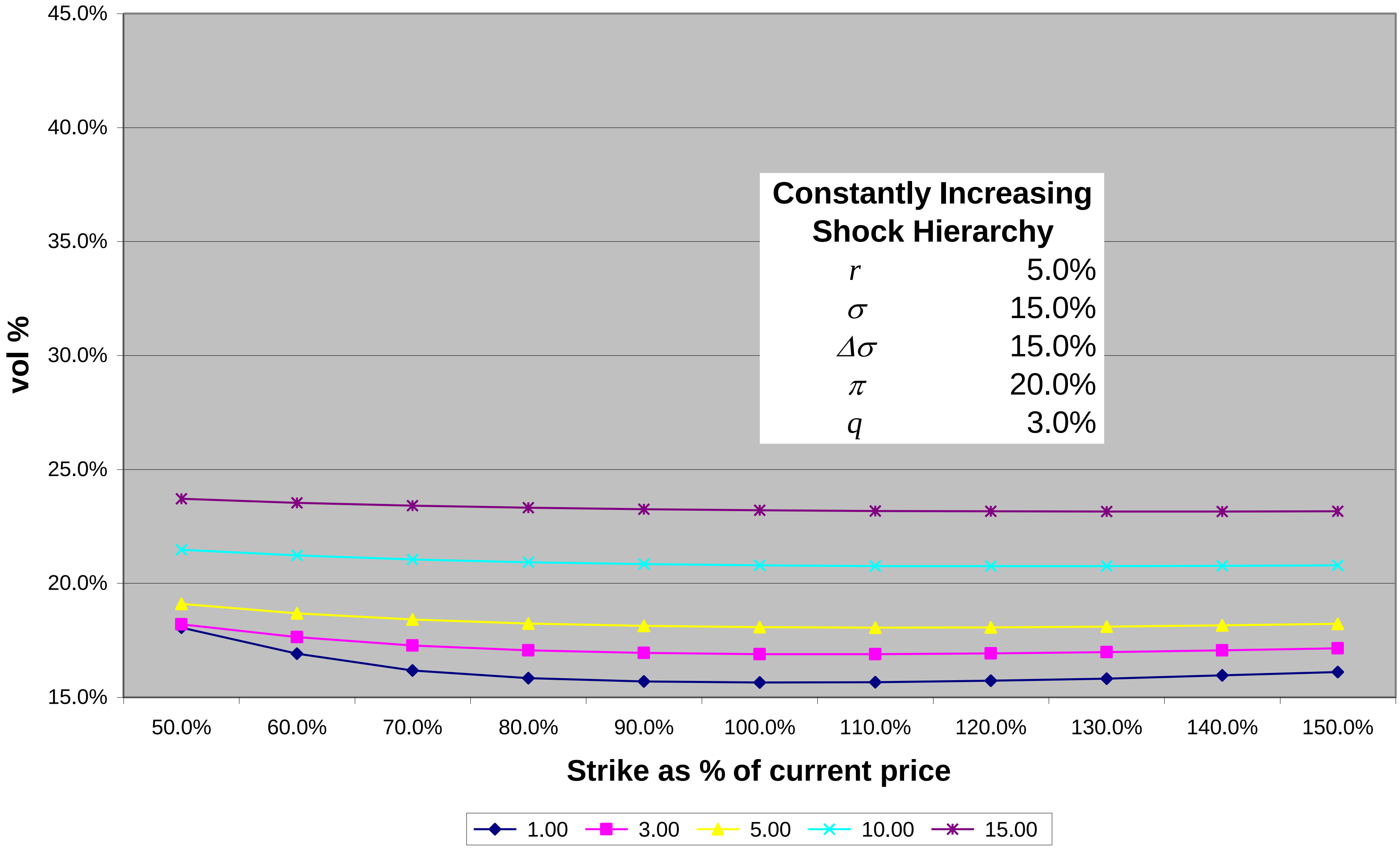
- Example: geometric hierarchy $\Delta\sigma_n^2 - \Delta{\sigma_{n-1}}^2 = \alpha^{n-1}\Delta\sigma^2$

- Formal solution is a stochastic volatility model where volatility jumps from one level to the next with transition intensity equal to cost of capital (Brute Force)

