On the Determination of Capital Charges in a Discounted Cash Flow Model

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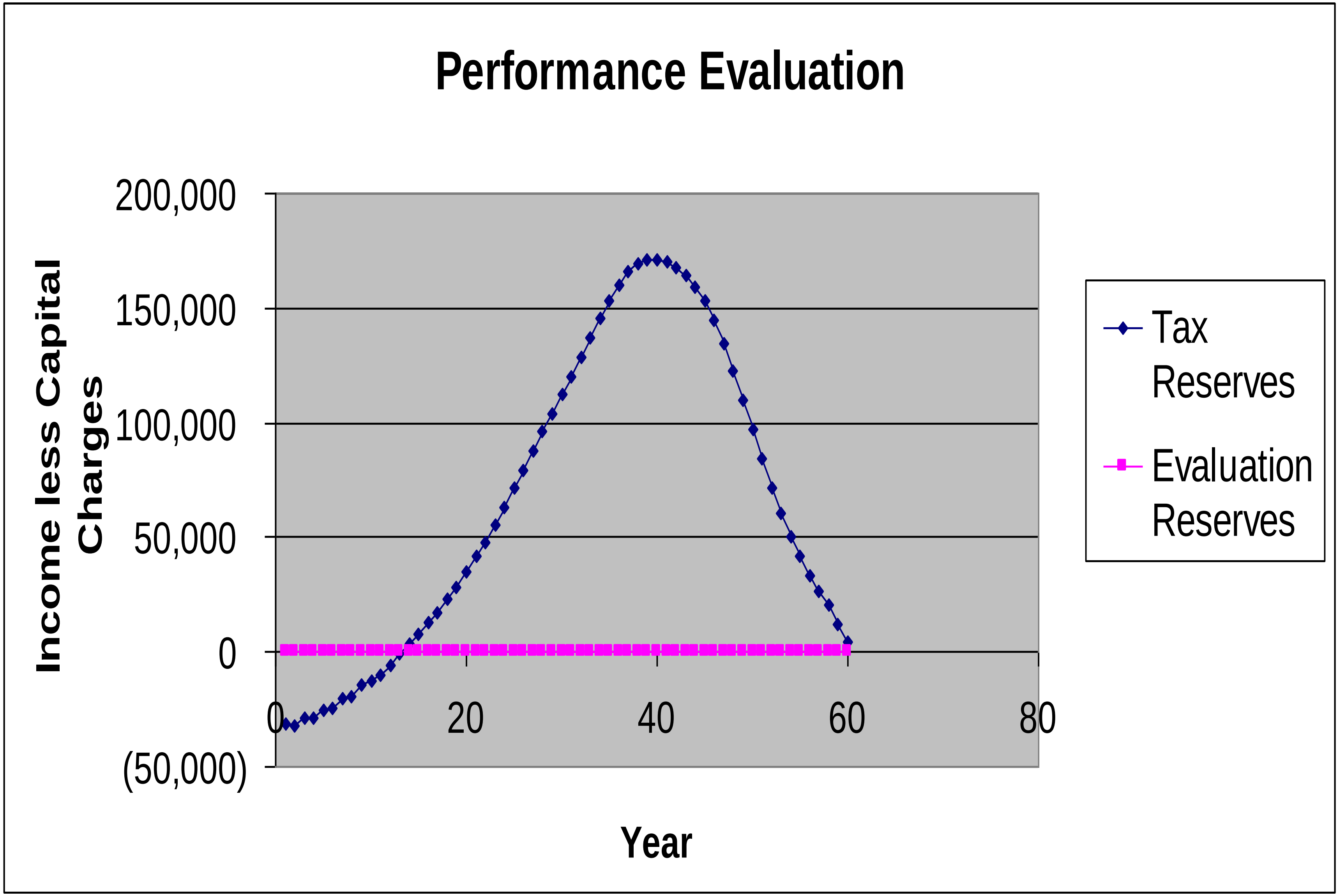
Motivation

- Solvency II
  - Required Assets determined on a consolidated basis
  - Assets allocated to the lines of business on a marginal basis
  - Division into “Reserves” and “Capital” is line by line
  - Do Capital Charges on capital and change in reserves cancel for performance analysis of line managers?

Motivation

- Multiple Candidates for Reserves:
  - U.S. Statutory Reserves;
  - U.S. GAAP Reserves;
  - U.S. Tax Reserves;
  - Fair Value of Liabilities;
  - Assets at a somewhat conservative solvency standard (Solvency II uses 75%);
  - Expected Loss under the realistic measure discounted at the risk-free rate.

