

Coherent Distortion Risk Measures in Portfolio Selection

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Abstract

The theme of this paper relates to solving portfolio selection problems using linear programming. We extend the well-known linear optimization framework for Conditional Value-at-Risk (CVaR)-based portfolio selection problems (see Rockafellar and Uryasev [30, 31]) to optimization over a more general class of risk measure known as the class of *Coherent Distortion Risk Measure (CDRM)*. In addition to CVaR, CDRM includes many other well-known risk measures including the Wang Transform measure, the Proportional Hazard measure, and the lookback measure. A case study is conducted to illustrate the flexibility of the linear optimization scheme, explore the efficiency of the $\frac{1}{n}$ -portfolio strategy, as well as compare and contrast optimal portfolios with respect to different CDRMs.

Keywords:

coherent risk measures, distortion risk measures, portfolio selection, conditional value-at-risk

1. Introduction

The problem of optimal portfolio selection is of paramount importance to investors, hedgers, fund managers, among others. Inspired by the seminal work of Markowitz [27], the research on optimal portfolio selection has been growing rapidly. Researchers and practitioners are constantly seeking better and more sophisticated risk and reward tradeoff in constructing optimal portfolios. The classical Markowitz model used variance as the measure of risk and this is perceived to be undesirable since it penalizes equally, regardless of downside risk or upside potential. Consequently, other measures of risk have been proposed in connection to portfolio optimization. These include semi-variance (see Markowitz [28]), partial moments (see Bawa and Lindenberg [5]), safety first principle (see Roy [32]), skewness and kurtosis (see Lai [26], Chunchachinda et al. [11], and Harvey et al. [21]).

More recently, both value-at-risk (VaR) and conditional value-at-risk (CVaR)³ risk mea-

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³CVaR is known as tail conditional expectation (TCE) in Artzner et al. [3], conditional tail expectation (CTE) in Wirch and Hardy [39], mean shortfall in Bertsimas et al. [9], and expected shortfall in Acerbi et al. [2]. Here we adopt the terminology of Rockafellar and Uryasev [30]. Readers are encouraged to refer to Rockafellar and Uryasev [31] for more comprehensive discussions on CVaR.

asures have been advocated in the context portfolio selection. See for example Gaivoronski and Pflug [18] and Rockafellar and Uryasev [30]. These two risk measures are widely accepted measure of financial risk among market participants and they have been adopted by the regulators for risk quantification. For example, VaR was adopted as the “first pillar” in Basel II, which are recommendations on banking laws and regulations mandated by Basel Committee on Banking Supervision (see Cooke [12]). CVaR is recommended by the National Association of Insurance Commissioners (NAIC) for setting regulatory risk-based capital requirements for variable annuities and similar products (see Tim Gaule [33]). See also Jorion [23] and Pritsker [29] which provided comprehensive discussions on risk management using VaR. Despite its popularity, a number of researchers cautioned the use of VaR as a measure of risk and instead supported CVaR. Basak and Shapiro [4] showed theoretically that optimal decisions based on VaR could result in higher risk exposure than when decisions were based on expected losses. Through an axiomatic approach, Artzner et al. [3] defined an important class of risk measures known as the coherent risk measure (CRM) and they shown that VaR fails to satisfy the sub-additivity property and hence it is not a CRM. CVaR, in contrast, satisfies all the properties of a CRM.

Another interesting class of risk measures is known as the distortion risk measure (DRM) which was studied by Wang [35, 36]. The axioms that were used to define DRM were originated from the insurance premium principles (see Goovaerts et al. [19] and Wang et al. [37]). Campana and Ferretti [10] considered the applications of DRM for discrete loss distributions. Gouieroux and Liu [20] provided a statistical framework for analyzing the sensitivities of DRMs with respect to various risk aversion parameters. Generally speaking, a DRM is the expectation of portfolio loss random variable under a distorted probability measure. Different distortions reflect different risk appetites of decision makers. From a mathematical point of view, DRM is a Choquet integral and all the standard results about Choquet integrals, such as those discussed in Denneberg [14], are applicable to DRM.

It is important to point out that both CRM and DRM are not subclasses of each other. Kusuoka [25] studied subclasses of CRM and proved a representation theorem for comonotonic law-invariant coherent risk measures. For continuous loss distributions, any comonotonic law-invariant coherent risk measure can be represented as a convex combination of CVaR’s at different confidence levels. Bertsimas and Brown [8] proved similar results for discrete loss distributions and strengthen their claim by proving only finite number of CVaR’s are needed in the representation. To the best of our knowledge, Bellini and Caperdoni [6] was the first to synchronize CRM and DRM and to study the intersection of both classes. They named the intersection of both CRM and DRM as the coherent distortion risk measures (CDRM) and CVaR is one of such members. They showed that the class of comonotonic law-invariant CRMs is precisely the class of CDRM. Acerbi and Simonetti [1] studied spectral measures of risk and applied them in portfolio selection problems. A spectral risk measure with admissible spectrum defined in their work can be viewed as a CDRM yet it overlooks the connection with its underlying distortion function and hence lack of interpretation of its risk appetite.

Parallel to the development of the more advanced portfolio models, there is also an

increased demand for more sophisticated mathematical programming methods in solving the resulting optimization problems. Special features of the optimization problems are exploited to facilitate the formulation and solving of the mathematical programming methods. To elaborate, it is well known that solving the CVaR-based portfolio problem directly can be quite challenging as it is formulated as a non-linear programming. On the other hand, as shown in Rockafellar and Uryasev [30, 31] that the CVaR-based optimization problem can be equivalently formulated as a linear program. Their results drastically stimulated the applicability of the optimization problems associated with CVaR. For example, Krokhmal et al. [24] investigated three different formulations of CVaR portfolio selection problems. Fábíán [17] considered CVaR objectives and constraints in two-stage stochastic models.

