ACTUARIAL AND FINANCIAL VALUATIONS OF GUARANTEED ANNUITY OPTIONS

R. W. Zhou AND H. X. Gao

School of Insurance
Southwestern University of Finance and Economics
Chengdu, China

Abstract

Guaranteed Annuity Options (GAOs) are options available to holders of certain pension policies. Under these contracts, policyholders contribute premiums into a fund managed by the insurer. At retirement, the policyholders buy life annuities at a guaranteed rate provided by the original insurer, or annuitize with another insurer. If the guaranteed annuity rates are better than the prevailing rates in the market, the insurer has to make up the difference. GAOs can be viewed as interest rate options, since retiring policyholders can choose to use the higher of the guaranteed annuity rate and the prevailing market rate. We study GAOs using two models for the interest rate; the Vasicek and the Cox-Ingersoll-Ross models. An actuarial approach is used to value the GAOs and compared with the value of a replicating portfolio.

1. Introduction

Guaranteed Annuity Options (GAOs) are options available to holders of certain pension policies. Under these contracts, policyholders contribute either single or regular premiums into a fund managed by the insurer. At retirement, the policyholders have the option to convert the maturity policy proceeds into life annuities at a guaranteed rate provided by the original insurer, or annuitize with another insurer. If the guaranteed annuity rates are more beneficial to the policyholders than the prevailing rates in the market, the insurer has to make up the difference. GAOs can be viewed as interest rate options, since retiring policyholders can choose to use the higher of the guaranteed annuity rate and the prevailing market rate.

GAOs have been designed to make the pension contract more attractive since the policyholder could count on a minimum annuitization rate. There is evidence of GAOs being issued in 1839 [Historic Records Working Party, 1972]. Today, GAO has become a common feature for many US tax sheltered insurance products. A survey conducted by the Government Actuary's Department in 1998 on life insurance companies' exposure to GAOs indicated that: the exposure to GAOs was relatively widespread within the industry and had the potential to have a significant financial effect on a number of companies [Treasury, 1998]. However, they were GAOs of UK retirement savings contracts sold in the 1970s and 1980s that drew most of the attention.

In the 1970s and 1980s, the most popular guaranteed rate for a males aged sixty five, was £111 annuity per annum per £1000 of maturity proceeds, or an annuity cash value ratio of 1:9. If the prevailing annuity rate provides an annual payment higher than £111 per £1000, a rational policyholder would choose the prevailing market rate. During these two decades, the average UK long-term interest rate was around 11% p.a. The break-even interest rate implicit in the GAOs based on the mortality basis used in the original calculations was in the region of 5%-6% p.a. [Guaranteed Annuity Options, Phelim Boyle & Mary Hardy]. Obviously, the GAOs were far out of the money. However, in the 1990's, as long-term interest rates fell, these GAOs began to move into the money. The inclusion of GAOs was discontinued in the UK by the end of 1980's. Unfortunately, the long-term nature of these pension policies still made GAO a significant risk management challenge for the life assurance industry and threatened the solvency of some UK insurance companies. The emerging liabilities under GAOs (near £2.6 billion) forced the closure of Equitable life (UK), the world's oldest mutual insurance company, to stop issuing new business in 2000.

Guaranteed annuity options have drawn considerable publicity in recent years. Bolton et al (1997) described the origin and nature of these guarantees. Boyle [Embedded Options in Insurance Contracts: Guaranteed Annuity Options] analyzed their pricing and risk management. O'Brien [Guaranteed Annuity Options: Five issues for resolution, 2001] discussed issues arising from GAOs in pension policies issued by U.K. life assurance companies and highlights the impact of improving mortality. Many researchers have applied either actuarial methods or no-arbitrage pricing theory to calculate the value of GAOs embedded in deferred annuity pension policies. In Pelsser's paper (2002), a market value for GAO was derived using martingale modelling techniques and a static replicating portfolio of vanilla interest rate swaptions that replicates the GAO was constructed. The replicating portfolio would have been extremely effective and fairly cheap as a hedge against the interest rate risk involved in the GAO based on the UK interest rate data from 1980 until 2000. Chu & Kwok [Valuation of guaranteed annuity options in affine term structure models] proposed three analytical approximation methods for the numerical valuation of GAOs: the stochastic duration approach, Edgeworth expansion and analytic approximation in affine diffusions. In Chu & Kwok's work, a two-factor affine interest rate term structure model was used. Ballotta and Haberman (2003) applied the one-factor Heath-Jarrow-Morton model to price GAO in unit-linked deferred annuity contracts with a single premium. In Boyle & Hardy's paper [Guaranteed annuity options, 2003] a simple on-factor interest rate model was used and the market price of the GAOs were obtained by option pricing approach. Boyle and Hardy also examined a number of conceptual and practical issues involved in dynamic hedging of the interest rate risk. Wilkie et al (2003) worked on unit-linked contracts and investigated two approaches to reserving and pricing. Their first approach is traditional actuarial approach: quantile, conditional tail expectation and reserves. The second approach is to use option pricing methodology to dynamically hedge a guaranteed annuity option. The 1984 and 1995 Wilkie models were used to depict the yield curve.

The work presented in this paper is based on Boyle & Hardy and Wilkie et al's papers. However, instead of the Wilkie models, the interest rate dynamics are modelled by Vasicek and Cox-Ingersoll-Ross (CIR) models. The rest of the paper is organized as follows. In the next section, the model setup of the GAO is presented. Vasicek and CIR models are introduced and estimated in Section 3. Maximum Likelihood estimator is applied for Vasicek model while estimated and exact Gaussian estimations are used for CIR model. The validity of these estimation methods is also examined in Section 3. In Section 4, the actuarial approach to value the GAOs is investigated. Monte Carlo simulation is used to derive the distribution of the guaranteed annuity options and the percentiles as well as VaRs (value at risk) are thus determined. The option pricing and hedging approach is studied in Section 5. A replicating portfolio consisting of equities and zero-coupon bonds is constructed to replicate the GAO and simulation results of delta hedging are presented. The sensitivity of the value of the guaranteed annuity option with respect to different parameters in the pricing model is also investigated.

2. Model setup of the GAO

For simplicity, only single premium equity-linked policies are considered in this paper. Like in Boyle & Hardy's paper (Boyle & Hardy, 2003), standard actuarial notations are used in setting up the GAO model. Assume a male purchase a single-premium equity-linked contact and pays the premium at time 0. The contract will mature at time *T*, say, at which date the policyholder will reach age 65. The premium is invested in an equity account with market value *S(t)* at time *t*, where *S(t)* is a random process and dividends reinvested. The policy offers a guaranteed annuity rate of $g = 9$, that is, \$1 of lump sum maturity value purchases \$1/*g* of annuity per annum. With *S(T)* maturity proceeds, the policyholder is able to purchase an annuity of $S(T)/g$, which has a market value of $\left(S(T)/g\right) \times a_{65}(T)$. A rational policyholder will choose whichever is higher: $\max\left(S(T)/g \times a_{65}(T), \quad S(T)\right)$. The insurer will cover the excess of the annuity cost over the maturity proceeds:

$$\max\left(S(T)/g \times a_{65}(T), \quad S(T)\right) - S(T)$$

or

$$S(T) \times \max\left(a_{65}(T)/g - 1, \quad 0\right). \tag{2.1}$$

In the paper, the expenses are ignored and the mortality risk is assumed to be fully diversified. The value of $a_{65}(T)$ is given by

$$a_{65}(T) = \sum_{n=1}^{\omega-65} {}_n p_{65} \times D_{T+n}(T) \tag{2.2}$$

where ${}_n p_r$ is the probability that a life aged r survives n years and $D_{T+n}(T)$ denotes the market value at time T of the unit par default-free zero-coupon bond with maturity date T+n. Note the limiting age of the policyholder is denoted by ω.

The policyholder's death may occur between time 0 and time T. It is assumed that on death the only benefit is a return of the value of the fund $S(t)$. As a result, the value of the GAO per initial life is reduced to ${}_T p_{65-T} \times S(T) \times \max\left(a_{65}(T)/g - 1, \quad 0 \right)$.