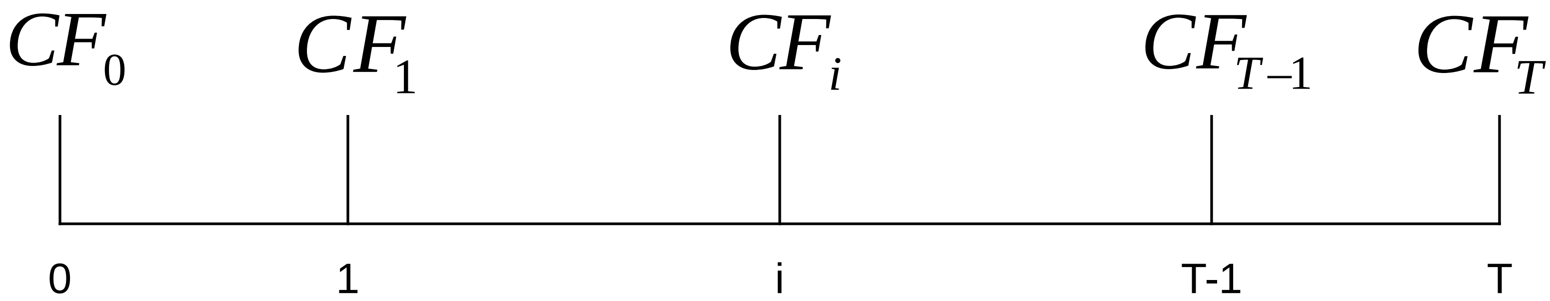


Performance Evaluation
Income less Capital Charges
200,000
150,000
100,000
50,000
0
(50,000)
0
20
40
60
80
Year
Tax Reserves
Evaluation Reserves

Discounted Cash Flow Model

- Myers and Cohn (1987)
- Cummins (1990)
- Taylor (1994)
  - Assumes reserves are “technical reserves”, i.e. discounted value of expected losses
  - Free parameter is “capital”, i.e. assets = capital + technical reserves.
- Overview in Cummins and Phillips (2000)

Examples
Single Premium / Single Loss



$$CF_0 = P - A_{0,T}$$

$$CF_1 = A_{0,T} - A_{1,T} - P\tau + A_{0,T} r_f (1-\tau) + V_{1,T}^{(\tau)} \tau$$

$$CF_i = A_{i-1,T} - A_{i,T} + A_{i-1,T} r_f (1-\tau) + \left(V_{i,T}^{(\tau)} - V_{i-1,T}^{(\tau)} \right) \tau$$

$$CF_T = A_{T-1,T} - L_T (1-\tau) + A_{T-1,T} r_f (1-\tau) - V_{T-1,T}^{(\tau)} \tau$$

Examples
Single Premium / Single Loss

$$P = \frac{E[L_T]}{(1+y)(1+x)^{T-1}} + \sum_{i=0}^{T-1} \frac{A_{i,T}(y - r_f)}{(1+y)(1+x)^i} - \tau \sum_{i=1}^{T-1} V_{i,T}^{(\tau)} \frac{y}{(1+y)(1+x)^i}$$

$$y \equiv \frac{x}{1-\tau}$$

– Equivalently

$$P = \frac{E[L_T]}{(1+r_f)^T} + \sum_{i=0}^{T-1} \frac{\Delta A_{i,T}(y - r_f)}{(1+y)(1+x)^i} - \tau \sum_{i=1}^{T-1} \frac{\Delta V_{i,T}^{(\tau)} y}{(1+y)(1+x)^i}$$

$$A_{i,T} = \frac{E[L_T]}{(1+r_f)^{T-i}} + \Delta A_{i,T} \qquad V_{i,T}^{(\tau)} = \frac{E[L_T]}{(1+r_f)^{T-i}} + \Delta V_{i,T}^{(\tau)}$$

Examples
Single Premium / Single Loss

- Evaluation Reserves $V^{(e)}_{i,T}$ and capital $A_{i,T} - V^{(e)}_{i,T}$

$$-E[L_T](1-\tau) + A_{t-1,t} r_f (1-\tau) + V^{(e)}_{t-1,t} - V^{(\tau)}_{t-1,t}\tau - (A_{t-1,t} - V^{(e)}_{t-1,t})x = 0$$

$$A_{i,T} r_f (1-\tau) + \left(V^{(e)}_{i,T} - V^{(e)}_{i+1,T}\right) - \left(V^{(\tau)}_{i,T} - V^{(\tau)}_{i+1,T}\right)\tau - (A_{i,T} - V^{(e)}_{i,T})x = 0$$

