Regime-Switching Portfolio Replication

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Long-Term Guarantees

Contract: Long-term Equity Guarantees/Options
Eg.

- Guaranteed Minimum Maturity Benefit
- Long-Term Stock Options

Example: Selling a 10-year European Put option on the S&P 500.

Due to the catastrophe nature of this risk, choose to hedge the contract.

Black-Scholes Hedging

Black-Scholes Put Option Price:

$$BSP_t = K \cdot e^{-r(T-t)} \cdot \Phi(-d_2) - S_t \cdot \Phi(-d_1)$$

$$\begin{aligned} d_1 &= \frac{log(S_t/K) + (T-t)(r+\sigma^2/2)}{\sqrt{T-t}\sigma} \\ d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

Hedge: Hold $H_t = -\Phi(-d_1)$ in stock.

One assumption of the framework: continuous re-balancing of the hedge.

Black-Scholes Hedging

Continuous re-balancing is obviously not feasible.

Monthly Re-balancing

- This will introduce Hedging Error
- $HE_{t+1} = BSP_{t+1} - (H_t \cdot S_{t+1} + B_t \cdot e^r)$

Another assumption: S_t follows a geometric Brownian Motion with constant variance σ^2.

Goal: Find a good σ for the S&P 500.

S&P 500

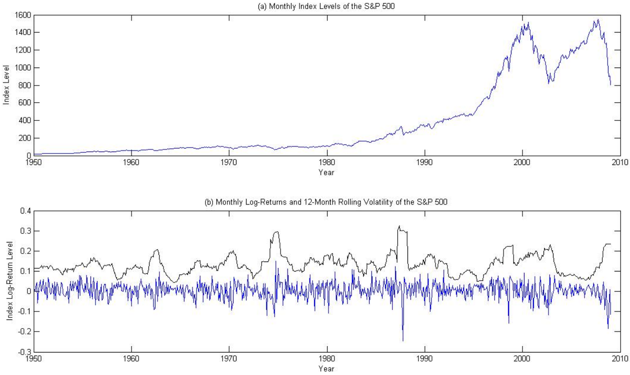


Figure: S&P 500 Monthly Index and Log-Return Levels

S&P 500 Volatility

- One could just estimate the volatility of the entire process
  - Such an approach would not capture the volatility clustering of the process.
- A better approach would be let the volatility parameter change over time, mimicking the volatility of the index.

Approach: Use a model that captures the volatility clustering of the index.

Hidden Markov Models

Hidden Markov Models

- First introduced in the 1960's by Baum.
- First applications were speech recognition in the 1970's

Suppose we have a time series that from $t = 1, 2, \ldots, t_0$ is governed by

$$y_t = \mu_1 + \sigma_1 \epsilon_t$$

At time t_0, there was a significant change in the parameters of the series. Over $t_0, \ldots, t_1$, the series behaves as

$$y_t = \mu_2 + \sigma_2 \epsilon_t$$

Then, at t_1, it changes back.

Hidden Markov Models in Finance

Hamilton (1989) proposed hidden Markov models for financial applications.

The idea being the market passes through different states:

- A stable normal market
- A high-volatility market
- Periods of uncertainty in transition between the above two states

Hidden Markov models can capture volatility clustering through the underlying state process.

Hidden Markov Models in Finance

Regime Switching Model Characteristics:

- The distribution of Y_t is only known conditional on $\rho_t \in \{1, 2, \ldots, K\}$, the regime of the process at time t.
- The unobserved regime process is Markov.
- The one-period transition probabilities are defined as

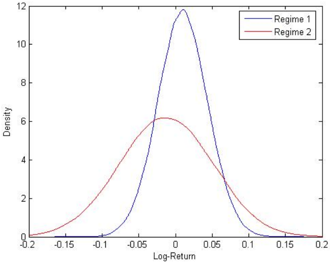
$$p_{i,j} = P[\rho_t = j | \rho_{t-1} = i] \qquad \forall i, j \in \{1, 2, \ldots, K\}, \forall t \in \{1, 2, \ldots, T\}$$