Let

$$V(T) \quad = \quad {}_T p_{65-T} \times S(T) \times \max\left(a_{65}(T)/g - 1, \quad 0 \right). \qquad (2.3)$$

be the value of the GAO per initial life at maturity time T, the GAO valuation problem thus becomes to find $V(t)$, the discounted value of $V(T)$ at time t, $0 \le t \le T$.

It must be noted that a few assumptions have been made:

1) The expenses are ignored when calculating $a_{65}(T)$

2) The premiums are invested and the annuities are purchased in US market

3) The equity and bonds are uncorrelated.

4) The mortality risk has been fully diversified and is independent of the financial risk.

3. Interest rate models and their estimations

The interest rate models employed are the Vasicek model and the Cox-Ingersoll-Ross (CIR) model. Both models are continuous-time one-factor short-rate models. Studies find that multifactor models generally outperform one-factor ones over longer forecast horizons. However, as suggested by Hull (2002), relatively simple one-factor models usually give reasonable prices for instruments if used carefully. Moreover, compared to multi-factor models, one-factor models lead to more straightforward closed-form formula for the GAO.

3.1 Maximum likelihood estimation for Vasicek model

The Vasicek model can be specified by either real-world parameters κ, μ, σ, λ, or risk-neutral parameters κ, θ, σ. In this paper, κ, μ, and σ are estimated using maximum likelihood methods with time-series data and λ is estimated by least square method using cross-sectional data [Nowman, 1997].

Note the maximum likelihood here is not a true maximum likelihood but "discretized maximum likelihood". As the length of sampling interval tends to zero, the sample paths of the discretization converge to the continuous path. However, as the data *r(t)* can only be recorded with certain minimum intervals, the "discretized maximum likelihood estimator" for κ, θ, and σ will not be consistent [Miscia, 2004]. A mall Monte Carlo study similar to Yu & Philips's (Yu & Philips, 2001) is conducted to test the validity of the estimation method. The experimental results suggest that the maximum likelihood method is able to produce very good estimates of σ and long-term mean. When it comes to the estimates of α and β, however, the results are far from satisfactory. A further study finds that the short-rate *r(t)*, when observed at the daily, weekly and even monthly frequencies, tends to have large autoregressive coefficients. The autocorrelation properties of the sequence $\{r_i\}$ are determined by the parameter β. It is well known that the ML estimate of the autocorrelation parameter for a sequence that almost has a "unit root" is downward biased (Andrew, 1993). Therefore, the ML estimate of β will have a downward bias which will result in an upward bias in the estimate of α. This is consistent with the experimental results.

3.2 Exact Gaussian estimation for CIR model

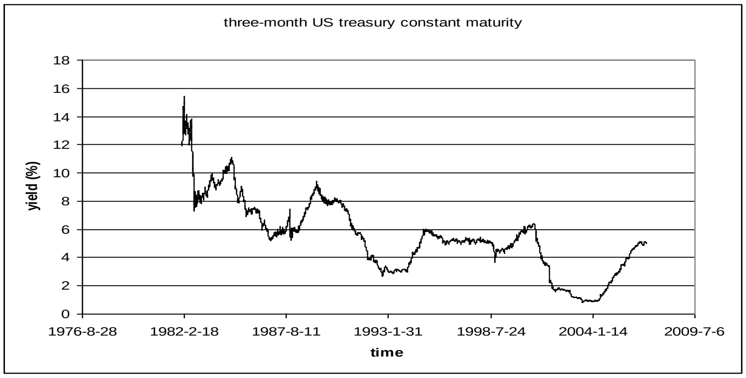
Yu & Philips [Yu & Philips, 2001] developed an exact Gaussian estimation for the CIR model by applying Dambis, Dubins-Schwarz theorem (hereafter DDB theorem) [Revuz & Yor, 1999]. The approach is based on the idea that any continuous time martingale can be written as a Brownian motion after a suitable time change and thus be estimated directly by maximum likelihood method.

Monte Carlo studies are carried out to validate the exact Gaussian estimation method. 2000 simulated daily interest rate data are generated and the CIR model is fitted. Experimental results indicate that the upward and downward biases for α and β are still present in the exact Gaussian estimates. However, they are smaller than that of Nowman's method. Similar improvements can also be found when weekly and monthly data are used. This suggests the efficiency of the exact Gaussian estimation method. In the following section, the models and approximations described in the above sections are employed and their estimates using US treasury rates are presented.

3.3 Model estimations using historical interest rate data

Three-month US treasury constant maturity yields obtained from Federal Reserve Statistical Release (http://www.federalreserve.gov/Release/h15/data.htm#fn10) is used in this paper. It contains 6231 daily observations as graphed in Figure 1. It can be observed from the figure that the interest rate goes down from about 15% in 1982 to 6% in 1987 and oscillates around 4% in 1990s. A long-term mean can be observed from the figure and the interest rate oscillates towards its long-term mean. This suggests the Vasicek and CIR model might be appropriate. The estimation results for the Vasicek and CIR models are presented in Table 1.

Figure 1: Three-month US treasury constant maturity yields



The estimates for κ and μ are obtained by: $\kappa = -\beta$, $\mu = \alpha / \beta$. The exact Gaussian estimation method produces estimate of σ which is similar to that of Nowman's, but leads to larger estimates of α and β. The long-term mean and the speed of reversion of the Vasicek model estimated by the maximum likelihood method are 4.29% and 0.239628 respectively. The Nowman method provides an estimate of the unconditional mean of 3.42%, while the exact Gaussian method gives a 2.97% estimate. The value of *a* used in the exact Gaussian estimation method is 0.0013. By choosing different values for *a*, different sample sequences are obtained. However, the resultant estimates are quite similar, as shown in Table 2. The observation confirms that the selection of parameter a in an acceptable range (not too big, not too small) will not have a considerable effect on the estimation results. The last column of Table 2 gives the number of sample data selected out of the 6231 observations. By setting *a* to 0.0013, the sample size is reduced to one tenth of its original size.

Table 1: Estimation results using three-month US treasury constant maturity rates

	α	β	σ	κ	μ
ML estimation for the Vasicek model	0.010475132	-0.239268	0.01257835	0.239268	0.04287734
Exact Gaussian estimation for the CIR model	0.005257795	-0.1768168	0.04673768	0.1768168	0.02973582

Table 2: The exact Gaussian estimation results with different values of *a*

a	α	β	κ	μ	# samples selected
0.0015	0.00597741	-0.19542	0.19542	0.030587	456
0.0014	0.00513331	-0.17503	0.17503	0.029328	488
0.0013	0.00525780	-0.17682	0.17682	0.029736	530
0.0011	0.00515702	-0.17294	0.17294	0.029820	633
0.00087	0.00558760	-0.18343	0.18343	0.030462	777
0.000655	0.00530892	-0.17846	0.17846	0.029748	1015
0.00044	0.00515327	-0.17187	0.17187	0.029984	1464
0.00022	0.00529800	-0.17295	0.17295	0.030631	2621

Figure 2: Time transformations for the 3-month US rates (01/01/1990 to 31/12/1995) with *a* = 0.0013

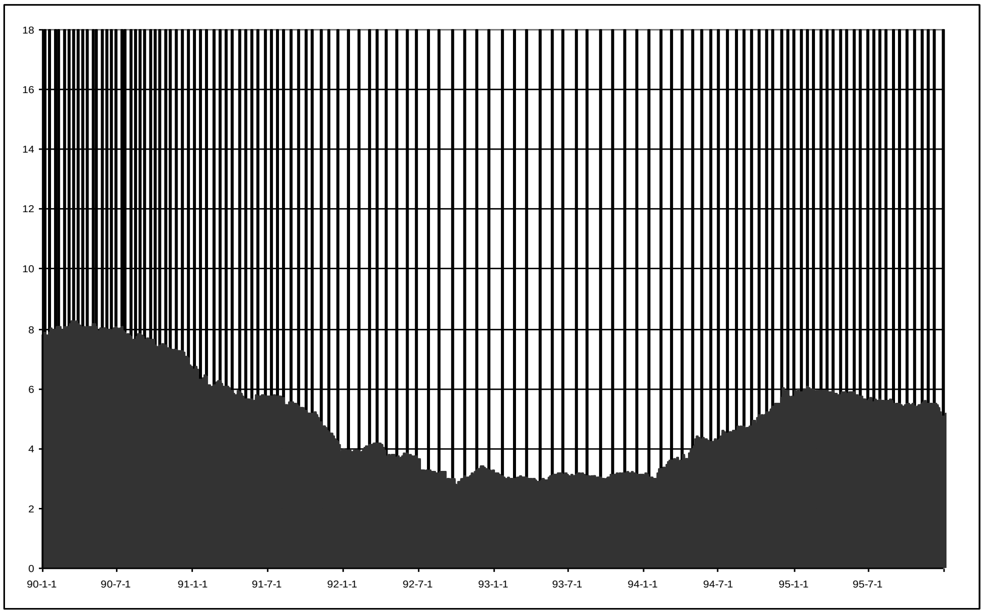


Figure 2 shows the sample selection resulting from the time transformation of the exact Gaussian method. The grey areas at the bottom of the graphs represent the interest rate, while the black vertical lines indicate the sample selected. From the graph, it can be easily observed that the sample selection frequency is higher in high interest rate region.