- solves for (by induction)

$$V^{(e)}_{T-1,T} = \frac{E[L_T]}{(1+x)} + \frac{A_{T-1,T}[x - r_f(1-\tau)]}{1+x} - \tau\frac{(E[L_T] - V^{(\tau)}_{T-1,T})}{1+x}$$

$$V^{(e)}_{i,T} = \frac{E[L_T]}{(1+x)^{T-i}} + \sum_{j=i}^{T-1}\frac{A_{j,T}[x - r_f(1-\tau)]}{(1+x)^{j-i+1}} - \tau\sum_{j=i}^{T-1}\frac{(V^{(\tau)}_{j+1,T} - V^{(\tau)}_{j,T})}{(1+x)^{j-i+1}}$$

$$V^{(e)}_{0,T} = P$$

Examples
Single Premium / Single Loss

- More intuitively ...

$$V_{i,T}^{(e)} = \frac{E[L_T]}{(1+r_f)^{T-i}} + \sum_{j=i}^{T-1} \frac{\Delta A_{j,T}[x - r_f(1-\tau)]}{(1+x)^{j-i+1}} - \tau \sum_{j=i}^{T-1} \frac{(\Delta V_{j+1,T}^{(\tau)} - \Delta V_{j,T}^{(\tau)})}{(1+x)^{j-i+1}}$$

Examples
Multiple Premium / Multiple Loss

- Nonstochastic P and A (i.e. losses are uncorrelated and premiums paid with certainty ...

$$\sum_{i=0}^{T-1} \frac{P_i}{(1+x)^i} = \sum_{i=1}^{T} \frac{E[L_i]}{(1+y)(1+x)^{i-1}} + \sum_{i=0}^{T-1} \frac{A_i(y-r_f)}{(1+y)(1+x)^i} - \tau \sum_{i=1}^{T-1} V_i^{(\tau)} \frac{y}{(1+y)(1+x)^i}$$

- Otherwise replace A_i with $E[A_i]$ in the premium equations, and A_j with $E[A_j \mid \Im_i]$ in the reserve equation. Make similar substitutions for P_i

Examples
Multiple Premium / Multiple Loss

- More intuitively ...

$$\sum_{i=0}^{T-1}\frac{P_i}{(1+r_f)^i}=\sum_{i=1}^{T}\frac{E[L_i]}{(1+r_f)^i}+\sum_{i=0}^{T-1}\frac{\Delta A_i(y-r_f)}{(1+y)(1+x)^i}-\tau\sum_{i=1}^{T-1}\frac{\Delta V_i^{(\tau)}y}{(1+y)(1+x)^i}$$

- Defining

$$A_i=\sum_{j=i+1}^{T}\frac{E[L_j]}{(1+r_f)^{j-i}}-\sum_{j=i+1}^{T-1}\frac{P_j}{(1+r_f)^{j-i}}+\Delta A_i$$

$$V_i^{(\tau)}=\sum_{j=i+1}^{T}\frac{E[L_j]}{(1+r_f)^{j-i}}-\sum_{j=i}^{T-1}\frac{P_j}{(1+r_f)^{j-i}}+\Delta V_i^{(\tau)}$$

- Practically, the ΔA_i and $\Delta V_i^{(\tau)}$ often depend on the premiums.

Examples
Multiple Premium / Multiple Loss

$$P_i(1-\tau) - E[L_{i+1}](1-\tau) + A_i r_f(1-\tau)$$
$$+ \left(V_i^{(e)} - V_{i+1}^{(e)}\right) - \left(V_i^{(\tau)} - V_{i+1}^{(\tau)}\right)\tau - (A_i - V_i^{(e)} - P_i)x = 0$$

$$V_i^{(e)} = \sum_{j=i+1}^{T} \frac{E[L_j]}{(1+x)^{j-i}} - \sum_{j=i}^{T-1} \frac{P_j}{(1+x)^{j-i}}$$
$$+ \sum_{j=i}^{T-1} \frac{A_j[x - r_f(1-\tau)]}{(1+x)^{j-i+1}} - \tau \sum_{j=i}^{T-1} \frac{[(V_{j+1}^{(\tau)} + E[L_{j+1}]) - [V_j^{(\tau)} + P_j]}{(1+x)^{j-i+1}}$$

$$V_i^{(e)} = \sum_{j=i+1}^{T} \frac{E[L_j]}{(1+r_f)^{j-i}} - \sum_{j=i}^{T-1} \frac{P_j}{(1+r_f)^{j-i}} + \sum_{j=i}^{T-1} \frac{\Delta A_j[x - r_f(1-\tau)]}{(1+x)^{j-i+1}} - \tau \sum_{j=i}^{T-1} \frac{(\Delta V_{j+1}^{(\tau)} - \Delta V_j^{(\tau)})}{(1+x)^{j-i+1}}$$

