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ACTUARIAL MODELS—LIFE CONTINGENCIES SEGMENT

SELECTED SECTIONS OF CHAPTER 5 FROM

INTRODUCTION TO PROBABILITY MODELS

by

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Sections 5.3.1, 5.3.2 (through Definition 5.1), 5.3.3, 5.3.4 (through Example 5.14 but excluding Example 5.13), Proposition 5.3 and the preceding paragraph, Example 5.18, 5.4.1(up to example 5.23), 5.4.2 (excluding Example 5.25), 5.4.3, and Exercise 40.

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5.3. The Poisson Process

5.3.1. Counting Processes

A stochastic process $\{N(t), t \geqslant 0\}$ is said to be a *counting process* if $N(t)$ represents the total number of "events" that occur by time t Some examples of counting processes are the following:

(a) If we let $N(t)$ equal the number of persons who enter a particular store at or prior to time t, then $\{N(t), t \geqslant 0\}$ is a counting process in which an event corresponds to a person entering the store. Note that if we had let $N(t)$ equal the number of persons in the store at time t, then $\{N(t), t \geqslant 0\}$ would *not* be a counting process (why not?)
(b) If we say that an event occurs whenever a child is born, then $\{N(t), t \geqslant 0\}$ is a counting process when $N(t)$ equals the total number of people who were born by time t [Does $N(t)$ include persons who have died by time t? Explain why it must.]
(c) If $N(t)$ equals the number of goals that a given soccer player scores by time t, then $\{N(t), t \geqslant 0\}$ is a counting process. An event of this process will occur whenever the soccer player scores a goal.

From its definition we see that for a counting process $N(t)$ must satisfy:

(i) $N(t) \geqslant 0$.
(ii) $N(t)$ is integer valued.
(iii) If $s < t$, then $N(s) \leqslant N(t)$.
(iv) For $s < t$, $N(t) - N(s)$ equals the number of events that occur in the interval $(s, t]$.

A counting process is said to possess *independent increments* if the numbers of events that occur in disjoint time intervals are independent. For example, this means that the number of events that occur by time 10 [that is, $N(10)$] must be independent of the number of events that occur between times 10 and 15 [that is, $N(15) - N(10)$].

The assumption of independent increments might be reasonable for example (a), but it probably would be unreasonable for example (b). The reason for this is that if in example (b) $N(t)$ is very large, then it is probable that there are many people alive at time t; this would lead us to believe that the number of new births between time t and time $t + s$ would also tend to be large [that is, it does not seem reasonable that $N(t)$ is independent of $N(t+s) - N(t)$, and so $\{N(t), t \geqslant 0\}$ would not have independent increments in example (b)]. The assumption of independent increments in example (c) would be justified if we believed that the soccer player's chances of scoring a goal today do not depend on "how he's been going." It would not be justified if we believed in "hot streaks" or "slumps."

A counting process is said to possess *stationary increments* if the distribution of the number of events that occur in any interval of time depends only on the length of the time interval. In other words, the process has stationary increments if the number of events in the interval $(s, s+t)$ has the same distribution for all s

The assumption of stationary increments would only be reasonable in example (a) if there were no times of day at which people were more likely to enter the store. Thus, for instance, if there was a rush hour (say, between 12 P.M. and 1 P.M.) each day, then the stationarity assumption would not be justified. If we believed that the earth's population is basically constant (a belief not held at present by most scientists), then the assumption of stationary increments might be reasonable in example (b). Stationary increments do not seem to be a reasonable assumption in example (c) since, for one thing, most people would agree that the soccer player would probably score more goals while in the age bracket 25–30 than he would while in the age bracket 35–40. It may, however, be reasonable over a smaller time horizon, such as one year.

5.3.2. Definition of the Poisson Process

One of the most important counting processes is the Poisson process which is defined as follows:

Definition 5.1 The counting process $\{N(t), t \geqslant 0\}$ is said to be a *Poisson process having rate* λ, $\lambda > 0$, if

(i) $N(0) = 0$.
(ii) The process has independent increments.
(iii) The number of events in any interval of length t is Poisson distributed with mean λt. That is, for all s, $t \geqslant 0$

$$P\{N(t+s) - N(s) = n\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \qquad n = 0, 1, \ldots$$

Note that it follows from condition (iii) that a Poisson process has stationary increments and also that

$$E[N(t)] = \lambda t$$

which explains why λ is called the rate of the process.

To determine if an arbitrary counting process is actually a Poisson process, we must show that conditions (i), (ii), and (iii) are satisfied. Condition (i), which simply states that the counting of events begins at time $t = 0$, and condition (ii) can usually be directly verified from our knowledge of the process. However, it is not at all clear how we would determine that condition (iii) is satisfied, and for this reason an equivalent definition of a Poisson process would be useful.

As a prelude to giving a second definition of a Poisson process we shall define the concept of a function $f(\cdot)$ being $o(h)$.

5.3.3. Interarrival and Waiting Time Distributions

Consider a Poisson process, and let us denote the time of the first event by T_1. Further, for $n > 1$, let T_n denote the elapsed time between the $(n-1)$st and the nth event. The sequence $\{T_n, n = 1, 2, \ldots\}$ is called the *sequence of interarrival times*. For instance, if $T_1 = 5$ and $T_2 = 10$, then the first event of the Poisson process would have occurred at time 5 and the second at time 15.

