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ACTUARIAL MODELS STUDY NOTE

SECTION 8.5 FROM THE SECOND PRINTING OF *ACTUARIAL MATHEMATICS*, SECOND EDITION

by

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8.5 Allocation of the Risk to Insurance Years

In Section 8.3 recursion relations for benefit reserves are developed by an analysis of the insurer's annual cash income and cash outflow. Now we extend this analysis to an accrual or incurred basis and develop allocations of the risk, as measured by the variance of the loss variables, to the insurance years. Figure 8.5.1 shows in a time diagram the insurer's annual cash incomes, cash outflows, and changes in liability for the general fully discrete insurance of (8.2.1). The random variable C_h is related to the cash flows of the policy year $(h, h+1)$. We now define a random variable related to the total change in liability, cash flow, and reserves.

FIGURE 8.5.1

Insurer's Cash Incomes, Outflows, and Changes in Liability for Fully Discrete General Insurance

Outflow		0	0	0	$b_{K(x)+1}$	0	
Income	π_0	π_1	π_2	$\pi_{K(x)}$	0	0	
ΔLiability		${}_1V$	${}_2V-(1+i)\,{}_1V$	${}_{K(x)}V-(1+i)_{K(x)-1}V$	$-(1+i)_{K(x)}V$	0	etc.
	0	1	2	$K(x)$	$K(x)+1$	$K(x)+2$	

Let Λ_h denote the present value at h (a non-negative integer) of the insurer's cash loss plus change in liability during the year $(h, h+1)$. If $(h, h+1)$ is before the year of death $[h < K(x)]$, then

$$\Lambda_h = C_h + v\ \Delta\text{Liability} = -\pi_h + v\ {}_{h+1}V - {}_hV.$$

If $(h, h+1)$ is the year of death $[h = K(x)]$, then

$$\Lambda_h = C_h + v\Delta\text{Liability} = v\ b_{h+1} - \pi_h - {}_hV.$$

And if $(h, h+1)$ is after the year of death, of course $\Lambda_h = 0$. Restating this definition as a function of $K(x)$, and rearranging the terms,

$$\Lambda_h = \begin{cases} 0 & K(x) = 0, 1, \ldots, h-1 \\ (v\ b_{h+1} - \pi_h) + \qquad (-\ {}_hV) & K(x) = h \\ (-\pi_h) \qquad + (v\ {}_{h+1}V - {}_hV) & K(x) = h+1, h+2, \ldots \end{cases} \tag{8.5.1}$$

The definition of Λ_h in (8.5.1) can be rewritten to display Λ_h as the loss variable for a 1-year term insurance with a benefit equal to the amount at risk on the basic policy. See Exercise 8.31.

It follows that

$$\text{E}[\Lambda_h | K(x) \geq h] = v\ b_{h+1}\ q_{x+h} + v\ {}_{h+1}V\ p_{x+h} - (\pi_h + {}_hV), \tag{8.5.2}$$

which is zero by (8.3.10).

Since the conditional distribution of Λ_h, given $K(x) = h, h+1, \ldots$, is a two-point distribution, then

$$\text{Var}[\Lambda_h | K(x) \geq h] = [v(b_{h+1} - {}_{h+1}V)]^2\ p_{x+h}\ q_{x+h}. \tag{8.5.3}$$

With $j \leq h$ we can use (2.2.10) and (2.2.11) to obtain

$$\text{E}[\Lambda_h | K(x) \geq j] = 0 \tag{8.5.4}$$

and

$$\text{Var}[\Lambda_h | K(x) \geq j] = \text{Var}[\Lambda_h | K(x) \geq h] \; {}_{h-j}p_{x+j} \tag{8.5.5}$$

Unlike the C_h's of Section 8.3, the Λ_h's are uncorrelated, an assertion that is proved in the following lemma This fact conveys some sense of the role of reserves in stabilizing financial reporting of insurance operations.

Lemma 8.5.1:

For non-negative integers satisfying $g \leq h < j$,

$$\text{Cov}[\Lambda_h, \Lambda_j | K(x) \geq g] = 0. \tag{8.5.6}$$

Proof:

From (8.5.4), $\text{E}[\Lambda_h | K(x) \geq g] = 0$; therefore,

$$\text{Cov}[\Lambda_h, \Lambda_j | K(x) \geq g] = \text{E}[\Lambda_h \Lambda_j | K(x) \geq g].$$