Solvency II Context
One Period

- In one year assets are $A_{0,1}[1+r_f(1-\tau)]-L_1(1-\tau)-P\tau$ and liabilities are 0. Solve

$$\Pr\{A_{0,1}[1+r_f(1-\tau)]-L_1(1-\tau)-P\tau \geq 0\} = 0.995$$

- to find

$$A_{0,1} = \frac{\xi_{L_1}^{0.995}(1-\tau)+P\tau}{[1+r_f(1-\tau)]}$$

- Premium is $P = \dfrac{E[L_1]+(\xi_{L_1}^{0.995}-E[L_1])R}{(1+r_f)}$ with

$$R \equiv \frac{x-r_f(1-\tau)}{1+x}$$

Solvency II Context
Multiple Period

- Last period is similar: $A_{T-1,T} = \dfrac{\xi_{L_T}^{0.995}(1-\tau) + V_{T-1,T}^{(\tau)}\tau}{[1 + r_f(1-\tau)]}$
- Other periods require the determination of

  $MV_0[L_{[T-i]}] = MV_i[L_T]$
- Key insight: $\tilde{P}_{i,T}$, the premium which would be charged at time i to cover the loss at time t, must be $MV_i[L_T]$ and this premium can be found from the previous analysis.
- Find the market values recursively.

Solvency II Context
Multiple Period

$$MV_j[L_t] = \frac{MV_{j+1}[L_T]}{(1+r_f)} + r_\tau \left[\frac{MV_{j+1}[L_T] - V^{(\tau)}_{j+1,t}}{(1+r_f)} \right] \qquad r_\tau \equiv \frac{\tau x}{(1-\tau)(1+x)} = \frac{1+y}{1+x} - 1$$

$$P = MV_0[L_T] = \left[\frac{E[L_T] + (\xi^{0.995}_{L_t} - E[L_T])R}{(1+r_f)^T} \right] (1+r_\tau)^{T-1} - r_\tau \sum_{i=1}^{T-1} \frac{V^{(\tau)}_{i,T}}{(1+r_f)^i} (1+r_\tau)^{T-1}$$

- Assets from

$$\Pr\left[A_{i,T}[1+r_f(1-\tau)] + (V^{(\tau)}_{i+1,T} - V^{(\tau)}_{i,T})\tau - \left\{ \left[\frac{E[L_T] + (\xi^{0.995}_{L_t} - E[L_T])R}{(1+r_f)^{T-i-1}} \right] (1+r_\tau)^{T-i-2} - r_\tau \sum_{j=i+2}^{T-1} \frac{V^{(\tau)}_{j,t}}{(1+r_f)^{j-i-1}} (1+r_\tau)^{j-i-2} \right\} \geq 0 \right] = 0.995$$

Solvency II Context
Multiple Period

- Assets

$$A_{i,t} = \frac{\left[\frac{E[L_T] + (\xi_{L_t}^{0.995} - E[L_T])R}{(1+r_f)^{T-i-1}} \right]}{[1 + r_f(1-\tau)]} (1 + r_\tau)^{T-i-2} - \frac{(V_{i+1,T}^{(\tau)} - V_{i,T}^{(\tau)})\tau}{[1 + r_f(1-\tau)]}$$

$$\frac{- r_\tau \sum_{j=i+2}^{T-1} \frac{V_{j,T}^{(\tau)}}{(1+r_f)^{j-i-1}} (1 + r_\tau)^{j-i-2}}{[1 + r_f(1-\tau)]}$$

Solvency II Context
Multiple Period

- Evaluation Reserves

$$V_{i,T}^{(e)} = \frac{E[L_T]}{(1+x)^{T-i-2}[1+r_f(1-\tau)](1+r_f)}\left[1+\frac{r_f\tau}{1+x}+r_\tau R+(1+r_\tau)R\ddot{s}_{\overline{T-i-2}|r_{ann}}\right]$$

$$+\frac{(\xi_{L_t}^{0.995}-E[L_T])R}{(1+x)^{T-i-2}[1+r_f(1-\tau)](1+r_f)}\left[1-\frac{\tau}{1+x}+R\ddot{s}_{\overline{T-i-2}|r_{ann}}\right]$$