We shall now determine the distribution of the T_n. To do so, we first note that the event $\{T_1 > t\}$ takes place if and only if no events of the Poisson process occur in the interval $[0, t]$ and thus,

$$P\{T_1 > t\} = P\{N(t) = 0\} = e^{-\lambda t}$$

Hence, T_1 has an exponential distribution with mean $1/\lambda$. Now,

$$P\{T_2 > t\} = E[P\{T_2 > t | T_1\}]$$

However,

$$\begin{aligned} P\{T_2 > t \mid T_1 = s\} &= P\{0 \text{ events in } (s, s+t] \mid T_1 = s\} \\ &= P\{0 \text{ events in } (s, s+t]\} \\ &= e^{-\lambda t} \end{aligned} \tag{5.12}$$

where the last two equations followed from independent and stationary increments. Therefore, from Equation (5.12) we conclude that T_2 is also an exponential random variable with mean $1/\lambda$ and, furthermore, that T_2 is independent of T_1. Repeating the same argument yields the following.

Proposition 5.1 $T_n, n = 1, 2, \ldots,$ are independent identically distributed exponential random variables having mean $1/\lambda$.

Remark The proposition should not surprise us. The assumption of stationary and independent increments is basically equivalent to asserting that, at any point in time, the process *probabilistically* restarts itself. That is, the process from any point on is independent of all that has previously occurred (by independent increments), and also has the same distribution as the original process (by stationary increments). In other words, the process has no *memory*, and hence exponential interarrival times are to be expected.

Another quantity of interest is S_n, the arrival time of the nth event, also called the *waiting time* until the nth event It is easily seen that

$$S_n = \sum_{i=1}^{n} T_i, \qquad n \geqslant 1$$

and hence from Proposition 5 1 and the results of Section 2.2 it follows that S_n has a gamma distribution with parameters n and λ That is, the probability density of S_n is given by

$$f_{S_n}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}, \qquad t \geqslant 0 \tag{5.13}$$

Equation (5 13) may also be derived by noting that the nth event will occur prior to or at time t if and only if the number of events occurring by time t is at least n. That is,

$$N(t) \geqslant n \quad \Leftrightarrow \quad S_n \leqslant t$$

Hence,

$$F_{S_n}(t) = P\{S_n \leqslant t\} = P\{N(t) \geqslant n\} = \sum_{j=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!}$$

which, upon differentiation, yields

$$\begin{aligned} f_{S_n}(t) &= -\sum_{j=n}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^j}{j!} + \sum_{j=n}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^{j-1}}{(j-1)!} \\ &= \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} + \sum_{j=n+1}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^{j-1}}{(j-1)!} - \sum_{j=n}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^j}{j!} \\ &= \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \end{aligned}$$

Example 5.11 Suppose that people immigrate into a territory at a Poisson rate $\lambda = 1$ per day.

(a) What is the expected time until the tenth immigrant arrives?

(b) What is the probability that the elapsed time between the tenth and the eleventh arrival exceeds two days?

Solution:
(a) $E[S_{10}] = 10/\lambda = 10$ days.
(b) $P\{T_{11} > 2\} = e^{-2\lambda} = e^{-2} \approx 0.133$ ■

Proposition 5.1 also gives us another way of defining a Poisson process. Suppose we start with a sequence $\{T_n, n \geqslant 1\}$ of independent identically distributed exponential random variables each having mean $1/\lambda$. Now let us define a counting process by saying that the nth event of this process occurs at time

$$S_n \equiv T_1 + T_2 + \cdots + T_n$$

The resultant counting process $\{N(t), t \geqslant 0\}$* will be Poisson with rate λ.

Remark Another way of obtaining the density function of S_n is to note that because S_n is the time of the nth event,

$$\begin{aligned} P\{t < S_n < t+h\} &= P\{N(t) = n-1, \text{ one event in } (t, t+h)\} + o(h) \\ &= P\{N(t) = n-1\}P\{\text{one event in } (t, t+h)\} + o(h) \\ &= e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}[\lambda h + o(h)] + o(h) \\ &= \lambda e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}h + o(h) \end{aligned}$$

where the first equality uses the fact that the probability of 2 or more events in $(t, t+h)$ is $o(h)$. If we now divide both sides of the preceding equation by h and then let $h \to 0$, we obtain

$$f_{S_n}(t) = \lambda e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}$$

5.3.4. Further Properties of Poisson Processes

Consider a Poisson process $\{N(t), t \geqslant 0\}$ having rate λ, and suppose that each time an event occurs it is classified as either a type I or a type II event. Suppose further that each event is classified as a type I event with probability p or a type II event with probability $1-p$, independently of all other events. For example, suppose that customers arrive at a store in accordance with a Poisson process having rate λ; and suppose that each arrival is male with probability $\frac{1}{2}$ and female with probability $\frac{1}{2}$.

*A formal definition of $N(t)$ is given by $N(t) \equiv \max\{n: S_n \leqslant t\}$ where $S_0 \equiv 0$

Then a type I event would correspond to a male arrival and a type II event to a female arrival.

Let $N_1(t)$ and $N_2(t)$ denote respectively the number of type I and type II events occurring in $[0, t]$. Note that $N(t) = N_1(t) + N_2(t)$

Proposition 5.2 $\{N_1(t), t \geqslant 0\}$ and $\{N_2(t), t \geqslant 0\}$ are both Poisson processes having respective rates λp and $\lambda(1-p)$. Furthermore, the two processes are independent.

Proof It is easy to verify that $\{N_1(t), t \geqslant 0\}$ is a Poisson process with rate λp by verifying that it satisfies Definition 5.3.