From (8.5.1) we see that Λ_h is equal to the constant $(v \; {}_{h+1}V - {}_hV - \pi_h)$ where Λ_j is nonzero. Thus,

$$\Lambda_h \Lambda_j = (v \; {}_{h+1}V - {}_hV - \pi_h)\Lambda_j \qquad \text{for all } K(x), \tag{8.5.7}$$

$$\text{E}[\Lambda_h \Lambda_j | K(x) \geq g] = (v \; {}_{h+1}V - {}_hV - \pi_h)\text{E}[\Lambda_j | K(x) \geq g] = 0,$$

and

$$\text{Cov}[\Lambda_k \Lambda_j | K(x) \geq g] = 0. \qquad \blacksquare$$

We now express the loss variables ${}_hL$ in terms of the Λ_h's From the definition of the Λ_h's and formula (8.3.6),

$$\sum_{j=h}^{\infty} v^{j-h} \Lambda_j = \sum_{j=h}^{\infty} v^{j-h}[C_j + v\Delta\text{Liability } (j, j+1)]$$

$$= {}_hL + \sum_{j=h}^{\infty} v^{j-h+1} \, \Delta\text{Liability } (j, j+1). \tag{8.5.8}$$

Conceptually the last term will be the present value of the final liability minus the liability at h, that is, $0 - {}_hV$. Thus we have the relationship

$${}_hL = \begin{cases} 0 & K(x) < h \\ \sum_{j=h}^{\infty} v^{j-h} \Lambda_j + {}_hV & K(x) \geq h, \end{cases} \tag{8.5.9}$$

which can be rewritten as

$${}_hL = \begin{cases} 0 & K(x) < h \\ \sum_{j=h}^{h+i-1} v^{j-h} \Lambda_j + \sum_{j=h+i}^{\infty} v^{j-h} \Lambda_j + {}_hV & K(x) \geq h. \end{cases} \tag{8.5.10}$$

These relationships can be interpreted as stating that the present value of future losses, measured at time h following issue, is equal to the present value of future cash flows, adjusted for changes in reserves, plus the reserve at h

Using the representation of L_h shown in (8.5.9), we have

$$\begin{aligned}\mathrm{Var}[{}_hL|K(x) \geq h] &= \sum_{j=h}^{\infty} v^{2(j-h)}\,\mathrm{Var}[\Lambda_j|K(x) \geq h] \\ &= \sum_{j=h}^{\infty} v^{2(j-h)}\,{}_{j-h}p_{x+h}\,\mathrm{Var}[\Lambda_j|K(x) \geq j] \\ &= \sum_{j=h}^{\infty} v^{2(j-h)}\,{}_{j-h}p_{x+h}\{[v(b_{j+1} - {}_{j+1}V)]^2\, p_{x+j}q_{x+j}\} \qquad (8.5.11)\end{aligned}$$

In this development the first line makes use of Lemma 8.5.1, the second (8.5.5), and the third (8.5.3).

Starting with (8.5.10) we can follow identical steps to obtain

$$\begin{aligned}\mathrm{Var}[{}_hL|K(x) \geq h] &= \sum_{j=h}^{h+i-1} v^{2(j-h)}\,\mathrm{Var}[\Lambda_j|K(x) \geq h] \\ &\quad + \sum_{j=h+i}^{\infty} v^{2(j-h)}\,\mathrm{Var}[\Lambda_j|K(x) \geq h] \\ &= \sum_{j=h}^{h+i-1} v^{2(j-h)}\,{}_{j-h}p_{x+h}\{[v(b_{j+1} - {}_{j+1}V)]^2\, p_{x+j}q_{x+j}\} \\ &\quad + \sum_{j=h+i}^{\infty} v^{2(j-h)}\,{}_{j-h}p_{x+h}\{[v(b_{j+1} - {}_{j+1}V)]^2\, p_{x+j}q_{x+j}\}. \qquad (8.5.12)\end{aligned}$$

The second summation can be rewritten by replacing the summation variable j by $l + h$ to obtain

$$\begin{aligned}&\sum_{l=i}^{\infty} v^{2l}\,{}_lp_{x+h}\{[v(b_{h+l+1} - {}_{h+l+1}V)]^2\, p_{x+h+l}q_{x+h+l}\} \\ &= v^{2i}\,{}_ip_{x+h} \sum_{l=i}^{\infty} v^{2(l-i)}\,{}_{l-i}p_{x+h+i}\,\{[v(b_{h+l+1} - {}_{h+l+1}V)]^2\, p_{x+h+l}q_{x+h+l}\} \\ &= v^{2i}\,{}_ip_{x+h}\,\mathrm{Var}[{}_{h+i}L|K(x) \geq h + i]. \qquad (8.5.13)\end{aligned}$$

The main results of these developments will be summarized as a theorem.