RSLN-2 Model: $Y_t = log(S_t / S_{t-1})$

$$\begin{aligned} Y_t | \rho_t &= \mu_{\rho_t} + \sigma_{\rho_t} \cdot \epsilon_t \\ \rho_t | \rho_{t-1} &= k \quad \text{w.p.} \quad p_{\rho_{t-1},k} \qquad k \in \{1, 2\} \end{aligned}$$

RSLN-2 Model for the S&P 500

Maximum Likelihood Parameters for the S&P 500:



Regime	μ	σ	Transition Parameters	Proportion
One	0.00990	0.03412	$p_{1,2} = 0.0475$	$\pi_1 = 0.809$
Two	-0.01286	0.06353	$p_{2,1} = 0.2017$	$\pi_2 = 0.191$

Generating a Volatility from the RSLN-2 Model

Static Unconditional Volatility

$$\begin{aligned}\sigma &= \sqrt{Var[Y_t]} \\ &= \sqrt{Var[E[Y_t|\rho_t]] + E[Var[Y_t|\rho_t]]}\end{aligned}$$

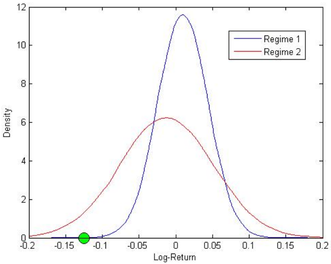
using the π_k's as regime weights.

This approach seems counterproductive:

- If one went to all the trouble of modeling volatility clustering, why use a static volatility?

Need to use the information in the data to more accurately select a volatility.

Data-Dependent Regime Probabilities



- The recent data observations provide insight into the current regime of the process.

Data-dependent Regime Probabilities:

$$p_k(t) = Pr(\rho_t = k | y_t, \ldots, y_1)$$

Data-Dependent Regime Probabilities

Future Data-dependent Regime Probabilities

$$
\begin{aligned}
p_k^+(t) &= Pr(\rho_{t+1} = k | y_t, \ldots, y_1) \\
&= p_1(t) \cdot p_{1,k} + p_2(t) \cdot p_{2,k}
\end{aligned}
$$

Question: How best can these probabilities be used in portfolio replication?

Generating a Volatility from the RSLN-2 Model

Dynamic Unconditional Volatility

$$\begin{aligned}\sigma &= \sqrt{Var[Y_t]} \\ &= \sqrt{Var[E[Y_t|\rho_t]] + E[Var[Y_t|\rho_t]]}\end{aligned}$$

using the $p_k^+(t)$'s as regime weights.

- If the model is 'correct', this is the unconditional volatility of the upcoming observation.
- The regime will be one or the other; the dynamic volatility will generally not be equal to either of the regime volatilities.
- But, you're somewhat covered against the less likely regime.

Regime Switching Optimization Methods

Indicator Volatility

$$\sigma = \sigma_k, \qquad \text{where } p_k^+(t) = \max(p_1^+(t), p_2^+(t))$$

- If the model is ‘correct’, this method will pick the correct volatility often.
- But, when you’ve picked the wrong regime, your volatility is significantly off.

Regime Switching Optimization Methods

One observation about the two methods:

- The change in hedging volatility significantly affects your monthly hedging error
- The Dynamic Volatility method has the largest number of significant jumps.
- The Indicator method has the biggest jumps. but less of them.

Question: Which of these hedging options is better?

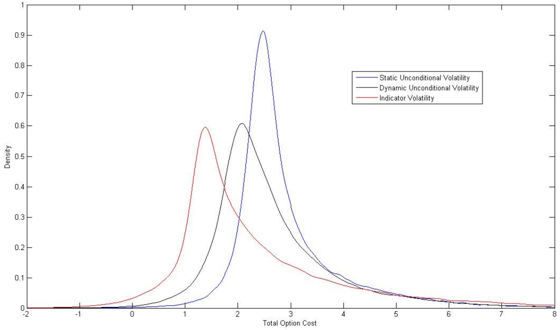
Regime Switching Optimization Methods

Answer: It's actually option dependent.