$$-\sum_{j=i}^{T-1}\frac{(V_{i+1,T}^{(\tau)}-V_{i,T}^{(\tau)})\tau}{[1+r_f(1-\tau)](1+x)^{j-i}}-r_\tau\sum_{j=i+2}^{T-1}\frac{V_{j,T}^{(\tau)}R\ddot{s}_{\overline{j-i-1}|r_{ann}}}{(1+x)^{j-i-1}[1+r_f(1-\tau)](1+r_\tau)}$$

$$r_{ann}=\frac{(1+r_\tau)(1+x)}{(1+r_f)}-1=\frac{1+y}{1+r_f}-1$$

Examples
Two Period Loss

$$E[L_1] = 400 \qquad \xi_{L_1}^{0.995} = 500$$

$$E[L_2] = 500 \qquad \xi_{L_2}^{0.995} = 700$$

$$r_f = 6\% \qquad x = 10\%$$

- Tax Reserves are Eq. Principle reserves at 7%.
- Guess $P_0 = P_1 = \Pi$
- Assets are $A_i = \dfrac{\xi_{L_{i+1}}^{0.995}(1-\tau) + \left(V_i^{(\tau)} - V_{i+1}^{(\tau)}\right)\tau + \tau P_i + MVL_{i+1}}{\left(1 + r_f(1-\tau)\right)}$

Examples
Two Period Loss

- Market Value of Liabilities are

$$MVL_i = \sum_{j=i+1}^{T} \frac{E[L_j -]}{(1+y)(1+x)^{j-i-1}} + \sum_{j=i}^{T-1} \frac{A_j(y-r_f)}{(1+y)(1+x)^{j-i}}$$

$$-\tau \sum_{j=i+1}^{T-1} V_j^{(\tau)} \frac{y}{(1+y)(1+x)^{j-i}} - \sum_{j=i}^{T-1} \frac{P_j}{(1+x)^{j-i}}$$

- $\Pi = 430.9106895$ sets $MVL_0 = 0$

Examples
Two Period Loss

Balance Sheet Items for Two Premium Two Loss Example

Time i	$V_i^{(\tau)}$	$\Delta V_i^{(\tau)}$	A_i	ΔA_i	$V_i^{(e)}$	Capital	MVL_i
0	0.00	0.00	491.69	75.85	0.00	60.78	0.00
1	48.31	7.52	601.13	129.43	50.22	120.00	51.07

Income Statement Items for Two Premium Two Loss Example

i	Cash Flow CF_i	Cash Income	Change in $V_i^{(e)}$	Capital Charges
0	(60.78)			
1	(53.15)	56.30	(50.22)	(6.08)
2	132.00	(38.22)	50.22	(12.00)

Examples
Two Year Term Life

- \$100,000 face, 1000 identical individuals

$$q_x = 0.020 \quad q_{x+1} = 0.025$$

$$P_0 = 1000\Pi \quad E[P_1] = 980\Pi$$

$$A_1(N_1) = \frac{\xi^{0.995}_{L_2|N_1}(1-\tau) + \tau N_1 \Pi}{[1 + r_f(1-\tau)]}$$

- $L_2 | N_1$ is binomial with probability 0.025
- N_1 is binomial with probability 0.02

$$E[A_1] = 2{,}412{,}312.05 + 320.5078876\Pi$$

Examples
Two Year Term Life

$$MVA_1 = A_0[1+r_f(1-\tau)] - 1000\,\Pi\,\tau - 100{,}000 * (1000 - N_1)(1-\tau)$$

$$MVL_1 = \frac{E[L_2 \mid N_1]}{(1+y)} - N_1\Pi + \frac{A_1[N_1](y-r_f)}{(1+y)}$$

- 99.5% solvency at $N_1 = 968$

$$A_0 = 4{,}237{,}501.48 - 579.87683598\,\Pi$$

- Determine $\Pi = 2{,}185.20$

$$E[V_1^{(e)}] = 161{,}338.82$$

Examples
Two Year Term Life

Balance Sheet Items for Term Life Example

i	$E[V_i^{(\tau)}]$	$E[\Delta V_i^{(\tau)}]$	$E[A_i]$	$E[\Delta A_i]$	$E[V_i^{(e)}]$	Expected Capital	$E[MVL_i]$
0	0.00	0.00	2,970,357.36	923,348.50	0.00	785,162.09	0.00
1	0.00	(169,829.39)	3,112,684.37	801,363.61	161,338.82	809,854.19	233,516.71