- $N_1(0) = 0$ follows from the fact that $N(0) = 0$.
- It is easy to see that $\{N_1(t), t \geqslant 0\}$ inherits the stationary and independent increment properties of the process $\{N(t), t \geqslant 0\}$. This is true because the distribution of the number of type I events in an interval can be obtained by conditioning on the number of events in that interval, and the distribution of this latter quantity depends only on the length of the interval and is independent of what has occurred in any nonoverlapping interval.
- $$\begin{aligned} P\{N_1(h) = 1\} &= P\{N_1(h) = 1 \mid N(h) = 1\}P\{N(h) = 1\} \\ &\quad + P\{N_1(h) = 1 \mid N(h) \geqslant 2\}P\{N(h) \geqslant 2\} \\ &= p(\lambda h + o(h)) + o(h) \\ &= \lambda p h + o(h) \end{aligned}$$
- $P\{N_1(h) \geqslant 2\} \leqslant P\{N(h) \geqslant 2\} = o(h)$

Thus we see that $\{N_1(t), t \geqslant 0\}$ is a Poisson process with rate λp and, by a similar argument, that $\{N_2(t), t \geqslant 0\}$ is a Poisson process with rate $\lambda(1-p)$. Because the probability of a type I event in the interval from t to $t + h$ is independent of all that occurs in intervals that do not overlap $(t, t+h)$, it is independent of knowledge of when type II events occur, showing that the two Poisson processes are independent. (For another way of proving independence, see Example 3.20.) ■

Example 5.12 If immigrants to area A arrive at a Poisson rate of ten per week, and if each immigrant is of English descent with probability $\frac{1}{12}$, then what is the probability that no people of English descent will emigrate to area A during the month of February?

Solution: By the previous proposition it follows that the number of Englishmen emigrating to area A during the month of February is Poisson distributed with mean $4 \cdot 10 \cdot \frac{1}{12} = \frac{10}{3}$. Hence the desired probability is $e^{-10/3}$. ■

It follows from Proposition 5.2 that if each of a Poisson number of individuals is independently classified into one of two possible groups with respective probabilities p and $1 - p$, then the number of individuals in each of the two groups will be independent Poisson random variables. Because this result easily generalizes to the case where the classification is into any one of r possible groups, we have the following application to a model of employees moving about in an organization

Example 5.14 Consider a system in which individuals at any time are classified as being in one of r possible states, and assume that an individual changes states in accordance with a Markov chain having transition probabilities $P_{ij}, i, j = 1, \ldots, r$ That is, if an individual is in state i during a time period then, independently of its previous states, it will be in state j during the next time period with probability P_{ij}. The individuals are assumed to move through the system independently of each other. Suppose that the numbers of people initially in states $1, 2, \ldots, r$ are independent Poisson random variables with respective means $\lambda_1, \lambda_2, \ldots, \lambda_r$. We are interested in determining the joint distribution of the numbers of individuals in states $1, 2, \ldots, r$ at some time n.

Solution: For fixed i, let $N_j(i), j = 1, \ldots, r$ denote the number of those individuals, initially in state i, that are in state j at time n. Now each of the (Poisson distributed) number of people initially in state i will, independently of each other, be in state j at time n with probability P_{ij}^n, where P_{ij}^n is the n-stage transition probability for the Markov chain having transition probabilities P_{ij} Hence, the $N_j(i), j = 1, \ldots, r$ will be independent Poisson random variables with respective means $\lambda_i P^n_{ij}$, $j = 1, \ldots, r$. Because the sum of independent Poisson random variables is itself a Poisson random variable, it follows that the number of individuals in state j at time n—namely $\sum_{i=1}^{r} N_j(i)$—will be independent Poisson random variables with respective means $\sum_i \lambda_i P^n_{ij}$, for $j = 1, \ldots, r$. ■

Application of Theorem 5.2 (Sampling a Poisson Process) In Proposition 5.2 we showed that if each event of a Poisson process is independently classified as a type I event with probability p and as a type II event with probability $1 - p$ then the counting processes of type I and type II events are independent Poisson processes with respective rates λp and $\lambda(1 - p)$. Suppose now, however, that there are k possible types of events and that the probability that an event is classified as a type i event, $i = 1, \ldots, k$, depends on the time the event occurs. Specifically, suppose that if an event occurs at time y then it will be classified as a type i event, independently of anything that has previously occurred, with probability $P_i(y)$, $i = 1, \ldots, k$ where $\sum_{i=1}^{k} P_i(y) = 1$. Upon using Theorem 5.2 we can prove the following useful proposition.

Proposition 5.3 If $N_i(t)$, $i = 1, \ldots, k$, represents the number of type i events occurring by time t then $N_i(t)$, $i = 1, \ldots, k$, are independent Poisson random variables having means

$$E[N_i(t)] = \lambda \int_0^t P_i(s)\, ds$$

Example 5.18 (Tracking the Number of HIV Infections) There is a relatively long incubation period from the time when an individual becomes infected with the HIV virus, which causes AIDS, until the symptoms of the disease appear. As a result, it is difficult for public health officials to be certain of the number of members of the population that are infected at any given time We will now present a first approximation model for this phenomenon, which can be used to obtain a rough estimate of the number of infected individuals.

Let us suppose that individuals contract the HIV virus in accordance with a Poisson process whose rate λ is unknown. Suppose that the time from when an individual becomes infected until symptoms of the disease appear is a random variable having a known distribution G. Suppose also that the incubation times of different infected individuals are independent.