S&P 500 10-Year Put Example

- Strike Price $= S_0 = 100$
- Monthly re-balancing.
- Bond: 5% per annum.
- Transaction Costs: 0.02% of change in stock position
- Using the described hedging methods, simulate from the model to determine which method generates the smaller total option costs (initial hedge + hedging error)

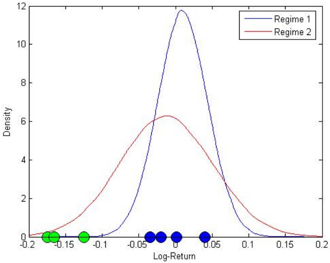
S&P 500 10-Year Put Example Results



Volatility	Static	Dynamic	Indicator
EPV[Total Option Cost]	2.9129	2.6406	2.3512

The Indictor method performs exceptionally well (19%!). But why?

S&P 500 10-Year Put Example Results



- The Dynamic and Indicator methods perform very similarly in most cases.
- When moving from Regime 2 to Regime 1, the Dynamic is too slow to react.

What About Parameter Uncertainty?

Parameter uncertainty is an important consideration

- Quite important for the example since I simulated from the fitted model to obtain results.
- Especially for Regime-switching models

Regime	μ	σ	Transition Parameters	Proportion
One	0.00990	0.03412	$p_{1,2} = 0.0475$	$\pi_1 = 0.809$
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Regime Switching Markov Chain Monte Carlo

Bayesian Modeling

- Treat each parameter as itself a random variable.
- Model beliefs about each parameter using prior distributions.
- Update your distributions based on the data to form posteriors.

For the RSLN-2, Metropolis-Hastings Algorithm was used

- Very quick simulation

RSLN-2 Parameter Comparison

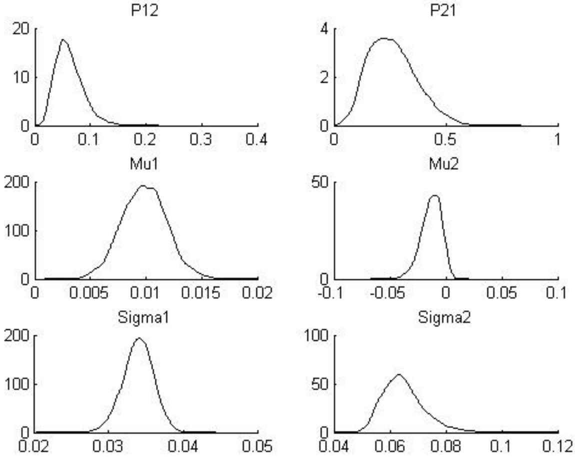
Maximum Likelihood Parameters for the S&P 500:

Regime	μ	σ	Transition Parameters
One	0.00990	0.03412	$p_{1,2} = 0.0475$
Two	-0.01286	0.06353	$p_{2,1} = 0.2017$

Bayesian Posterior-Means for the S&P 500:

Regime	μ	σ	Transition Parameters
One	0.0099	0.0340	$p_{1,2} = 0.0620$
Two	-0.0129	0.0652	$p_{2,1} = 0.2631$

RSLN-2 Parameter Posterior Distributions

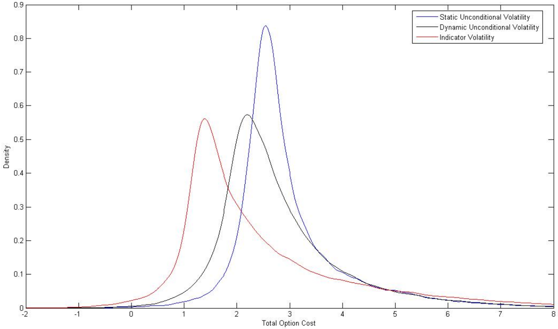


10-Year S&P Put Example

S&P 500 10-Year Put Example Revisited

- Use the posterior parameter distributions to generate the model simulations
- Still use the MLE parameter estimates for hedging decisions.

10-Year S&P Put Example Revisited



Volatility	Static	Dynamic	Indicator
EPV[Total Option Cost]	3.0107	2.8290	2.5306

The Indicator still performs best, but by less of a margin (16%).

Conclusions & Future Work

Summary of Results:

- Regime-switching portfolio replication can be worth it.
- Best type of method depends on the option you're hedging.
- Often, you want hedging strategies that react quickly.
- Parameter uncertainty can play a role.

Future Work

- More complicated Regime-Switching or Hybrid Models (RSGARCH)
- Relax the fixed interest rate assumption