Once the estimates of α, β and σ are obtained, the parameter λ is calculated using the least square error method on cross-sectional data. The results are summarized in Table 3. Note the estimates in Table 3 will be used in the actuarial and financial valuations of the GAOs, which are presented in the following sections.

Table 3: Estimation results for the Vasicek and CIR models

	κ	μ	σ	λ
Vasicek model	0.239268	0.04287734	0.01257835	-0.58025
CIR model (Exact Gaussian)	0.1768168	0.02973582	0.04673768	-0.12269

4. Actuarial valuation of guaranteed annuity options

The 3-month US treasury rate is used as a proxy of the short-rate $r(t)$ in this paper. It can be observed from historical data that the term structure of the interest rate is quite complicated. Most of the time, the long-term rate is above the short-term rate. However, at some points of time, e.g. January 1982, March 1989 and October 2000, the long-term interest rate is smaller than the short-term interest rate. Therefore, it's quite natural for one to question whether the one-factor Vasicek and CIR models are capable of modelling such a complicated term structure. To test the validities of the estimated Vasicek and CIR models, comparisons are made between the historical long-term interest rates and the long-tem rates which are calculated by using the estimated models. Based on comparison results, downward adjustments are made to parameter κ as the estimates of κ are usually upward biased for both the Vasicek and CIR models. It must be noted that the ML estimated values for the parameters μ and σ are kept unchanged since they proved to be quite accurate in previous simulation results.

Four different mortality tables are considered when studying the cost of the guarantee:

(1) 1971 Group Annuity Mortality sex-distinct table (GAM71). It was developed specifically for use in the valuation of pension plans before the GAM83 tables were introduced in August 1983. Life expectancy under GAM71 for a male aged 65 is 14.6 years.

(2) 1983 Group Annuity Mortality Table (GAM 83). GAM83 is based on group annuitant experience from 1964 to 1968. GAM93 is probably the most common mortality table used by pension actuaries; 75% of the plans in a 2003 Watson Wyatt survey of actuarial assumptions and funding used GAM83 for funding calculations. Under GAM83, the life expectancy for a male aged 65 is 16.2 years.

(3) The 1994 Uninsured Pensioner Mortality Table (UP94). The UP94 table is based on uninsured pensioner experience projected to 1994. It was developed based on a study of 1985 to 1989 mortality experience of 29 retirement systems. UP94 is one of the first mortality table to factor in generational mortality, which recognizes the trend of mortality improvement and dynamically projects and incorporates those improvements. Under UP94, the life expectancy for a male aged 65 is 16.76 years.

(4) The Retired Pensioners Mortality Tale (RP2000). The RP2000 table was based on mortality experience from 1990 to 1994, which is then projected to 2000. It is the only table whose underlying rates are based solely on retirement plan mortality experience. It was developed by the SOA specifically for current liability calculations. Under RP2000, the life expectancy for a male aged 65 is 17.1 years

Figure 3: Mortality improvements

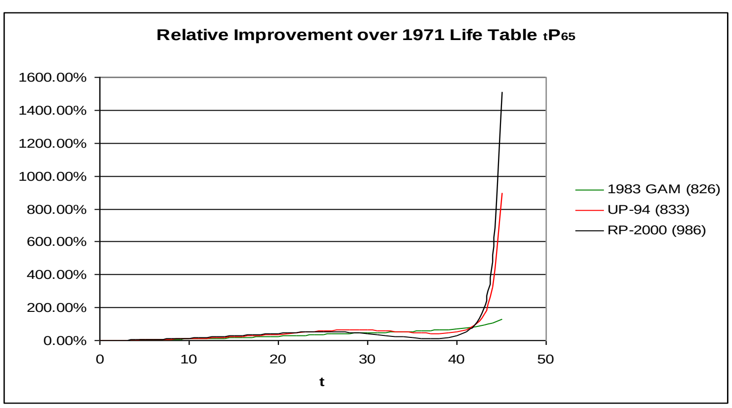


Figure 3 gives the relative improvement (in percentage) of GAM83, UP94, and RP2000 over the GAM71 table. From the graph, it can be easily observe that there have been substantial improvements in male mortality since the publication of the GAM71 table. The increase in longevity is quite dramatic over the period covered by these four tables. The expectations of life for a male aged 65 are 14.6, 16.2, 16.76, and 17.1 years using GAM71, GAM83, UP94, and RP2000 respectively. Thus the expected future lifetime of a male aged 65 increased by 2.5 years from the GAM71 table to the RP2000 table.

Figure 4: Cost of GAO per $100 maturity proceedings calculated by using the Vasicek model with adjusted parameters and the historical short-term rates

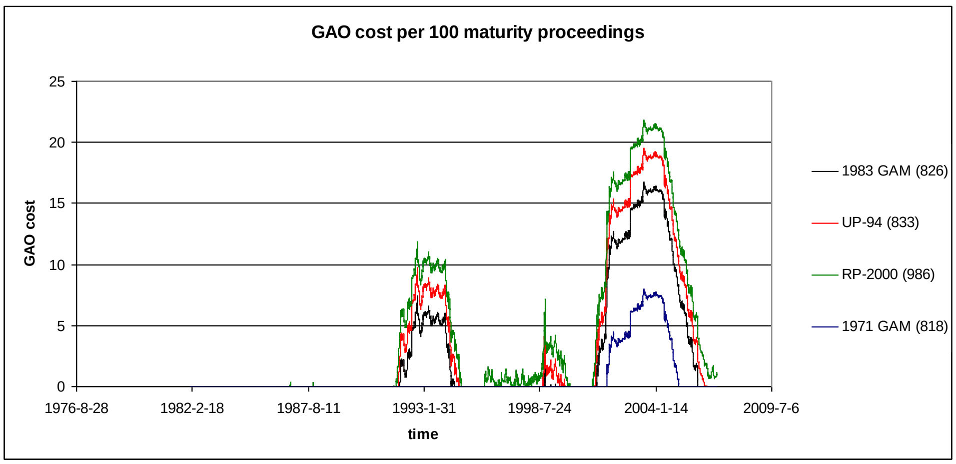
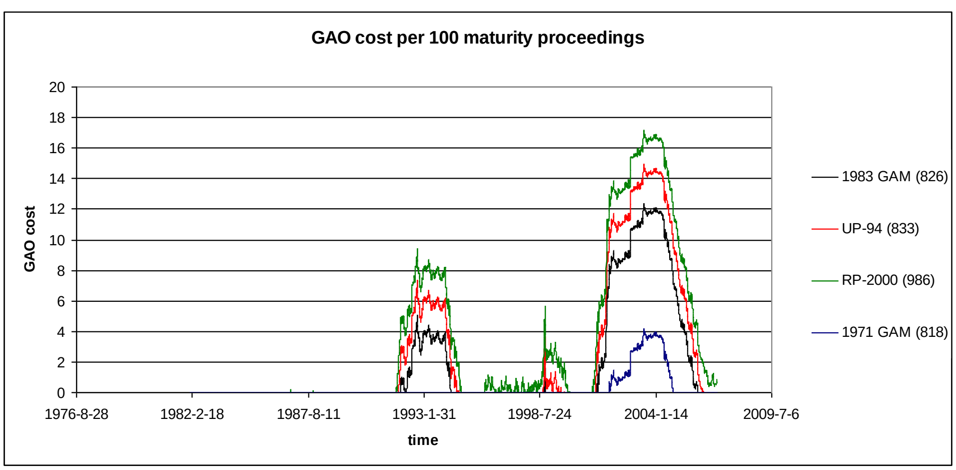