Let $N_1(t)$ denote the number of individuals who have shown symptoms of the disease by time t. Also, let $N_2(t)$ denote the number who are HIV positive but have not yet shown any symptoms by time t. Now, since an individual who contracts the virus at time s will have symptoms by time t with probability $G(t-s)$ and will not with probability $\bar{G}(t-s)$, it follows from Proposition 5.3 that $N_1(t)$ and $N_2(t)$ are independent Poisson random variables with respective means

$$E[N_1(t)] = \lambda \int_0^t G(t-s)\,ds = \lambda \int_0^t G(y)\,dy$$

and

$$E[N_2(t)] = \lambda \int_0^t \bar{G}(t-s)\,ds = \lambda \int_0^t \bar{G}(y)\,dy$$

Now, if we knew λ, then we could use it to estimate $N_2(t)$, the number of individuals infected but without any outward symptoms at time t, by its mean value $E[N_2(t)]$. However, since λ is unknown, we must first estimate it. Now, we will presumably know the value of $N_1(t)$, and so we can use its known value as an estimate of its mean $E[N_1(t)]$. That is, if the number of individuals who have exhibited symptoms by time t is n_1, then we can estimate that

$$n_1 \approx E[N_1(t)] = \lambda \int_0^t G(y)\,dy$$

Therefore, we can estimate λ by the quantity $\hat{\lambda}$ given by

$$\hat{\lambda} = n_1 \Big/ \int_0^t G(y)\,dy$$

Using this estimate of λ, we can estimate the number of infected but symptomless individuals at time t by

$$\begin{aligned}\text{estimate of } N_2(t) &= \hat{\lambda} \int_0^t \bar{G}(y)\,dy \\ &= \frac{n_1 \int_0^t \bar{G}(y)\,dy}{\int_0^t G(y)\,dy}\end{aligned}$$

For example, suppose that G is exponential with mean μ. Then $\bar{G}(y) = e^{-y/\mu}$, and a simple integration gives that

$$\text{estimate of } N_2(t) = \frac{n_1 \mu (1 - e^{-t/\mu})}{t - \mu(1 - e^{-t/\mu})}$$

If we suppose that $t = 16$ years, $\mu = 10$ years, and $n_1 = 220$ thousand, then the estimate of the number of infected but symptomless individuals at time 16 is

$$\text{estimate} = \frac{2{,}200(1 - e^{-1.6})}{16 - 10(1 - e^{-1.6})} = 218.96$$

That is, if we suppose that the foregoing model is approximately correct (and we should be aware that the assumption of a constant infection rate λ that is unchanging over time is almost certainly a weak point of the model), then if the incubation period is exponential with mean 10 years and if the total number of individuals who have exhibited AIDS symptoms during the first 16 years of the epidemic is 220 thousand, then we can expect that approximately 219 thousand individuals are HIV positive though symptomless at time 16. ■

5.4. Generalizations of the Poisson Process

5.4.1. Nonhomogeneous Poisson Process

In this section we consider two generalizations of the Poisson process The first of these is the nonhomogeneous, also called the nonstationary, Poisson process, which is obtained by allowing the arrival rate at time t to be a function of t.

Definition 5.4 The counting process $\{N(t), t \geqslant 0\}$ is said to be a *nonhomogeneous Poisson process with intensity function* $\lambda(t),\ t \geqslant 0$, if

(i) $N(0) = 0$.
(ii) $\{N(t),\ t \geqslant 0\}$ has independent increments.
(iii) $P\{N(t+h) - N(t) \geqslant 2\} = o(h)$.
(iv) $P\{N(t+h) - N(t) = 1\} = \lambda(t)h + o(h)$.

If we let

$$m(t) = \int_0^t \lambda(y)\,dy$$

then it can be shown that

$$P\{N(s+t)-N(s) = n\} = e^{-[m(s+t)-m(s)]}\frac{[m(s+t) - m(s)]^n}{n!}, \quad n \geqslant 0 \qquad (5.23)$$

That is, $N(s+t) - N(s)$ is a Poisson random variable with mean $m(s+t) - m(s)$. Since this implies that $N(t)$ is Poisson with mean $m(t)$, we call $m(t)$ the *mean value function* of the nonhomogeneous Poisson process.

The proof of Equation (5.23) follows along the lines of the proof of Theorem 5.1, with a slight modification. Fix nonnegative values s and u, let

$$N_s(t) = N(s+t) - N(s)$$

and define

$$g(t) = E[\exp\{-uN_s(t)\}]$$

Then,

$$\begin{aligned} g(t+h) &= E[\exp\{-uN_s(t+h)\}] \\ &= E[\exp\{-uN_s(t)\}\exp\{-u(N_s(t+h) - N_s(t))\}] \\ &= g(t)\ E[\exp\{-u(N_s(t+h) - N_s(t))\}] \qquad \text{by independent increments} \end{aligned}$$

Conditioning on $N(s+t+h) - N(s+t)$ yields

$$\begin{aligned} E[\exp\{-u(N_s(t+h) - N_s(t))\}] &= E[\exp\{-u(N(s+t+h) - N(s+t))\}] \\ &= 1 - \lambda(s+t)h + e^{-u}\lambda(s+t)h + o(h) \end{aligned}$$