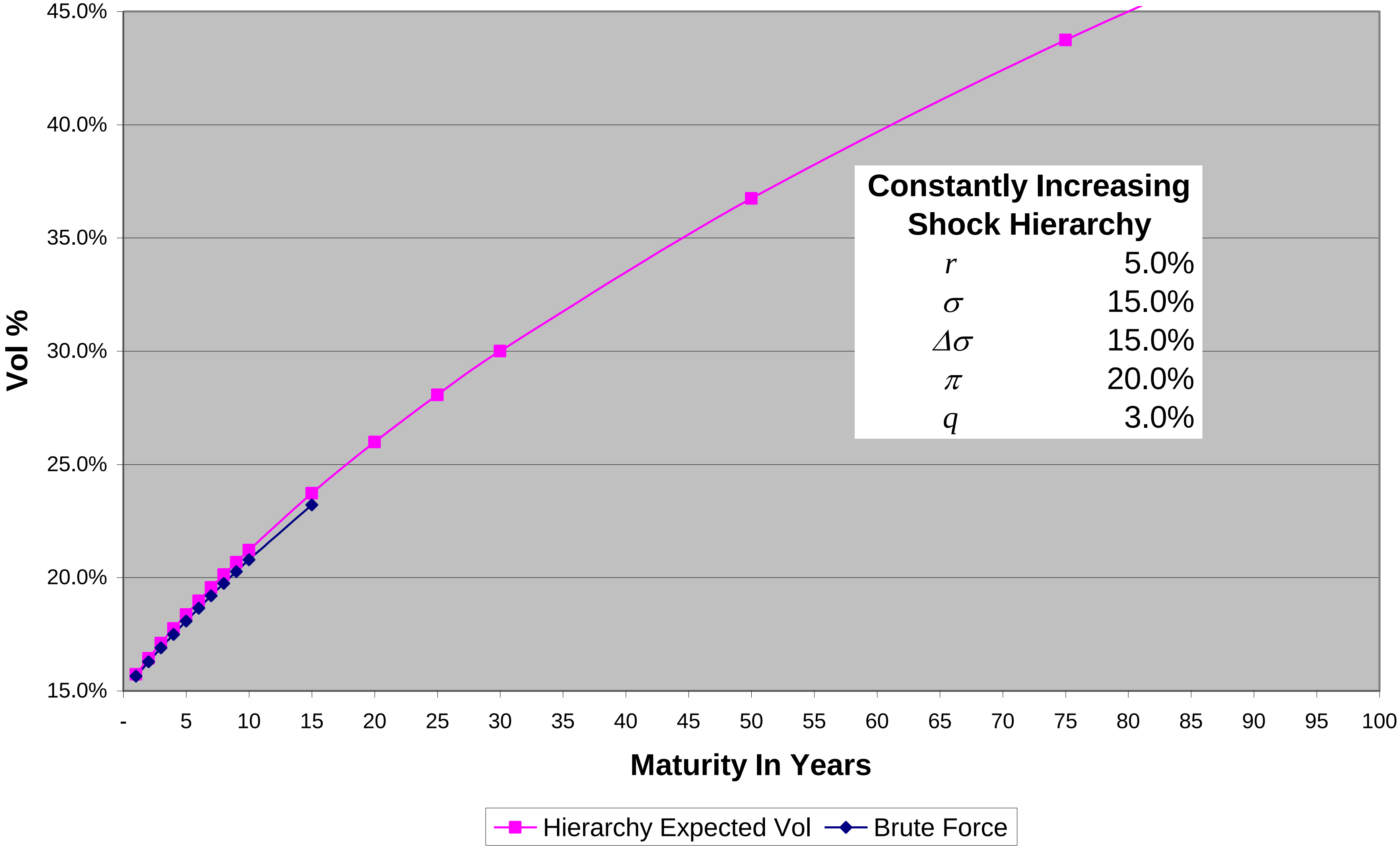
$$\sigma^2 \xrightarrow{\pi} \sigma^2 + \Delta\sigma^2 \xrightarrow{\pi} \sigma^2 + \Delta{\sigma_2}^2 \xrightarrow{\pi} \ldots$$

- Closed form solutions for some special cases e.g. $\alpha = 1$ or $\alpha = 0$.