Figure 5: The cost of GAO per $100 maturity proceedings using the CIR model with adjusted parameters and the historical short-rates



As a result of the mortality improvement and the impact of falling long-term interest rate, the cost of the GAO at maturity increases significantly over the last decade. The evolution of the emerging liability under the GAO are graphed in Figure 4 and 5 using the estimated Vasciek and CIR models together with the historical short-term interest rates and the four different mortality tables. Note that at this stage no option pricing formula or stochastic analysis is involved when calculating the cost of the GAO at maturity. The policy proceeds at maturity are assumed to be held constant at $100 and the cost reported is thus the cost % of the policy maturity cash value. Comparing Figure 4 and Figure 5, one can notice that the costs calculated by the Vasicek model and the CIR model have similar shapes but slightly different scales.

As previously discussed, the valuation problem of GAOs is actually to find the discounted value of *V(T)* at time 0, *V(0)*. If the premium is invested in a share portfolio which has market value *S(t)* at time *t*, the value of the GAO when the contract is issued at time 0 is

$$V(0) \quad = \quad {}_T p_{65-T} \times S(0) \times \max\left(a_{65}(T)/g - 1, \quad 0 \right).$$

With the assumptions of fully diversified mortality and no mortality improvement over the lifetimes of the pensioners, the survival probability ${}_t p_x$ can be considered as fixed. The value of $a_{65}(T)$ is determined by the market values of the unit par default-free zero-coupon bonds at time *T*. However, the time *T* value of a zero-coupon bond is unknown at time 0. It has a distribution that depends on the movement of interest rates between time 0 and *T*. Therefore, at time 0, $a_{65}(T)$ is a random variable with a complicated distribution which is determined by the interest rate dynamics.

Like in Wilkie's paper, simulation technique is employed to find the distribution of *V(0)*. Given *r(0)*, the short-rate at time 0, 10,000 values of *r(T)* are simulated by using the Vasicek or CIR model estimated in the previous sections. The market prices of the unit par zero-coupon bonds are calculated and the value of $a_{65}(T)$ is then derived for each simulation. Consequently, 10,000 values of *V(0)* are obtained. The distribution of *V(0)* can be approximated by the histogram of these *V(0)* values.

Table 4: Present value of cost of GAO per $100 single premium: Vasicek model, mortality RP2000, r(0) = 5%

Term to Maturity	Mean	Q_{90}	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.5}$	$Q_{99.9}$
10	11.616	30.085	38.198	45.659	55.109	62.097	74.878
15	13.572	36.071	45.522	55.247	65.518	75.979	99.085
20	14.364	38.170	49.497	60.098	72.619	83.247	106.340
25	15.786	43.098	55.591	66.484	80.431	93.184	119.151
30	16.746	45.510	58.007	69.709	85.907	99.983	134.263
35	17.511	48.506	62.252	73.979	91.535	103.912	136.769
40	18.149	49.143	63.477	78.014	98.675	111.661	145.011

Simulation results are shown in Tables 4 to 9. Table 4 is based on the Vasicek model estimated in the previous sections. That is, the present value of cost of GAO is calculated with initial conditions as at the end of December 2006. The initial short-rate *r(0)* is 5%, and μ = 4.2877%, κ = 0.047854, σ = 1.258%, λ = -0.23891 in the Vasicek model. The mortality table used is RP2000. Policy terms of 10, 15, 20, 25, 30, 35, and 40 are assumed. The quantiles for 90%, 95%, 97.5%, 99%, 99.5%, and 99.9%, are denoted as Q_{90}, Q_{95}, $Q_{97.5}$, Q_{99}, $Q_{99.5}$, and $Q_{99.9}$ respectively in the table.

Table 5: Present value of cost of GAO per $100 single premium: CIR model, mortality RP2000, and r(0) = 5%

Term to Maturity	Mean	Q_{90}	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.5}$	$Q_{99.9}$
10	22.836	36.291	39.208	41.291	43.328	44.677	46.854
15	24.786	38.108	40.644	42.561	44.482	45.394	46.688
20	25.575	38.473	41.048	42.933	44.833	45.949	47.366
25	26.297	39.255	41.761	43.541	45.125	46.044	47.560
30	26.547	39.322	41.566	43.205	45.197	46.048	47.955
35	27.034	39.826	42.430	44.260	45.846	46.735	49.090
40	27.299	40.376	42.752	44.627	46.250	47.179	48.902

Table 5 is based on the CIR model estimated in the previous sections. The initial short-rate rate *r(0)* is 5%, and μ = 2.974%, κ = 0.132613, σ = 4.674%, λ = -0.10054 in the CIR model. From Tables 4 and 5, it can be observed that the discounted present values of the cost of GAOs are not negligible with an initial interest rate of 5%. In addition, the quantiles are quite substantial. There are cases where the quantiles are even higher than the $100 single premium. The mean values of the present value of GAO costs are generally higher when the CIR model is used. A possible reason is that the CIR model has a lower long-term mean than the Vasicek model. The quantiles, however, are another story. All the quantiles obtained by the Vasicek model, except Q_{90}, Q_{95} with T =10 and Q_{90} with T =15 and 20, are larger than those obtained by the CIR model. The higher the quantile, the bigger is the difference. This indicates that the Vasicek model is more volatile than the CIR model.

From the figures, it can also be observed that as the term to maturity increases, the cost of the GAO becomes higher. However, this is not always true. When the initial interest rate is high the cost increases with the term; when the initial interest rate is low, the cost may reduce with the term, or reduce first and increase later, as shown in Tables 6 and 7. When the initial interest rate r(0) is low, there is a greater chance that the interest rate in the short term will also be low, and thus the discounted present value of the cost of GAO will be higher. Comparing Table 4 with Table 6 and Table 5 with Table 7, one can notice that the GAO cost for shorter terms varies much more than that of longer terms when the initial interest rate changes.

Table 6: Present value of cost of GAO per $100 single premium: Vasicek model, mortality RP2000, r(0) = 2%

Term to Maturity	Mean	Q_{90}	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.5}$	$Q_{99.9}$
10	22.640	46.819	56.426	66.049	76.518	84.405	102.468
15	22.162	49.999	60.590	69.845	82.921	94.151	115.953
20	21.187	50.505	61.676	74.453	88.911	97.951	118.237
25	20.608	50.723	63.806	76.002	91.375	102.677	130.136
30	20.321	51.462	65.456	77.921	94.544	108.357	137.301
35	20.841	54.097	69.069	83.790	100.261	114.777	144.632
40	20.405	53.669	68.978	84.621	101.490	116.875	147.089

Table 7: Present value of cost of GAO per $100 single premium: CIR model, mortality RP2000, and r(0) = 2%

Term to Maturity	Mean	Q_{90}	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.5}$	$Q_{99.9}$
10	28.126	39.913	42.061	43.654	45.072	46.001	47.165
15	27.519	39.580	41.905	43.535	44.896	45.799	47.207
20	27.076	39.267	41.559	43.268	45.060	46.079	47.204
25	26.942	39.473	41.743	43.696	45.027	46.035	47.540
30	27.018	39.876	42.168	44.044	45.759	46.617	47.986
35	27.381	40.174	42.597	44.226	46.010	46.912	48.587
40	27.287	40.344	42.754	44.444	46.189	47.383	48.966

The quantiles calculated above was an alternative name for the concept also described as "Value at Risk' or VaR for simplicity. VaR has, however, been criticised for being 'incoherent'. It is possible for a quantile to be smaller than the mean value of a risk. This is unsatisfactory. Another problem is that, when risks are combined into a portfolio, it is possible for the quantile for the portfolio to be greater than the sum of the corresponding quantiles for the individual risks [Wilkie et al, 2003].