Hence,

$$g(t+h) = g(t)(1 - \lambda(s+t)h + e^{-u}\lambda(s+t)h) + o(h)$$

or

$$\frac{g(t+h) - g(t)}{h} = g(t)\,\lambda(s+t)(e^{-u} - 1) + \frac{o(h)}{h}$$

Letting $h \to 0$ gives

$$g'(t) = g(t)\,\lambda(s+t)(e^{-u} - 1)$$

or, equivalently,

$$\frac{g'(t)}{g(t)} = \lambda(s+t)(e^{-u} - 1)$$

Consequently,

$$\int_0^t \frac{g'(y)}{g(y)}\,dy = (e^{-u} - 1)\int_0^t \lambda(s+y)\,dy$$

Since $g(0) = 1$, the preceding yields that

$$\log g(t) = (e^{-u} - 1)\int_0^t \lambda(s+y)\,dy$$

or

$$g(t) = \exp\left\{\int_0^t \lambda(s+y)\,dy(e^{-u} - 1)\right\}$$

Since the right side is the Laplace transform function of a Poisson variable with mean $\int_0^t \lambda(s+y)\,dy = m(s+t) - m(s)$, the result follows from the fact that the Laplace transform uniquely determines the distribution. ■

Remark That $N(s+t) - N(s)$ has a Poisson distribution with mean $\int_s^{s+t} \lambda(y)\,dy$ is a consequence of the Poisson limit of the sum of independent Bernoulli random variables (see Example 2.47). To see why, subdivide the interval $[s, s+t]$ into n subintervals of length $\frac{t}{n}$, where subinterval i goes from $s + (i-1)\frac{t}{n}$

to $s + i\frac{t}{n}$, $i = 1, \ldots, n$. Let $N_i = N(s + i\frac{t}{n}) - N(s + (i-1)\frac{t}{n})$ be the number of events that occur in subinterval i, and note that

$$\begin{aligned} P\{\geqslant 2 \text{ events in some subinterval}\} &= P\left(\bigcup_{i=1}^{n}\{N_i \geqslant 2\}\right) \\ &\leqslant \sum_{i=1}^{n} P\{N_i \geqslant 2\} \\ &= n\,o(t/n) \qquad \text{by Axiom } d \end{aligned}$$

Because

$$\lim_{n\to\infty} no(t/n) = \lim_{n\to\infty} t\,\frac{o(t/n)}{t/n} = 0$$

it follows that, as n increases to ∞, the probability of having two or more events in any of the n subintervals goes to 0. Consequently, with a probability going to 1, $N(t)$ will equal the number of subintervals in which an event occurs. Because the probability of an event in subinterval i is $\lambda(s + i\frac{t}{n})\frac{t}{n} + o(\frac{t}{n})$, it follows, because the number of events in different subintervals are independent, that when n is large the number of subintervals that contain an event is approximately a Poisson random variable with mean

$$\sum_{i=1}^{n} \lambda\left(s + i\frac{t}{n}\right)\frac{t}{n} + no(t/n)$$

But,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \lambda\left(s + i\frac{t}{n}\right)\frac{t}{n} + no(t/n) = \int_{s}^{s+t} \lambda(y)\,dy$$

and the result follows. ■

The importance of the nonhomogeneous Poisson process resides in the fact that we no longer require the condition of stationary increments. Thus we now allow for the possibility that events may be more likely to occur at certain times than during other times.

Example 5.22 Siegbert runs a hot dog stand that opens at 8 A.M. From 8 until 11 A.M. customers seem to arrive, on the average, at a steadily increasing rate that starts with an initial rate of 5 customers per hour at 8 A.M. and reaches a maximum of 20 customers per hour at 11 A.M. From 11 A.M. until 1 P.M. the (average) rate seems to remain constant at 20 customers per hour. However, the (average) arrival rate then drops steadily from 1 P.M. until closing time at 5 P.M. at which time it has the value of 12 customers per hour. If we assume that the numbers of customers arriving at Siegbert's stand during disjoint time periods are independent, then what is a good probability model for the preceding? What is the probability that no customers arrive between 8:30 A.M. and 9:30 A.M. on Monday morning? What is the expected number of arrivals in this period?

Solution: A good model for the preceding would be to assume that arrivals constitute a nonhomogeneous Poisson process with intensity function $\lambda(t)$ given by

$$\lambda(t) = \begin{cases} 5+5t, & 0 \leqslant t \leqslant 3 \\ 20, & 3 \leqslant t \leqslant 5 \\ 20-2(t-5), & 5 \leqslant t \leqslant 9 \end{cases}$$

and

$$\lambda(t) = \lambda(t-9) \qquad \text{for } t > 9$$

Note that $N(t)$ represents the number of arrivals during the first t hours that the store is open. That is, we do not count the hours between 5 P.M. and 8 A.M. If for some reasons we wanted $N(t)$ to represent the number of arrivals during the first t hours regardless of whether the store was open or not, then, assuming that the process begins at midnight we would let

$$\lambda(t) = \begin{cases} 0, & 0 \leqslant t \leqslant 8 \\ 5+5(t-8), & 8 \leqslant t \leqslant 11 \\ 20, & 11 \leqslant t \leqslant 13 \\ 20-2(t-13), & 13 \leqslant t \leqslant 17 \\ 0, & 17 < t \leqslant 24 \end{cases}$$

and

$$\lambda(t) = \lambda(t-24) \qquad \text{for } t > 24$$

As the number of arrivals between 8:30 A.M. and 9:30 A.M. will be Poisson with mean $m(\frac{3}{2}) - m(\frac{1}{2})$ in the first representation [and $m(\frac{19}{2}) - m(\frac{17}{2})$ in the second representation], we have that the probability that this number is zero is