Implied Volatility - Brute Force Parameter Risk



Parameter Risk: At the Money Implied Volatility



Parameter Risk

- Good News! Parameter Risk is actually fairly easy to do in practice
- Can replace shock hierarchy with a deterministic model (mean of the hierarchy) $\sigma^2 + \beta\Delta\sigma^2$
- Final valuation model

$$\frac{\partial V}{\partial t} + (r-q)S\frac{\partial V}{\partial S} + \frac{(\sigma^2 + \beta\Delta\sigma^2)S^2}{2}\frac{\partial^2 V}{\partial S^2} - rV = -\pi[1-(1-\alpha)\beta]\frac{\partial V}{\partial \beta}$$

- Has convenient closed form solutions

Put the pieces together

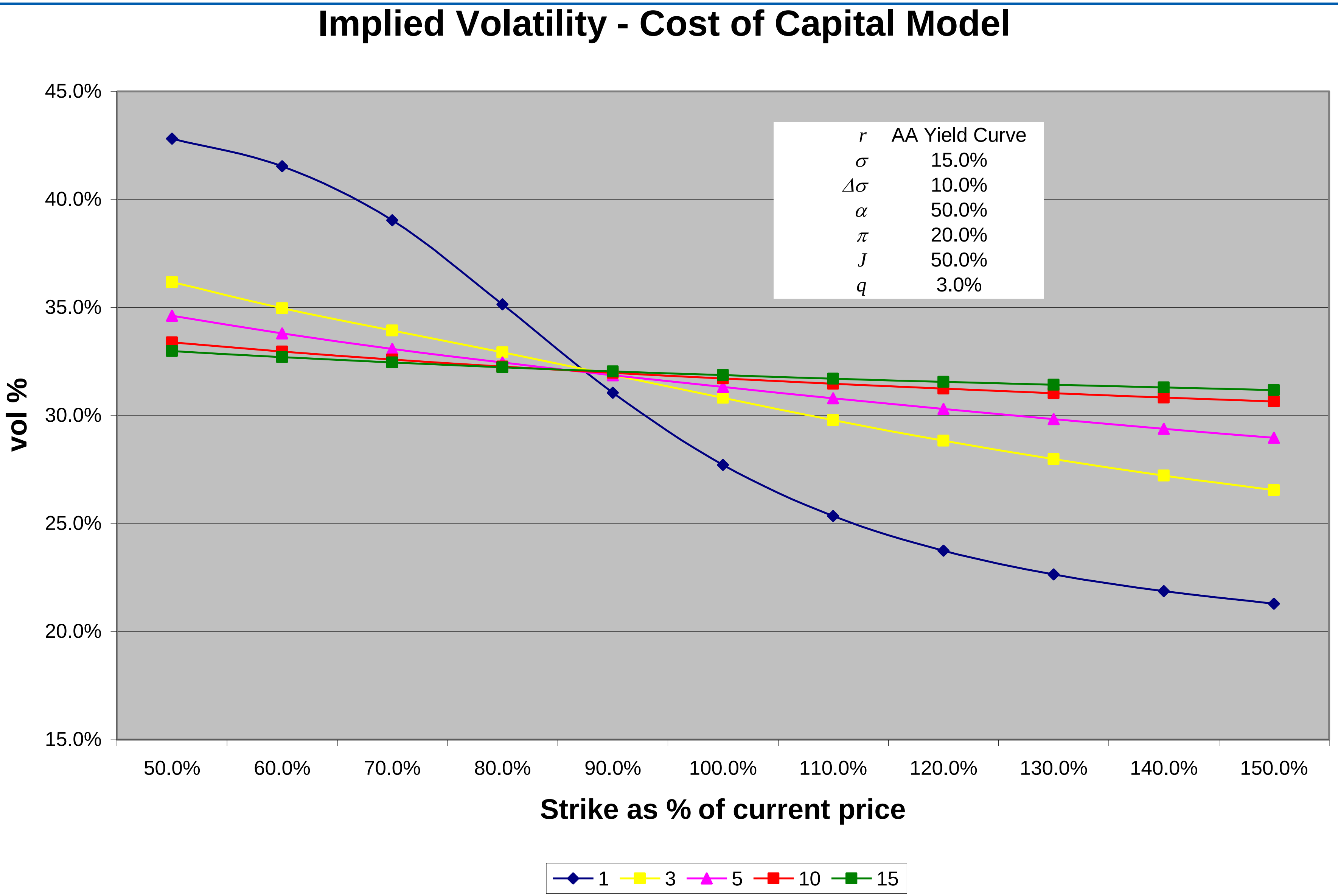
- Put parameter and contagion risk together

$$\frac{\partial V}{\partial t}+(r-q)S\frac{\partial V}{\partial S}+\pi[1-(1-\alpha)\beta]\frac{\partial V}{\partial \beta}+\frac{(\sigma^2+\beta\Delta\sigma^2)S^2}{2}\frac{\partial^2 V}{\partial S^2}-rV$$

$$=-\pi\left[V(t,JS)-V(t,S)-(J-1)S\frac{\partial V}{\partial S}\right].$$

- If we want to fit June 30, 2009 S&P 500 market data can use
  - J = 50%, π= 20%, q= 3.0%
  - $\Delta\sigma^2$ = 10%, α = 50%
- Reasonable fit for first 15 years