Income Statement Items for Term Life Example

i	Cash Flow $E[CF_i]$	Expected Cash Income	Change in $E[V_i^{(e)}]$	Expected Capital Charges
0	(785,162.09)			
1	53,824.11	239,855.03	(161,338.82)	(78,516.21)
2	890,839.61	(80,353.40)	161,338.82	(80,985.42)

Examples
Whole Life

- We need $\xi^{0.995}_{D_i|N_i}$, $\Pr[N_j \mid N_i]$
- Assume $MVL_{i+1}[N_{i+1}] = c_{i+1}\left(N_{i+1}\right) + d_{i+1}\left(N_{i+1}\right)\Pi$
- Solve for

$$A_i[N_i] = \frac{100{,}000\,\xi^{0.995}_{D_i|N_i}(1-\tau) - \tau\left(N_i - \xi^{0.995}_{D_i|N_i}\right)\left(\tilde{V}^{(\tau)}_{i+1} - N_i\tilde{V}^{(\tau)}_i\right) + c_{i+1}\left(N_i - \xi^{0.995}_{D_i|N_i}\right)}{\left(1 + r_f(1-\tau)\right)}$$

$$+ \frac{\tau N_i + d_{i+1}\left(N_i - \xi^{0.995}_{D_i|N_i}\right)}{\left(1 + r_f(1-\tau)\right)}\Pi = a_i[N_i] + b_i[N_i]\Pi$$

$$E_k\left[A_i\left(N_k\right)\right] = \sum_{N_i=0}^{1000} \Pr[N_i \mid N_k]\, A_i\left(N_i\right) = e_{ki}\left(N_k\right) + f_{ki}\left(N_k\right)\Pi$$

Examples
Whole Life

- Finally (whew ...)

$$MVL_i[N_i] = \left\{ \sum_{j=i+1}^{t} \frac{100{,}000\, N_i \left({}_{j-i-1}p_{x+i} \right) q_{x+j-1}}{(1+y)(1+x)^{j-i-1}} + \sum_{j=i}^{t-1} \frac{e_{ij}[N_i](y-r_f)}{(1+y)(1+x)^{j-i}} \right.$$

$$\left. - \tau \sum_{j=i}^{t-1} \frac{y N_i \left({}_{j-i}p_{x+i} \right) \tilde{V}_j^{(\tau)}}{(1+y)(1+x)^{j-i}} \right\} + \left\{ \sum_{j=i}^{t-1} \frac{f_{ij}[N_i](y-r_f)}{(1+y)(1+x)^{j-i}} - \sum_{j=i}^{t-1} \frac{N_i \left({}_{j-i}p_{x+i} \right)}{(1+x)^{j-i}} \right\} \Pi$$

$$= c_i[N_i] + d_i[N_i]\Pi$$

- Then
- $MVL_0[1000] = 0$ gives $\Pi = \dfrac{-c_0[1000]}{d_0[1000]}$

Examples
Whole Life

- 1980 CSO on 40 year old.