$$\exp\left\{-\int_{1/2}^{3/2} (5+5t)\,dt\right\} = e^{-10}$$

and the mean number of arrivals is

$$\int_{1/2}^{3/2} (5+5t)\,dt = 10 \quad \blacksquare$$

Suppose that events occur according to a Poisson process with rate λ, and suppose that, independent of what has previously occurred, an event at time s is a type 1 event with probability $P_1(s)$ or a type 2 event with probability $P_2(s) = 1 - P_1(s)$. If $N_i(t), t \geqslant 0$, denotes the number of type i events by time t, then it easily follows from Definition 5.4 that $\{N_1(t), t \geqslant 0\}$ and $\{N_2(t), t \geqslant 0\}$ are independent nonhomogeneous Poisson processes with respective intensity functions $\lambda_i(t) = \lambda P_i(t), i = 1, 2$. (The proof mimics that of Proposition 5.2.) This result gives us another way of understanding (or of proving) the time sampling Poisson process result of Proposition 5.3 which states that $N_1(t)$ and $N_2(t)$ are independent Poisson random variables with means $E[N_i(t)] = \lambda \int_0^t P_i(s)\, ds, i = 1, 2$.

5.4.2. Compound Poisson Process

A stochastic process $\{X(t), t \geqslant 0\}$ is said to be a *compound Poisson process* if it can be represented as

$$X(t) = \sum_{i=1}^{N(t)} Y_i, \qquad t \geqslant 0 \tag{5.24}$$

where $\{N(t), t \geqslant 0\}$ is a Poisson process, and $\{Y_i, i \geqslant 1\}$ is a family of independent and identically distributed random variables that is also independent of $\{N(t), t \geqslant 0\}$. As noted in Chapter 3, the random variable $X(t)$ is said to be a compound Poisson random variable.

Examples of Compound Poisson Processes

(i) If $Y_i \equiv 1$, then $X(t) = N(t)$, and so we have the usual Poisson process.

(ii) Suppose that buses arrive at a sporting event in accordance with a Poisson process, and suppose that the numbers of fans in each bus are assumed to be independent and identically distributed. Then $\{X(t), t \geqslant 0\}$ is a compound Poisson process where $X(t)$ denotes the number of fans who have arrived by t. In Equation (5.24) Y_i represents the number of fans in the ith bus.

(iii) Suppose customers leave a supermarket in accordance with a Poisson process. If Y_i, the amount spent by the ith customer, $i = 1, 2, \ldots$, are independent and identically distributed, then $\{X(t), t \geqslant 0\}$ is a compound Poisson process when $X(t)$ denotes the total amount of money spent by time t. ■

Because $X(t)$ is a compound Poisson random variable with Poisson parameter λt, we have from Examples 3.10 and 3.17 that

$$E[X(t)] = \lambda t E[Y_1] \tag{5.25}$$

and

$$\mathrm{Var}(X(t)) = \lambda t E\left[Y_1^2\right] \tag{5.26}$$

Example 5.24 Suppose that families migrate to an area at a Poisson rate $\lambda = 2$ per week. If the number of people in each family is independent and takes on the values 1, 2, 3, 4 with respective probabilities $\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}$, then what is the expected value and variance of the number of individuals migrating to this area during a fixed five-week period?

Solution: Letting Y_i denote the number of people in the ith family, we have that

$$E[Y_i] = 1 \cdot \tfrac{1}{6} + 2 \cdot \tfrac{1}{3} + 3 \cdot \tfrac{1}{3} + 4 \cdot \tfrac{1}{6} = \tfrac{5}{2},$$
$$E\left[Y_i^2\right] = 1^2 \cdot \tfrac{1}{6} + 2^2 \cdot \tfrac{1}{3} + 3^2 \cdot \tfrac{1}{3} + 4^2 \cdot \tfrac{1}{6} = \tfrac{43}{6}$$

Hence, letting $X(5)$ denote the number of immigrants during a five-week period, we obtain from Equations (5.25) and (5.26) that

$$E[X(5)] = 2 \cdot 5 \cdot \tfrac{5}{2} = 25$$

and

$$\mathrm{Var}[X(5)] = 2 \cdot 5 \cdot \tfrac{43}{6} = \tfrac{215}{3} \quad ■$$

There is a very nice representation of the compound Poisson process when the set of possible values of the Y_i is finite or countably infinite. So let us suppose that there are numbers α_j, $j \geqslant 1$, such that

$$P\{Y_1 = \alpha_j\} = p_j, \qquad \sum_j p_j = 1$$

Now, a compound Poisson process arises when events occur according to a Poisson process and each event results in a random amount Y being added to the cumulative sum. Let us say that the event is a type j event whenever it results in adding the amount α_j, $j \geqslant 1$. That is, the ith event of the Poisson process is a type j event if $Y_i = \alpha_j$. If we let $N_j(t)$ denote the number of type j events by time t, then it follows from Proposition 5.2 that the random variables $N_j(t)$, $j \geqslant 1$, are independent Poisson random variables with respective means

$$E[N_j(t)] = \lambda p_j t$$

Since, for each j, the amount α_j is added to the cumulative sum a total of $N_j(t)$ times by time t, it follows that the cumulative sum at time t can be expressed as

$$X(t) = \sum_j \alpha_j N_j(t) \tag{5.27}$$

As a check of Equation (5.27), let us use it to compute the mean and variance of $X(t)$. This yields

$$\begin{aligned} E[X(t)] &= E\left[\sum_j \alpha_j N_j(t)\right] \\ &= \sum_j \alpha_j E[N_j(t)] \\ &= \sum_j \alpha_j \lambda p_j t \\ &= \lambda t\, E[Y_1] \end{aligned}$$

Also,

$$\begin{aligned} \text{Var}[X(t)] &= \text{Var}\left[\sum_j \alpha_j N_j(t)\right] \\ &= \sum_j \alpha_j^2 \text{Var}[N_j(t)] \qquad \text{by the independence of the } N_j(t),\ j \geqslant 1 \\ &= \sum_j \alpha_j^2 \lambda p_j t \\ &= \lambda t\, E[Y_1^2] \end{aligned}$$

where the next to last equality follows since the variance of the Poisson random variable $N_j(t)$ is equal to its mean.