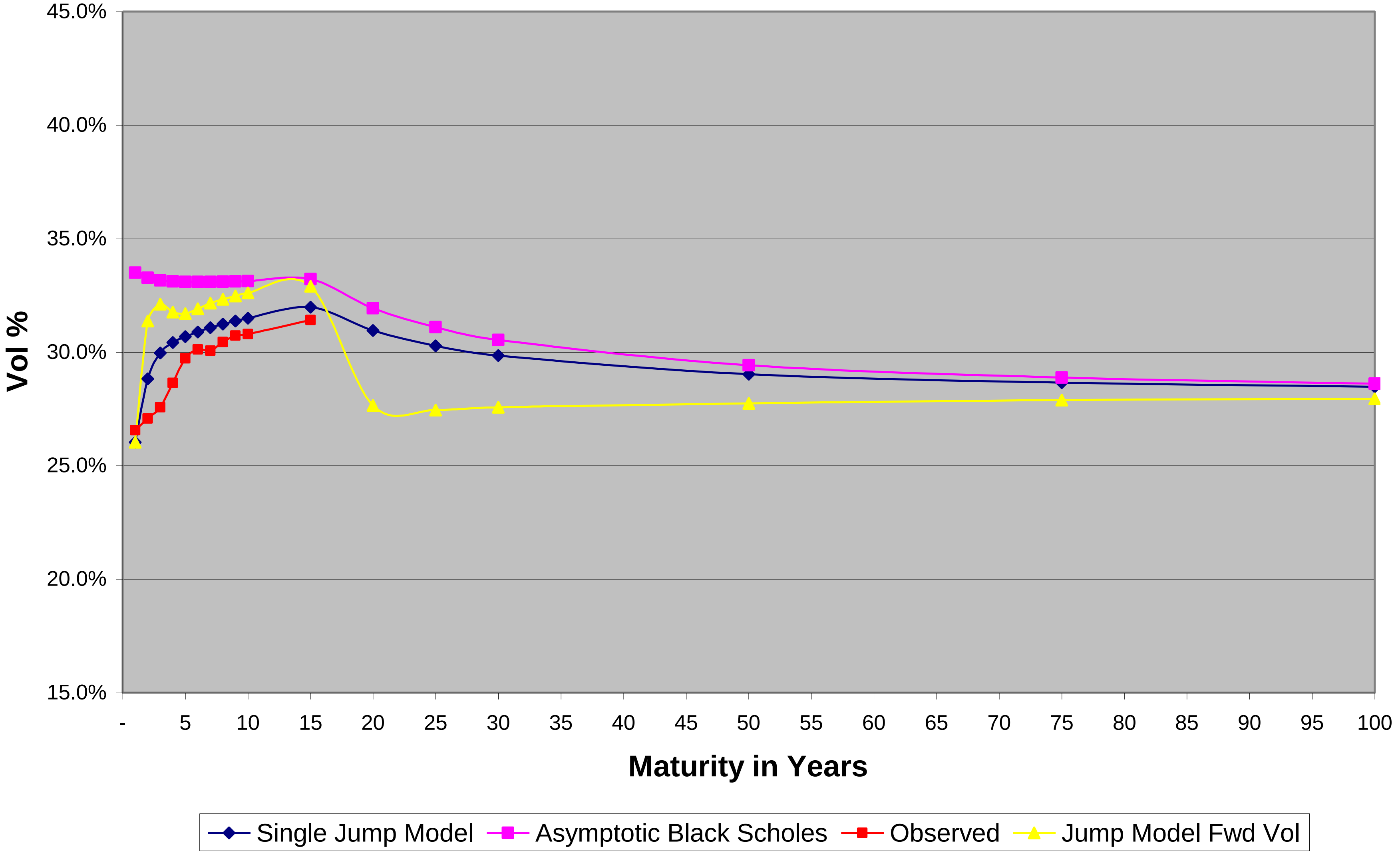
Put the pieces together



Final Step: Extrapolation

- Fit not perfect but appears to capture major risk issues
- As of June 30 ,2009 we are still in financial crisis mode
- Conclusion: must respect market data for first 15 years but can use more “reasonable” parameters after that time
- Example: Assume π goes to 10% after 15 years

At the Money Implied Volatility Extrapolation Assumptions



Asymptotic Black Scholes Implied Volatility