Theorem 8.5.1

$$\mathrm{Var}({}_hL|K(x) \geq h)$$

a. $$= \sum_{j=h}^{\infty} v^{2(j-h)}\,\mathrm{Var}[\Lambda_j|K(x) \geq h] \qquad (8.5.14)$$

b. $$= \sum_{j=h}^{\infty} v^{2(j-h)}\,{}_{j-h}p_{x+h}\,\{[v(b_{j+1} - {}_{j+1}V)]^2\, p_{x+j}q_{x+j}\} \qquad (8.5.15)$$

c. $$= \sum_{j=h}^{h+i-1} v^{2(j-h)}\,{}_{j-h}p_{x+h}\{[v(b_{j+1} - {}_{j+1}V)]^2\, p_{x+j}q_{x+j}\} + v^{2i}\,{}_ip_{x+h}\,\mathrm{Var}[{}_{h+i}L|K(x) \geq h + i]. \qquad (8.5.16)$$

Proof:

(a) follows from (8.5.11), first line
(b) follows from (8.5.11), third line
(c) follows from (8.5.12) and (8.5.13).

We refer to this theorem as the Hattendorf theorem, and we illustrate its application in the following two examples. Items (b) and (c) of the theorem can be used as backward recursion formulas that are useful for understanding the duration allocation of risk and, perhaps, for computing.

Just as the random variables C, introduced in (8.3.1), allocate each loss to insurance years, and the random variables Λ, introduced in (8.5.1), allocate cash loss and liability adjustment to insurance years, the Hattendorf theorem facilitates the allocation of mortality risk, as measured by $\mathrm{Var}[{}_hL|K(x) \geq h]$ to insurance years. This allocation facilitates risk management planning for a limited number of future insurance years rather than for the entire insurance period. This option permits sequential risk management decisions.

The formula $\mathrm{Var}[\Lambda_h|K(x) \geq h] = [v(b_{h+1} - {}_{h+1}V)]^2 p_{x+h} q_{x+h}$ confirms that the amount at risk $(b_{h+1} - {}_{h+1}V)$ is a major determinate of mortality risk, as measured by the variance. In fact if $b_{h+1} = {}_{h+1}V$ for all non-negative integer values of h, mortality risk drops to zero.

Example 8.5.1

Consider an insured from Example 7.4.3 who has survived to the end of the second policy year. For this insured, evaluate

a. $\mathrm{Var}[{}_2L|K(50) \geq 2]$ directly
b. $\mathrm{Var}[{}_2L|K(50) \geq 2]$ by means of the Hattendorf theorem
c. $\mathrm{Var}[{}_3L|K(50) \geq 3]$
d. $\mathrm{Var}[{}_4L|K(50) \geq 4]$.

Solution:

a. For the direct calculation, we need a table of values for ${}_2L$.

Outcome of $K(50) - 2 = j$	${}_2L$	Conditional Probability of Outcome
0	$1{,}000v - 6.55692\,\ddot{a}_{\overline{1}|} = 936.84$	${}_{0|}q_{52} = 0.0069724$
1	$1{,}000v^2 - 6.55692\,\ddot{a}_{\overline{2}|} = 877.25$	${}_{1|}q_{52} = 0.0075227$
2	$1{,}000v^3 - 6.55692\,\ddot{a}_{\overline{3}|} = 821.04$	${}_{2|}q_{52} = 0.0081170$
≥ 3	$0 - 6.55692\,\ddot{a}_{\overline{3}|} = -18.58$	${}_3p_{52} = 0.9773879$

Then $\mathrm{E}[{}_2L|K(50) \geq 2] = 1.64$, in agreement with the value shown in Example 7.4.3 and

$$\text{Var}[{}_2L|K(50) \geq 2] = \text{E}[{}_2L^2|K(50) \geq 2] - (\text{E}[{}_2L|K(50) \geq 2])^2$$
$$= 17{,}717.82 - (1.64)^2$$
$$= 17{,}715.1.$$

b. To apply the Hattendorf theorem, we can use the benefit reserves from Example 7.4.3 to calculate the variances of the losses associated with the 1-year term insurances.

j	q_{52+j}	$v^2\,(1{,}000 - 1{,}000\ {}_{2+j+1}V^1_{50:\overline{5}|})^2\ p_{52+j}\ q_{52+j}$
0	0.0069724	6 140.842
1	0.0075755	6 674.910
2	0.0082364	7 269.991

Then by (8.5.15),

$$\text{Var}[{}_2L|K(50) \geq 2] = 6{,}140.842 + (1.06)^{-2}(6{,}674.910)p_{52}$$
$$+ (1.06)^{-4}(7{,}269.991)\,{}_2p_{52} = 17{,}715.1,$$

which agrees with the value found by the direct calculation in part (a).

Note that in the direct method it was necessary to consider the gain in the event of survival to age 55; but for the Hattendorf theorem, we need to consider only the losses associated with the 1-year term insurances for the net amounts at risk in the remaining policy years. Thereafter, the net amount at risk is 0, and the corresponding terms in (8.5.15) vanish.

Also note that the standard deviation, $\sqrt{17{,}751.1} = 133.1$, for a single policy is more than 80 times the benefit reserve, $\text{E}[{}_2L|K(50)=2, 3, \ldots] = 1.64$.