Thus, we see that the representation (5.27) results in the same expressions for the mean and variance of $X(t)$ as were previously derived.

One of the uses of the representation (5.27) is that it enables us to conclude that as t grows large, the distribution of $X(t)$ converges to the normal distribution. To see why, note first that it follows by the central limit theorem that the distribution of a Poisson random variable converges to a normal distribution as its mean increases. (Why is this?) Therefore, each of the random variables $N_j(t)$ converges to a normal random variable as t increases. Because they are independent, and because the sum of independent normal random variables is also normal, it follows that $X(t)$ also approaches a normal distribution as t increases.

Example 5.26 In Example 5.24, find the approximate probability that at least 240 people migrate to the area within the next 50 weeks.

Solution: Since $\lambda = 2$, $E[Y_i] = 5/2$, $E[Y_i^2] = 43/6$, we see that

$$E[X(50)] = 250, \qquad \mathrm{Var}[X(50)] = 4300/6$$

Now, the desired probability is

$$\begin{aligned} P\{X(50) \geqslant 240\} &= P\{X(50) \geqslant 239.5\} \\ &= P\left\{\frac{X(50) - 250}{\sqrt{4300/6}} \geqslant \frac{239.5 - 250}{\sqrt{4300/6}}\right\} \\ &= 1 - \phi(-0.3922) \\ &= \phi(0.3922) \\ &= 0.6525 \end{aligned}$$

where Table 2.3 was used to determine $\phi(0.3922)$, the probability that a standard normal is less than 0.3922. ■

Another useful result is that if $\{X(t),\ t \geqslant 0\}$ and $\{Y(t),\ t \geqslant 0\}$ are independent compound Poisson processes with respective Poisson parameters and distributions λ_1, F_1 and λ_2, F_2, then $\{X(t)+Y(t),\ t \geqslant 0\}$ is also a compound Poisson process. This is true because in this combined process events will occur according to a Poisson process with rate $\lambda_1 + \lambda_2$, and each event independently will be from the first compound Poisson process with probability $\lambda_1/(\lambda_1 + \lambda_2)$. Consequently, the combined process will be a compound Poisson process with Poisson parameter $\lambda_1 + \lambda_2$, and with distribution function F given by

$$F(x) = \frac{\lambda_1}{\lambda_1 + \lambda_2} F_1(x) + \frac{\lambda_2}{\lambda_1 + \lambda_2} F_2(x)$$

5.4.3. Conditional or Mixed Poisson Processes

Let $\{N(t),\ t \geqslant 0\}$ be a counting process whose probabilities are defined as follows. There is a positive random variable L such that, conditional on $L = \lambda$, the counting process is a Poisson process with rate λ Such a counting process is called a *conditional* or a *mixed* Poisson process.

Suppose that L is continuous with density function g Because

$$
\begin{aligned}
P\{N(t+s) - N(s) = n\} &= \int_0^\infty P\{N(t+s) - N(s) = n \mid L = \lambda\} g(\lambda)\, d\lambda \\
&= \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} g(\lambda)\, d\lambda \qquad (5.28)
\end{aligned}
$$

we see that a conditional Poisson process has stationary increments. However, because knowing how many events occur in an interval gives information about the possible value of L, which affects the distribution of the number of events in any other interval, it follows that a conditional Poisson process does not generally have independent increments. Consequently, a conditional Poisson process is not generally a Poisson process.

Example 5.27 If g is the gamma density with parameters m and θ,

$$
g(\lambda) = \theta e^{-\theta\lambda} \frac{(\theta\lambda)^{m-1}}{(m-1)!}, \qquad \lambda > 0
$$

then

$$
\begin{aligned}
P\{N(t) = n\} &= \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} \theta e^{-\theta\lambda} \frac{(\theta\lambda)^{m-1}}{(m-1)!}\, d\lambda \\
&= \frac{t^n \theta^m}{n!(m-1)!} \int_0^\infty e^{-(t+\theta)\lambda} \lambda^{n+m-1}\, d\lambda
\end{aligned}
$$

Multiplying and dividing by $\frac{(n+m-1)!}{(t+\theta)^{n+m}}$ gives

$$
P\{N(t) = n\} = \frac{t^n \theta^m (n+m-1)!}{n!(m-1)!(t+\theta)^{n+m}} \int_0^\infty (t+\theta) e^{-(t+\theta)\lambda} \frac{((t+\theta)\lambda)^{n+m-1}}{(n+m-1)!}\, d\lambda
$$

Because $(t+\theta)e^{-(t+\theta)\lambda}((t+\theta)\lambda)^{n+m-1}/(n+m-1)!$ is the density function of a gamma $(n+m,\ t+\theta)$ random variable, its integral is 1, giving the result

$$
P\{N(t) = n\} = \binom{n+m-1}{n} \left(\frac{\theta}{t+\theta}\right)^m \left(\frac{t}{t+\theta}\right)^n
$$