Table 8: Present value of cost of GAO per $100 single premium: Vasicek model, mortality RP2000, r(0) = 5%

Term to Maturity	Mean	T_{90}	T_{95}	$T_{97.5}$	T_{99}	$T_{99.5}$	$T_{99.9}$
10	11.616	41.155	48.592	55.493	64.166	70.009	82.380
15	13.572	49.534	58.701	67.667	80.367	91.421	111.304
20	14.364	53.702	64.089	74.130	86.780	96.358	115.220
25	15.786	59.869	71.164	82.100	96.652	107.575	129.057
30	16.746	63.321	75.611	87.668	104.705	117.588	145.949
35	17.511	67.207	79.744	92.022	108.976	121.893	153.085
40	18.149	70.077	84.725	99.332	117.942	131.589	169.430

To solve this problem, 'conditional tail expectation' (CTE) can be used. As the α% quantile Q_α of a risk X is defined as $\Pr(X < Q_\alpha) = \alpha\%$, the CTE at level α (denoted by T_α)

is defined as $T_\alpha = \mathrm{E}[X \mid X \geq Q_\alpha]$. It is easily calculated during the simulations. For the sorted 10,000 simulation results of *V(0)*, the CTE at level 99% equals to the average of the 100 largest values of *V(0)*, from $V(0)_{9901}$ to $V(0)_{10000}$ inclusive. The CTEs of the cost of a GAO by using the Vasicek and CIR model are presented in Tables 8 and 9 respectively.

Table 9: Present value of cost of GAO per $100 single premium: CIR model, mortality RP2000, and r(0) = 5%

Term to Maturity	Mean	T_{90}	T_{95}	$T_{97.5}$	T_{99}	$T_{99.5}$	$T_{99.9}$
10	22.836	39.722	41.764	43.312	44.905	45.931	47.418
15	24.786	41.117	42.928	44.299	45.567	46.202	47.380
20	25.575	41.493	43.303	44.715	46.072	46.752	47.821
25	26.297	42.134	43.829	45.057	46.273	46.963	48.531
30	26.547	42.013	43.641	44.984	46.355	47.113	48.401
35	27.034	42.798	44.544	45.790	47.112	48.005	49.440
40	27.299	43.216	44.921	46.184	47.425	48.145	49.379

From the tables, it can be observed that the CTE values are always greater than the corresponding quantiles, since $T_\alpha = \mathrm{E}[X \mid X \geq Q_\alpha] \geq Q_\alpha$. Besides, the value of CTE is larger than the mean, since Q_0 is itself equal to the mean, and $Q_\alpha > Q_\beta$ if $\alpha > \beta$. And it was shown (Artzner, 1998) that, when risks are combined into a portfolio, the portfolio CTE cannot be greater than the sum of the individual CTEs.

5. Financial pricing and hedging of guaranteed annuity options

The similarities between the guaranteed annuity options and other types of financial option have been pointed out by many researchers. Among them are: Wilkie et al. (2003), Boyle & Hardy (2003), Bolton et al. (1997), and Pelsser (2002). The work presented here is mainly based on Boyle & Hardy (1998), in which modern option pricing and dynamic hedging techniques are used.

5.1 Option pricing

The value of a GAO at maturity time T is $S(T) \times \max\left(a_{65}(T)/g - 1, \quad 0 \right)$ where $a_{65}(T) = \sum_{n=1}^{\omega-65} {}_n p_{65} \times D_{T+n}(T)$. From the formula, it's quite obvious that a GAO is similar to a call option on a coupon bond with the annuity payments and survival probabilities being incorporated in the notional coupons.

The price at maturity time T of a zero-coupon bond with unit maturity value, maturing at $T+n$ ($n \geq 1$) is denoted as $D_{T+n}(T)$ or $P(T,T+n)$. The value of $D_{T+n}(T)$ at time T depends on the term structure which is assumed to be known at time T. At time $t < T$, however, $D_{T+n}(T)$ is a random variable under stochastic interest rate model. By the no-arbitrage theorem, the value of a risk at time t ($t < T$) which has payoff $V(T)$ at time T is

$$V(t) = P(t,T)E_t^{Q_T}[V(T) \mid F_t]$$

where Q_T is the *forward-risk* adjusted measure. The market value of the share portfolio $S(T)$ is also a random variable and is assumed to be independent of interest rates. Note this is a very strong assumption but it simplifies the analysis. The value of GAO at time 0 thus becomes:

$$V(0) = \frac{{}_T p_{65-T}}{g} S(0) E^{Q_T}\left[\left(\sum_{n=1}^{\omega-65} {}_n p_{65} \times P(T,T+n) - g\right)^+\right]$$

The expression inside the expectation on the right hand side corresponds to a call option on a coupon paying bond where the 'coupon' payment at time (T+n) is ${}_n p_{65}$ and the expiration date is time T. This 'coupon bond' has value at time T:

$$\sum_{n=1}^{\omega-65} {}_n p_{65} \times P(T,T+n).$$

The market value at time t of this coupon bond is

$$P(t) = \sum_{n=1}^{\omega-65} {}_n p_{65} \times P(t,T+n).$$

So $P(t)$ is the value of a deferred annuity, but without allowance for mortality before retirement. With notation $P(t)$, we have

$$V(0) = \frac{{}_T p_{65-T}}{g} S(0) E^{Q_T}\left[\left(P(T) - g\right)^+\right].$$

Jamshidian (1989) showed that if the interest rate follows a one-factor process, then the market price of the option on the coupon bond with strike price g is equal to the price of a portfolio of options on the individual zero-coupon bonds with strike prices K_n, where $\{K_n\}$ are equal to the notional zero-coupon bond prices to give an annuity $a_{65}(T)$ with market price g at T. That is, let r_T^* denote the value of the short-rate at time T for which

$$\sum_{n=1}^{\omega-65} {}_n p_{65} D^*(T,T+n) = g$$

where the asterisk is used to indicate that each zero-coupon bond is evaluated using the short-rate r_T^*. K_n is then set as $K_n = D^*(T,T+n)$. The call option with strike g and expiration date T on the coupon bond $P(t)$ can be valued as

$$C\left[P(t),g,t\right]=\sum_{n=1}^{\omega-65}{}_{n}p_{65}C\left[P(t,T+n),K_n,t\right]$$

where $C\left[P(t,T+n),K_n,t\right]$ is the price at time t of a call option on the zero-coupon bond with maturity (T+n), strike price K_n and expiration date T. Under the forward-risk measure,

$$C[P(t),g,t]=P(t,T)E^{Q_T}\left[\left(P(T)-g\right)^{+}\mid F_t\right].$$

Thus

$$E^{Q_T}\left[\left(P(T)-g\right)^{+}\mid F_t\right]=\frac{\sum_{n=1}^{\omega-65}{}_{n}p_{65}C\left[P(t,T+n),K_n,t\right]}{P(t,T)}$$

and we have

$$V(0)=\frac{{}_{T}p_{65-T}}{g}S(0)\frac{\sum_{n=1}^{\omega-65}{}_{n}p_{65}C\left[P(0,T+n),K_n,0\right]}{P(0,T)}.$$

Two interest rate models have been estimated and studies in the previous chapters. They are the Vasicek model and the CIR model. Experimental results show that with proper parameter settings, both models can generate reasonable cost of GAOs. However, as the closed-form solution of option price is very complicated under the CIR model, only the Vasicek model is considered in this chapter.