Similarly, we use (8.5.15) to calculate

c. $\text{Var}[{}_3L|K(50) \geq 3] = 6{,}674.910 + (1.06)^{-2}(7{,}269.991)\ p_{53} = 13{,}096.2$

d. $\text{Var}[{}_4L|K(50) \geq 4] = 7{,}269.991$, or after rounding, 7,270.0. ▼

Example 8.5.2

Consider a portfolio of 1,500 policies of the type described in Example 7.4.3 and discussed in Example 8.5.1. Assume all policies have annual premiums due immediately. Further, assume 750 policies are at duration 2, 500 are at duration 3, and 250 are at duration 4, and that the policies in each group are evenly divided between those with 1,000 face amount and those with 3,000 face amount

a. Calculate the aggregate benefit reserve.
b. Calculate the variance of the prospective losses over the remaining periods of coverage of the policies assuming such losses are independent. Also, calculate the amount which, on the basis of the normal approximation, will give the insurer a probability of 0.95 of meeting the future obligations to this block of business.
c. Calculate the variance of the losses associated with the 1-year term insurances for the net amounts at risk under the policies and the amount of supplement to

the aggregate benefit reserve that, on the basis of the normal approximation, will give the insurer a probability of 0.95 of meeting the obligations to this block of business for the 1-year period.

d Redo (b) and (c) with each set of policies increased 100-fold in number.

Solution:

a Let Z be the sum of the prospective losses on the 1,500 policies. The symbols E[Z] and Var(Z) used below for the mean and variance of the portfolio of 1,500 policies are abridged, for in both cases the expectations are to be computed with respect to the set of conditions given above for the insureds. Using the results of Example 7.4.3, we have for the aggregate benefit reserve

$$\begin{aligned} E[Z] &= [375(1) + 375(3)](1.64) + [250(1) + 250(3)](1.73) \\ &\quad + [125(1) + 125(3)](1.21) \\ &= 4{,}795. \end{aligned}$$

b. From Example 8.5.1, we have

$$\begin{aligned} \mathrm{Var}(Z) &= [375(1) + 375(9)](17{,}715.1) \\ &\quad + [250(1) + 250(9)](13{,}096.2) \\ &\quad + [125(1) + 125(9)](7{,}270.0) \\ &= (1.0825962) \times 10^8 \end{aligned}$$

and $\sigma_Z = 10{,}404.8$

Then, if

$$0.05 = \Pr(Z > c) = \Pr\left(\frac{Z - 4{,}795.0}{10{,}404.8} > \frac{c - 4{,}795.0}{10{,}404.8}\right),$$

the normal approximation would imply

$$\frac{c - 4{,}795.0}{10{,}404.8} = 1.645,$$

or

$$c = 21{,}911,$$

which is 4.6 times the aggregate benefit reserve, E[Z].

c Here we take account of only the next year's risk. For each policy, we consider a variable equal to the loss associated with a 1-year term insurance for the net amount at risk. Let Z_1 be the sum of these loss variables. The expected loss for each of the 1-year term insurances is 0, hence $E[Z_1] = 0$.

From the table in part (b) of Example 8.5.1 we can obtain the variances of the losses in regard to the 1-year term insurances, and hence

$$\begin{aligned}\mathrm{Var}(Z_1) &= [375(1) + 375(9)](6{,}140.8) + [250(1) + 250(9)](6{,}674.9) \\ &\quad + [125(1) + 125(9)](7{,}270.0) \\ &= (4.880275) \times 10^7\end{aligned}$$

and $\sigma_{Z_1} = 6985.9$.

If c_1 is the required supplement to the aggregate benefit reserve, then

$$0.05 = \Pr(Z_1 > c_1) = \Pr\left(\frac{Z_1 - 0}{6{,}985.9} > \frac{c_1 - 0}{6{,}985.9}\right),$$

and we determine, again by the normal approximation,

$$c_1 = (1.645)(6{,}985.9) = 11{,}492,$$

which is 2.4 times the aggregate benefit reserve 4,795.

d. In this case, $E[Z] = 479{,}500$ and $\mathrm{Var}(Z) = (1.0825962) \times 10^{10}$. By the normal approximation the amount c required to provide a probability of 0.95 that all future obligations will be met is

$$479{,}500 + 1.645\sqrt{1.0825962} \times 10^5 = 650{,}659,$$

which is 1.36 times the aggregate benefit reserve $E[Z]$.

Also, $\mathrm{Var}(Z_1)$ is now $(4.880275) \times 10^9$. The amount c_1 of supplement to the aggregate benefit reserve required to give a 0.95 probability that the insurer can meet policy obligations for the next year is $1.645\sqrt{4.880275} \times 10^{4.5} = 114{,}918$, or 24% of the aggregate benefit reserve. ▼