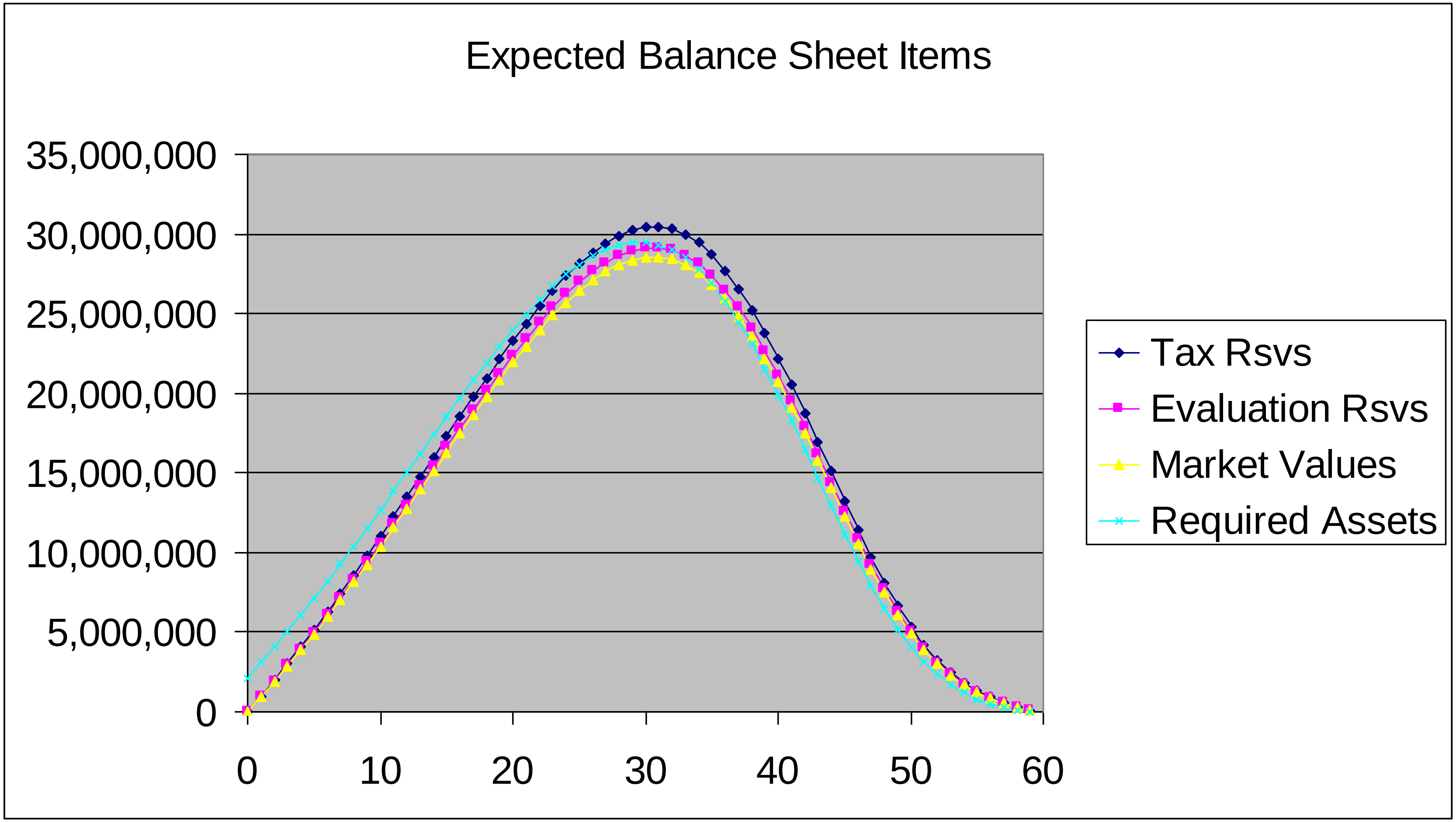
Assumptions	Premium
Equivalence Principle	$1203.30
Sol 99.5%, Tax EP 6%	$1234.95
Sol 99.5%, Tax EP 6.5%	$1272.80
Sol 99.5%, Tax CRVM 6.5%	$1301.37
Sol 99%, Tax EP 6%	$1233.50
Sol 95%, Tax EP 6%	$1229.28