Under the Vasicek model, the price at time t of a call option on a zero-coupon bond with strike price g, maturity date T+n, and expiration date T is given as follows.

$$C\left[P(t,T+n),K_n,t\right]=P(t,T+n)N(h_1(n,t))-K_nP(t,T)N(h_2(n,t))$$

Where

$$h_1(n,t)=\frac{\ln P(t,T+n)-\ln P(t,T)-\ln K_n}{\sigma_P(n,t)}+\frac{\sigma_P(n,t)}{2}$$

$$h_2(n,t)=\frac{\ln P(t,T+n)-\ln P(t,T)-\ln K_n}{\sigma_P(n,t)}-\frac{\sigma_P(n,t)}{2}=h_1(n,t)-\sigma_P(n,t)$$

and

$$\sigma_P(n,t)=\sigma\sqrt{\frac{1-e^{-2\kappa(T-t)}}{2\kappa}}\frac{(1-e^{-\kappa n})}{\kappa}.$$

The parameters of the Vasicek model are set to: $r(0)$ = 5%, μ = 4.2877%, κ = 0.047854, σ = 1.258%, and λ = -0.23891. Table 10 shows, for terms 10, 15, 20, 25, 30, 35, and 40, the initial value of the GAO per $100 single premium, using mortality table RP2000. Compared with Table 4, it can be observed that the option pricing costs are smaller for all terms than the mean costs calculated using actuarial method. With actuarial method as described in Section 4, all the premiums are invested into an equity account. In option pricing, however, the premiums are invested into equities and bonds with different maturities according to certain proportions. As a result, the maturity values of GAOs should be discounted back with different rates and hence the $V(0)$ values would be different in these two methods as shown in Table 4 and Table 10. Note in Table 4, the values all increase considerable with term. Now in Table 10, the values of longer terms are not much greater than those of shorter terms. Of course with different parameters different results might be obtained.

Table 10: The cost of GAO at time 0 obtained by option pricing with r(0) = 0.05

	T=10	T=15	T=20	T=25	T=30	T=35	T=40
V(0)	6.585170	6.729368	6.883999	7.052726	7.228360	7.413138	7.613177

By varying the initial conditions, the following results as given in Table 11 are obtained. The first row gives the values of $V(0)$ with doubled κ. Comparing them to Table 10, one can observe that $V(0)$ with $T = 10$ increases while $V(0)$ with all the other terms decrease when κ is doubled, and the effect of doubled κ is more obvious for longer terms. The results for doubling the volatility σ are tabulated in the second row. As the interest rates become more volatile, the probability that the insurer has to pay for the GAO becomes higher and the cost of the GAO becomes bigger, as shown in Table 11. The last two rows of the table give the effect of changing the initial interest rate $r(0)$. When $r(0)$ is lower, the chance that $r(t)$ will remain low becomes higher, and the value of $V(0)$ will thus be larger.

Table 21: The cost of GAO at time 0, V(0), calculated with different parameter settings

	T=10	T=15	T=20	T=25	T=30	T=35	T=40
$r(0)$ = 5%, κ = 0.09571 σ = 1.258% λ = -0.23891	6.834641	6.624155	6.491909	6.422260	6.398067	6.413455	6.467144
$r(0)$ = 5%, κ = 0.047854 σ = 2.516% λ = -0.23891	12.21211	15.34684	18.69096	22.15480	25.60556	28.95980	32.18586
$r(0)$ = 2%, κ = 0.047854 σ = 1.258% λ = -0.23891	13.731644	11.760524	10.595791	9.868453	9.401490	9.110786	8.952066
$r(0)$ = 8%, κ = 0.047854 σ = 1.258% λ = -0.23891	2.599921	3.487202	4.232036	4.880734	5.451819	5.962125	6.428779

5.2 Dynamic hedging

In order for the above theoretical option price to be taken as the true or 'fair' value of the GAO, a hedging strategy must exist such that the results of the investments according to the hedging strategy replicate the desired payoff of the GAO. To hedge the GAO against both the equity and interest risks, we would need to invest in the following securities: an equity index with market value $S(t)$ at time t, zero-coupon bond maturing at time T, and zero-coupon bonds maturing at time $T+n$ (n = 1, …, ω-65). Note the hedging strategy employed here is delta hedging strategy. The hedging ratios are thus the partial derivatives of $V(t)$ over the corresponding underlying securities. The number of units invested in the index at time t is denoted by $H_S(t)$ where

$$H_s(t) = \frac{{}_T p_{65-T}}{g} \frac{\sum_{n=1}^{\omega-65} {}_n p_{65} C\left(P(t,T+n), K_n, t\right)}{P(t,T)}$$
$$= \frac{{}_T p_{65-T}}{g} \sum_{n=1}^{\omega-65} {}_n p_{65} \left[\frac{P(t,T+n)}{P(t,T)} N(h_1(n,t)) - K_n N(h_2(n,t)) \right]$$

The second consists of an investment at time t of $H_0(t)$ units of the zero-coupon bond which matures at time T, where

$$H_0(t) = -\frac{{}_T p_{65-T}}{g} S(t) \sum_{n=1}^{\omega-65} {}_n p_{65} \frac{P(t,T+n)}{P(t,T)^2} N(h_1(n,t)).$$

The replicating portfolio also consists of investments of $H_n(t)$ units of the zero-coupon bonds which matures at time $T+n$ (n = 1, …, ω-65), where

$$H_n(t) = \frac{{}_T p_{65-T}}{g} S(t) {}_n p_{65} \frac{N(h_1(n,t))}{P(t,T)} \qquad n = 1, \ldots, \omega - 65.$$

If the limiting age of the policyholder is 110, we have to invest at all times in the 47 securities according to the above hedging proportions. Note that the value of the initial hedge is

$$H_s(0)S(0) + H_0(0)P(0,T) + \sum_{n=1}^{\omega-65} H_n(0)P(0,T+n)$$
$$= \frac{{}_T p_{65-T}}{g} S(0) \sum_{n=1}^{\omega-65} {}_n p_{65} \left[\frac{P(0,T+n)}{P(0,T)} N(h_1(n,t)) - K_n N(h_2(n,t)) \right]$$
$$= H_s(0)S(0) = V(0)$$

which is equal to the value of the GAO at time 0. At maturity time T, $P(T,T)$ = 1, $\sigma_P(n,T)$ = 0, $h_1(n,T) = h_2(n,T) = +\infty$, $N(h_1(n,T)) = N(h_2(n,T))$ = 1, and hence the value of the hedge portfolio is

$$
\begin{aligned}
& H_s(T)S(T) + H_0(T)P(T,T) + \sum_{n=1}^{\omega-65} H_n(T)P(T,T+n) \\
&= \frac{{}_T p_{65-T}}{g} S(T) \sum_{n=1}^{\omega-65} {}_n p_{65} \left[P(T,T+n) - K_n \right] \\
&\quad - \frac{{}_T p_{65-T}}{g} S(T) \sum_{n=1}^{\omega-65} {}_n p_{65} P(T,T+n) \\
&\quad + \frac{{}_T p_{65-T}}{g} S(T) \sum_{n=1}^{\omega-65} {}_n p_{65} P(T,T+n) \\
&= \frac{{}_T p_{65-T}}{g} S(T) \sum_{n=1}^{\omega-65} {}_n p_{65} \left[P(T,T+n) - K_n \right] = V(T) .
\end{aligned}
$$

That is, the result of the investment process matches exactly the required payoff of the GAO at maturity time *T*. Suppose the hedge is to be rebalanced at time *t*+*h*, just before rebalancing the value of the hedge portfolio is

$$
H_s(t)S(t+h) + H_0(t)P(t+h,T) + \sum_{n=1}^{\omega-65} H_n(t)P(t+h,T+n)
$$

where *S*(*t*+*h*), P(*t*+*h*,T), and P(*t*+*h*,T+n) denote the market prices at time *t*+*h* of the hedge assets. The new hedging weights $H_S(t+h)$, $H_0(t+h)$, and $H_n(t+h)$ are computed based on these new asset prices and the value of the revised hedge is

$$
G(t+h) = H_s(t+h)S(t+h) + H_0(t+h)P(t+h,T) + \sum_{n=1}^{\omega-65} H_n(t+h)P(t+h,T+n) .
$$

Note, the value of the hedge portfolio at time t is denoted by *G*(*t*).