Therefore, the number of events in an interval of length t has the same distribution of the number of failures that occur before a total of m successes are amassed, when each trial is a success with probability $\frac{\theta}{t+\theta}$. ■

To compute the mean and variance of $N(t)$, condition on L. Because, conditional on L, $N(t)$ is Poisson with mean Lt, we obtain

$$E[N(t)|L] = Lt$$
$$\mathrm{Var}(N(t)|L) = Lt$$

where the final equality used that the variance of a Poisson random variable is equal to its mean. Consequently, the conditional variance formula yields

$$\begin{aligned}\mathrm{Var}(N(t)) &= E[Lt] + \mathrm{Var}(Lt) \\ &= tE[L] + t^2\,\mathrm{Var}(L)\end{aligned}$$

We can compute the conditional distribution function of L, given that $N(t) = n$, as follows.

$$\begin{aligned}P\{L \leqslant x | N(t) = n\} &= \frac{P\{L \leqslant x,\ N(t) = n\}}{P\{N(t) = n\}} \\ &= \frac{\int_0^\infty P\{L \leqslant x,\ N(t) = n | L = \lambda\}\, g(\lambda)\, d\lambda}{P\{N(t) = n\}} \\ &= \frac{\int_0^x P\{N(t) = n | L = \lambda\}\, g(\lambda)\, d\lambda}{P\{N(t) = n\}} \\ &= \frac{\int_0^x e^{-\lambda t}(\lambda t)^n\, g(\lambda)\, d\lambda}{\int_0^\infty e^{-\lambda t}(\lambda t)^n\, g(\lambda)\, d\lambda}\end{aligned}$$

where the final equality used Equation (5.28). In other words, the conditional density function of L given that $N(t) = n$ is

$$f_{L|N(t)}(\lambda \mid n) = \frac{e^{-\lambda t}\lambda^n\, g(\lambda)}{\int_0^\infty e^{-\lambda t}\lambda^n\, g(\lambda)\, d\lambda}, \qquad \lambda \geqslant 0 \tag{5.29}$$

Example 5.28 An insurance company feels that each of its policyholders has a rating value and that a policyholder having rating value λ will make claims at times distributed according to a Poisson process with rate λ, when time is measured in years The firm also believes that rating values vary from policyholder to policyholder, with the probability distribution of the value of a new policyholder being uniformly distributed over $(0, 1)$. Given that a policyholder has made n claims in his or her first t years, what is the conditional distribution of the time until the policyholder's next claim?

Solution: If T is the time until the next claim, then we want to compute $P\{T > x \mid N(t) = n\}$ Conditioning on the policyholder's rating value gives, upon using Equation (5 29),

$$P\{T > x \mid N(t) = n\} = \int_0^{\infty} P\{T > x \mid L = \lambda,\ N(t) = n\} f_{L|N(t)}(\lambda \mid n)\, d\lambda$$

$$= \frac{\int_0^1 e^{-\lambda x} e^{-\lambda t} \lambda^n \, d\lambda}{\int_0^1 e^{-\lambda t} \lambda^n \, d\lambda} \quad ■$$

There is a nice formula for the probability that more than n events occur in an interval of length t. In deriving it we will use the identity

$$\sum_{j=n+1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!} = \int_0^t \lambda e^{-\lambda x} \frac{(\lambda x)^n}{n!}\, dx \tag{5.30}$$

which follows by noting that it equates the probability that the number of events by time t of a Poisson process with rate λ is greater than n with the probability that the time of the $(n+1)$st event of this process (which has a gamma $(n+1, \lambda)$ distribution) is less than t. Interchanging λ and t in Equation (5.30) yields the equivalent identity

$$\sum_{j=n+1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!} = \int_0^{\lambda} t e^{-tx} \frac{(tx)^n}{n!}\, dx \tag{5.31}$$

Using Equation (5.28) we now have

$$\begin{aligned}
P\{N(t) > n\} &= \sum_{j=n+1}^{\infty} \int_0^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!} g(\lambda)\, d\lambda \\
&= \int_0^{\infty} \sum_{j=n+1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!} g(\lambda)\, d\lambda && \text{(by interchanging)} \\
&= \int_0^{\infty} \int_0^{\lambda} t e^{-tx} \frac{(tx)^n}{n!}\, dx\, g(\lambda)\, d\lambda && \text{(using (5.31))} \\
&= \int_0^{\infty} \int_x^{\infty} g(\lambda)\, d\lambda\, t e^{-tx} \frac{(tx)^n}{n!}\, dx && \text{(by interchanging)} \\
&= \int_0^{\infty} \bar{G}(x)\, t e^{-tx} \frac{(tx)^n}{n!}\, dx
\end{aligned}$$

Exercise

*40. Show that if $\{N_i(t),\ t \geqslant 0\}$ are independent Poisson processes with rate λ_i, $i = 1, 2$, then $\{N(t),\ t \geqslant 0\}$ is a Poisson process with rate $\lambda_1 + \lambda_2$ where $N(t) = N_1(t) + N_2(t)$.

Solution

40. The easiest way is to use Definition 3 1. It is easy to see that $\{N(t), t \geqslant 0\}$ will also possess stationary and independent increments Since the sum of two independent Poisson random variables is also Poisson, it follows that $N(t)$ is a Poisson random variable with mean $(\lambda_1 + \lambda_2)t$