Examples
Whole Life

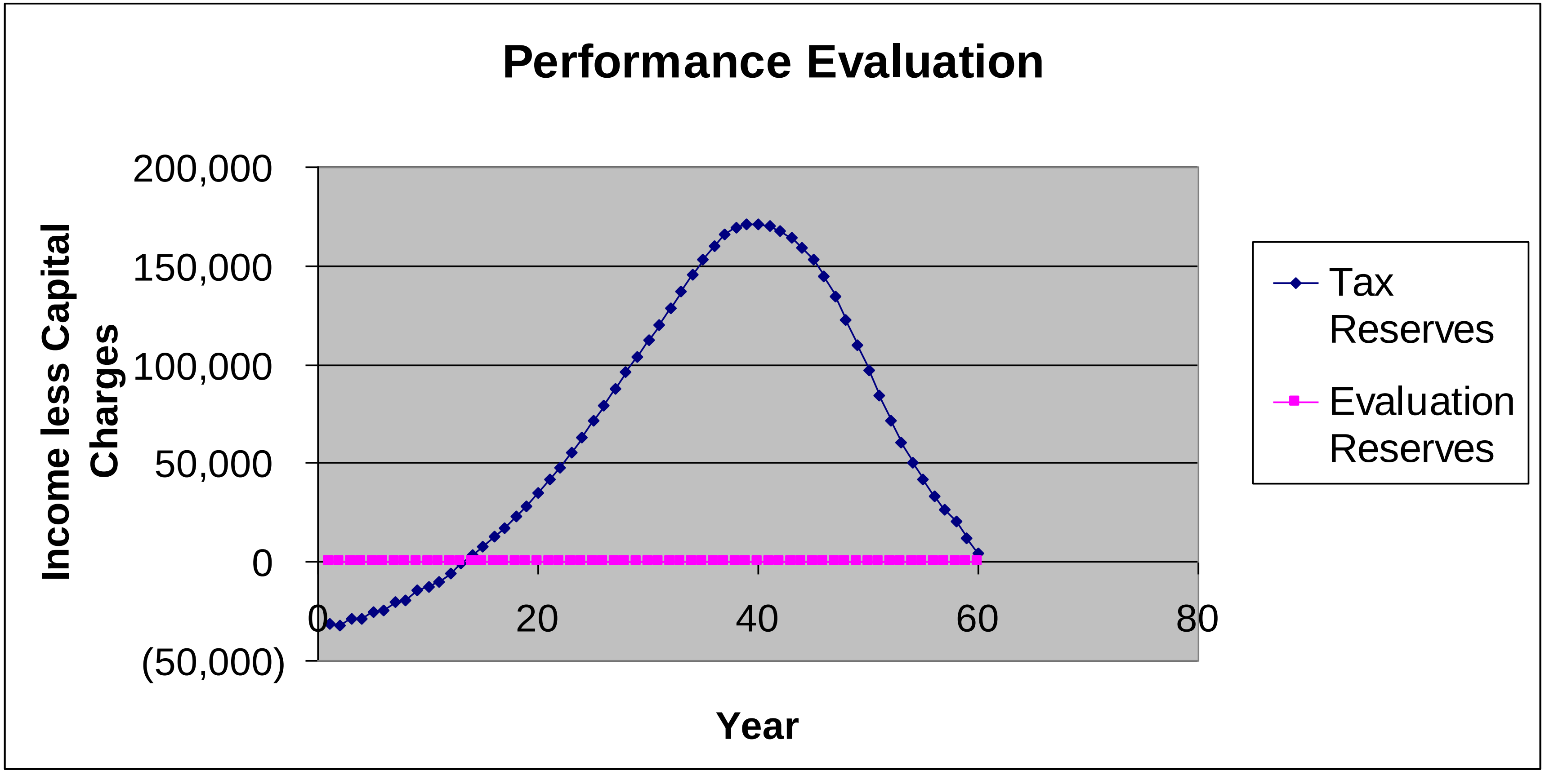
i	Cash Flow $E[CF_i]$	Expected Cash Income	Change in $E[V_i^{(e)}]$	Expected Capital Charges
0	(883,845)			
1	48,144	1,030,638	(942,253)	(88,385)
2	110,037	1,059,478	(967,070)	(92,409)
3	57,235	1,087,511	(996,865)	(90,646)

i	Expected Cash Income	Change in $E[V_i^{(\tau)}]$	Adjusted Expected Capital Charges	Adjusted Income
0				
1	1,030,638	(973,497)	(88,385)	(31,244)
2	1,059,478	(1,002,048)	(89,284)	(31,854)
3	1,087,511	(1,032,237)	(84,024)	(28,749)

Examples
Whole Life



Examples
Whole Life



Examples
GMDB

$$MVL(S_i) = \sum_{t=1}^{\infty} q(1 - q - \lambda)^{t-1} \left[e^{-r_f t} N(-d_2) - S_i (1-\Pi)^t N(-d_1) \right]$$

$$- \sum_{t=0}^{\infty} (1 - q - \lambda)^t \, \Pi (1-\Pi)^t S_i$$

$$d_1 = \frac{\ln[S_i (1-\Pi)^t] + \left(r + \frac{\sigma^2}{2} \right) t}{\sigma \sqrt{t}}$$

$$d_2 = \frac{\ln[S_i (1-\Pi)^t] + \left(r - \frac{\sigma^2}{2} \right) t}{\sigma \sqrt{t}}$$

Examples
GMDB

- Solvency criterion gives

$$\xi_{s_{i+1}}^{0.995} = S_i e^{\mu - \sigma N^{-1}(0.995)} (1 - \Pi)$$

$$A(S_i) = \frac{(1 - q - \lambda) MVL(\xi_{s_{i+1}}^{0.995}) + q Max(1 - \xi_{s_{i+1}}^{0.995}, 0)(1 - \tau)}{[1 + r_f (1 - \tau)]}$$

$$+ \frac{[\widetilde{V}^{(\tau)}(S_i) - (1 - q - \lambda)\widetilde{V}^{(\tau)}(\xi_{s_{i+1}}^{0.995})] + S_i \Pi \tau}{[1 + r_f (1 - \tau)]}$$

Examples
GMDB

- Assume $r_f = 6\%$ $\tau = 34\%$

  $q = 2\%$ $\lambda = 10\%$

- $\Pi = 0.0009422$ from option pricing theory.
- "x" is the free parameter, and equations give: $x = 12.07\%$

Examples
GMDB