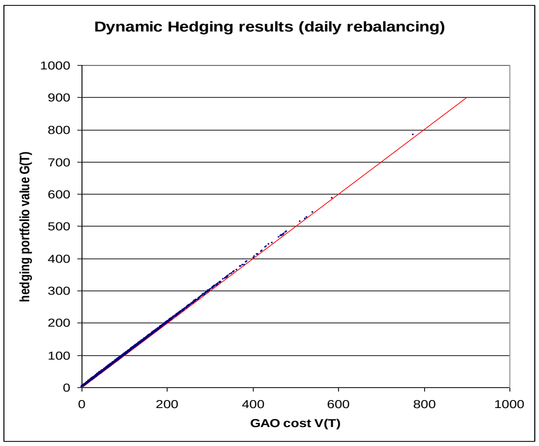
If the ‘real world’ model is the same as the model used for option pricing, and if hedging is carried out continuously, free of transaction costs, then the hedge portfolio is self-financing since the hedging proportions are actually the partial derivatives of *G*(*t*) over the corresponding assets. However, the rebalancing is done discretely in practice. In addition, there are transactions costs and the market movements can deviate significantly from those implied by the model. All these can lead to considerable hedging errors.

From the above discussion, it can be concluded that the value of the replicating portfolio, *G*(*t*), will almost certainly not exactly match the value of *V*(*t*). The following investment strategy is then considered: invest the correct amounts in the portfolio, and invest the balance if there’s any in the zero-coupon bond, or borrow the shortage by shorting the zero-coupon bond. The assumption implied in the above investment strategy is that we can borrow or lend at the risk-free rate, which is of course not true in the real world.

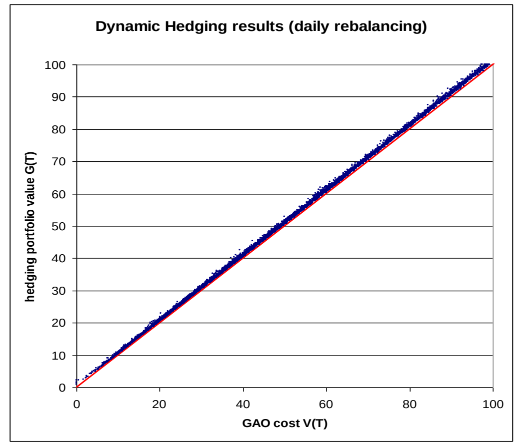
The hedging results of 10,000 simulations with *h*=1/250 (daily rebalancing) are shown in Figure 6. The hedging is carried out according to the investment strategy described in the previous paragraphs. *S*(*t*) is assumed to follow the stochastic equation d*S*(*t*) = $\mu_s S(t)$dt +

$\sigma_s S(t)\mathrm{d}W_t$, where μ_S = 0.1, σ_S = 0.2, and $\mathrm{d}W_t$ is the standard Brownian motion. Note that $S(t)$ is independent of $r(t)$ under the assumption. The initial short-rate r(0) varies between 3% and 20% and $S(0)$ equals to $100 in the 10,000 simulations.

Figure 6: Hedging results of 10,000 simulations (daily rebalancing) (a) plots in normal scale; (b) partial view



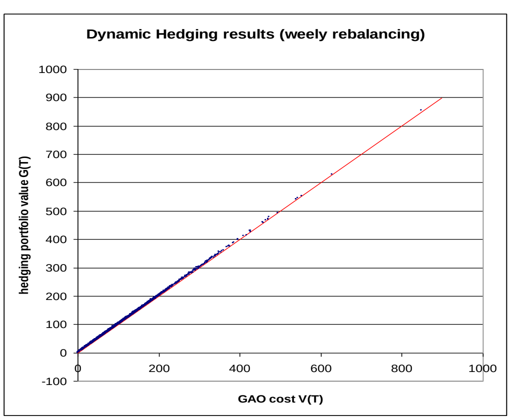
(a)



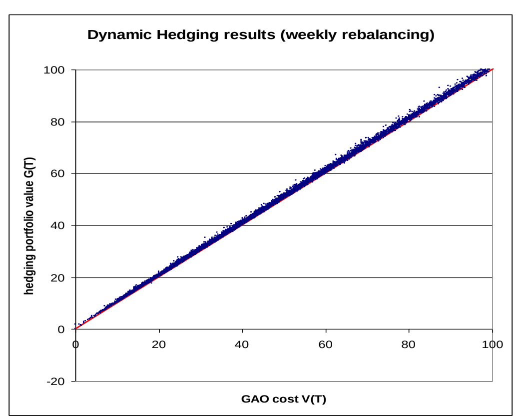
(b)

In Figure 6, the values of $G(T)$, the investment proceeds at maturity time T, are plotted against the values of $V(T)$, the amount required to pay off the option at maturity. One can see that in general the investment proceeds correspond with the amounts required very closely. Surpluses at maturity (although not significant) can be observed when the hedging strategy was followed. Figures 7 and 8 shows the simulation results with the same variables but hedging weekly and monthly respectively. From the figures, it can be observed that although the investment results cluster around the 45-degree line, the correspondence between the $V(t)$ and $G(T)$ is by no means perfect. Proportionately large profits and deficits can be observed especially in monthly rebalanced cases.

Figure 1: Hedging results of 10,000 simulations (weekly rebalancing) (a) plots in normal scale; (b) partial view



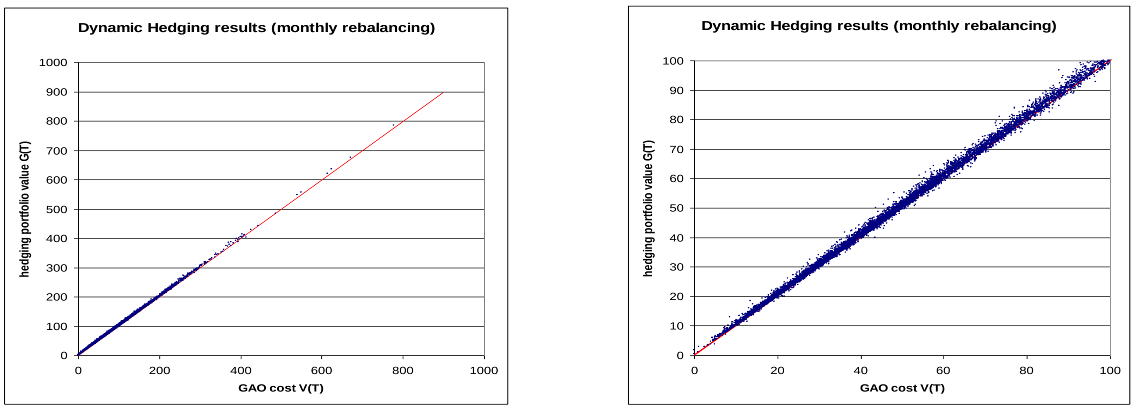
(a)



(b)

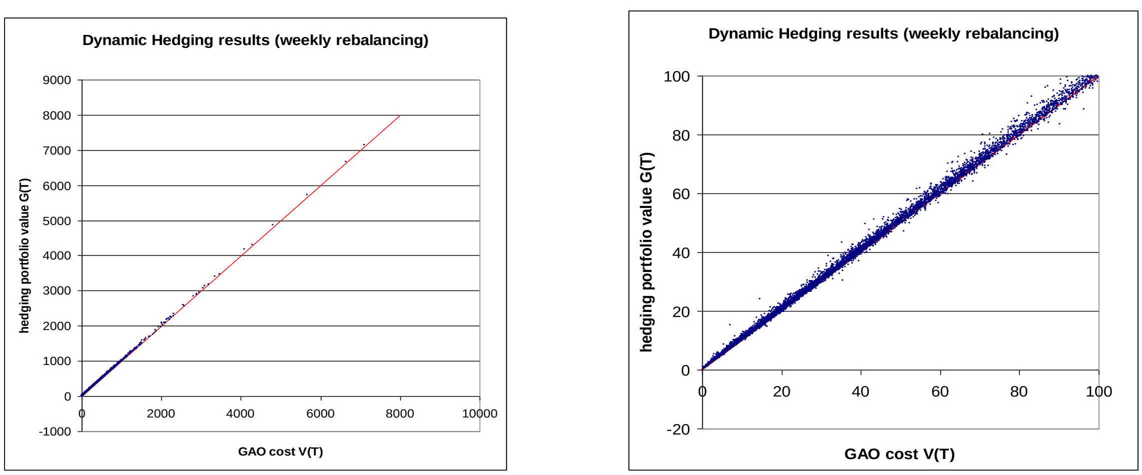
As the results depend strongly on the value of $S(T)$, the effect of changing $S(t)$ model (varying the volatility σ_S) is investigated and presented in Figure 9 and 10. One can see that as the $S(t)$ process becomes more volatile, the correspondence between $G(t)$ and $V(t)$ becomes less close. The extreme values are far more extreme when σ_S = 0.4 than when σ_S = 0.2.

Figure 2: Hedging results of 10,000 simulations (monthly rebalancing) (a) plots in normal scale; (b) partial view



(a) (b)

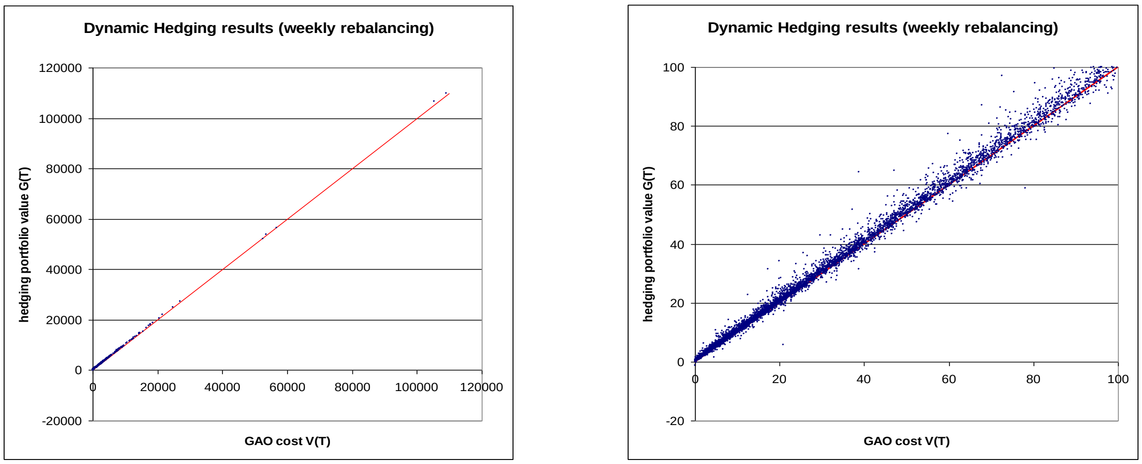
Figure 3: Hedging results of 10,000 simulations (weekly rebalancing, σ_s = 0.3) (a) plots in normal scale; (b) partial view



(a) (b)

The results demonstrate that the investment strategy we have described above, if hedging is sufficiently frequent, and if the real-world model is as assumed, does give results that correspond with the required payoff quite closely. This validates the option and hedging formula is appropriate for the guaranteed annuity options. In this paper, it is assumed that the real world interest rates in fact behave in accordance with the Vasicek model that has been defined and used for the calculation of option values and hedging quantities. The true behaviour of the real world interest rates may actually be very different. In addition, $S(t)$ is assumed to have a log-normal distribution while it is now well established in the empirical literature that equity prices do not follow a simple lognormal process [Hardy, 2003]. All these and many other frictional factors such as transaction costs which are not considered in this work will almost certainly widen the hedging errors in real-world applications.

Figure 4: Hedging results of 10,000 simulations (weekly rebalancing, σ_s = 0.4) (a) plots in normal scale; (b) partial view



(a) (b)

6. Conclusions

In this paper, the value of Guaranteed Annuity Options (GAOs) is investigated in an environment of stochastic interest rates. The maturity value of GAO at time T is modelled by a mathematical model and its discounted value at time t is calculated under the following assumptions: 1) fully diversified mortality; 2) no mortality improvement; 3) no expense; 4) US market; 5) mortality risk is independent of the financial risk and 6) equities and bonds are uncorrelated.

Two methods are employed to find the discounted value of GAO at time t. They are actuarial method and financial pricing method. In actuarial method, all the premiums are investigated into an equity account and the discount value of GAO at time 0 can be modelled by an option on the deferred annuity $a_{65}(T)$. To find the value of $a_{65}(T)$, two one-factor stochastic interest rate models are introduced: Vasicek model and CIR model. Their estimation methods are described and tested with the simulated data. The experiment results suggest that the risk-neutral long-term mean and the volatility can be estimated quite accurately while the estimate of the mean-reversion rate is upward biased. The estimated Vasicek model and CIR model are then calibrated using the cross sectional data. The maturity costs of the GAOs are calculated using the calibrated models and are compared against the costs calculated using the historical long-term interest rates.

Experimental results show that these calibrated models can give reasonable results and thus can be used to find the discounted value of the GAO at time 0, $V(0)$. Means, variances, percentiles, and conditional tail expectations of $V(0)$ with different maturities are found by the Monte Carlo simulation.

The second method is financial pricing method. Under this method, the value of $V(0)$ can be calculated using the option pricing formulas. Note only the Vasicek model is employed as it is difficult to find a closed-form option price solution for the CIR model. The values of V(0) with different maturities are calculated and presented. A replicating portfolio is constructed and the delta hedging strategy is employed to make sure that the results of the investments replicate the desired payoff of the GAO. The experiment results demonstrate that, if the delta hedging is sufficiently frequent, the investment process matches with the required payoff quite closely.

This work shows that both the actuarial and financial methods are able to give reasonable valuations of the GAOs under proper assumptions. However, the results obtained by these two methods are not the same as $V(0)$s are invested and discounted differently. Therefore, a possible direction for future work might be analyzing the relationships between these two sets of results. Delta hedging strategy is employed in this work. When the volatility is high, however, other hedging strategies such as gamma hedging might be more appropriate. Moreover, no mortality improvement is considered in this work. This is obviously not the case in real world. The valuation models can thus be made more realistic by including mortality improvements.

References

Andersen, T.G., and J. Lund, 1997, "Estimating Continuous Time Stochastic Volatility Models of the Short-Term Interest Rate," Journal of Econometrics, 77, 343-377

Andrews, Donald W. K. 1993, "Exactly-unbiased estimation of first order autoregressive/unit root models", Econometrica, 61, 139-166, 1993

Artzner, P., 1998, "Coherent measures of risk", Mathematical Finance, 9(3), 203-228

Bergstrom, Albert R., 1984, "Continuous time stochastic models and issues of aggregation over time", in Z. Griliches and M. D. Intriligator, Eds.: Handbook of Econometrics, Vol. II (Elsevier Science, Amsterdam).

Chan, K. C., G. A. Karolyi, F. A. Longstaff, and a. B. Sanders, 1992, "An Empirical Comparison of Alternative Models of the Short-Term Interest Rate," Journal of Finance, 47, 1209-1227

Chapman, D.A., Long, J.B. and Pearson, N.D., 1999, "Using proxies for the short-rate: when are three months like an instant?", Review of Financial Studies, vol. 12, No. 4, 763-806

Cox, John, Jonathan Ingersoll, and Stephen Ross, 1985, “A Theory of the Term Structure of Interest Rates”, Econometrica, vol. 53, No. 2, pp. 385-407

Hull, J., 2002, “Options Futures and Other Derivatives”, Prentice Hall.

Jamshidian, F., 1991, “Bond and Option Evaluation in the Gaussian Interest Rate Model”, Research in Finance, 9, 131-170

Maghsoodi, Y., 2000, “estimation and empirical evaluation of the time-dependent extended-CIR term structure model”, IMA journal of Mathematics applied in business and industry (2000) 11, 161-176

Miscia, O.D., 2004, “Term structure of interest models: concept and estimation problem in a continuous-time setting”

Nowman, K.B., 1997, “Gaussian Estimation of Single-Factor Continuous Time Models of The Term Structure of Interest Rates”, The Journal of Finance, vol. LII, No. 4, Sept. 1997

Revuz, Daniel and Marc Yor. 1999, “Continuous Martingales and Brownian Motion”, Springer-Verlag, 1999

Stanton, R., 1997, “A Nonparametric Model of Term Structure Dynamics and the Market Price of Interest Rate Risk,” Journal of Finance, 52, 1973-2002

Wilkie, A.D., Waters, H. R.and Yang, S., 2003, “Reserving, pricing and hedging for policies with Guaranteed Annuity Options”

Yu, J. & Philips, Peter C.B., 2001, “Gaussian estimation of continuous time models of the short term interest rate, July 2001