The rest of the paper is organized as follows. Section 2 and 3 review, respectively, the CVaR minimization approach developed in Rockafellar and Uryasev [30, 31] and the classes of CRM and DRM. Main contributions of the paper are collected in Section 4. We begin the section by first listing some properties associated with CDRM. We then generalize the finite generation theorem Bertsimas and Brown [8] for CDRM. We also show that any CDRM can be defined as a convex combination of ordered portfolio losses and equivalently a convex combination of CVaRs. We solve the CDRM-based portfolio optimization via linear programming and thus generalize the results of Rockafellar and Uryasev [30, 31]. Section 5 complements the paper by providing some numerical examples to illustrate the applicability of our proposed optimization problems. In particular, we implement four different CDRM-based portfolio models and these results are compared to the naive $\frac{1}{n}$ -portfolio strategy. Section 6 concludes the paper.

2. CVaR-Based Portfolio Optimization Model

The purpose of this section is to introduce two of the most popular risk measures known as the Value-at-Risk (VaR) and the Conditional Value-at-Risk (CVaR). The connection to portfolio optimization, particularly the convex formulation and linearization scheme for the CVaR minimization problems of Rockafellar and Uryasev [30, 31] is highlighted.

Let $l = f(\mathbf{x}, \mathbf{y})$ be portfolio losses associated with the decision vector \mathbf{x} , to be chosen from a set $\mathbf{S} \subseteq \mathbb{R}^n$, and the random vector $\mathbf{y} \in \mathbb{R}^m$. The vector \mathbf{x} represents what we may generally call a portfolio, with \mathbf{S} capturing the set of all feasible portfolios subject to certain portfolio constraints. For every \mathbf{x} , the loss $f(\mathbf{x}, \mathbf{y})$ is a random variable having a distribution in \mathbb{R} induced by the distribution of $\mathbf{y} \in \mathbb{R}^m$. The underlying probability distribution of \mathbf{y} is assumed to be discrete with probability masses \mathbf{p} , i.e., $\Pr[l = l(\mathbf{x}, y_i)] = p_i$ for $i = 1, \dots, m$.

Note that in many cases it is assumed that $p_i = \frac{1}{m}$, i.e., the portfolio loss has a discrete uniform distribution. This is not a very limiting assumption if we restrict ourselves to discrete portfolio loss distributions, which is typically the case if we are obtaining distributional information via scenario generation or from historical data samples. In addition, given any arbitrary discrete distribution representable with rational numbers, we may always convert it to discrete uniform distribution for some large enough m . While we impose this assumption in our numerical example (see Section 5), we emphasize that we do not rely on such assumption in our proposed risk measure based optimization framework to be discussed in Section 4.

For every portfolio \mathbf{x} , denoted by $\Psi(\mathbf{x}, \cdot)$ the cumulative distribution function for the portfolio loss $\mathbf{l} = f(\mathbf{x}, \mathbf{y})$,

$$\Psi(\mathbf{x}, \zeta) = \sum_{i=1}^m p_i \mathbf{1}_{\{l_i \leq \zeta\}} \quad (1)$$

Then α -VaR and α -CVaR are defined as follows (see Rockafellar and Uryasev [31, Proposition 8]):

Definition 2.1. Suppose for each $\mathbf{x} \in \mathbf{S}$, the distribution of the portfolio loss $\mathbf{l} = f(\mathbf{x}, \mathbf{y})$ is concentrated in $m < \infty$ points, and $\Psi(\mathbf{x}, \cdot)$ is a step function with jumps at these points. Now fixing \mathbf{x} and let $l_{(1)} \leq l_{(2)} \leq \dots \leq l_{(m)}$ denote the corresponding ordered portfolio loss points and $p_{(i)} > 0, i = 1, \dots, m$, represent the probability of realizing loss $l_{(i)}$. If i_α denotes the unique index satisfying

$$\sum_{i=1}^{i_\alpha} p_{(i)} \geq \alpha > \sum_{i=1}^{i_\alpha-1} p_{(i)}, \quad (2)$$

then α -VaR and α -CVaR of the portfolio loss are given, respectively, by

$$\zeta_\alpha(\mathbf{x}) = l_{(i_\alpha)}, \quad (3)$$

and

$$\phi_\alpha(\mathbf{x}) = \frac{1}{1-\alpha} \left[\left(\sum_{i=1}^{i_\alpha} p_{(i)} - \alpha \right) l_{i_\alpha} + \sum_{i=i_\alpha+1}^m p_{(i)} l_{(i)} \right]. \quad (4)$$

Essentially, VaR is a quantile risk measure which measures the potential loss over a defined period for a given confidence level while CVaR captures the average losses from extreme event. These risk measures are used by banks and insurance companies, respectively, for the calculation of regulatory capital risk charge. In addition to been adopted by the regulators for quantifying risk, these risk measures have also been exploited in the context of portfolio selection. In fact a general portfolio optimization model can be formulated as follows:

$$\min_{\mathbf{x} \in \mathbf{S}} \rho(\mathbf{x}) \quad (5)$$

for appropriately defined feasible portfolios \mathbf{S} and appropriately chosen risk measure ρ .

Note that if ρ captures the variance of the return of portfolio \mathbf{x} , then the above optimization problem recovers the standard Markowitz model. Similarly, if ρ is replaced by VaR, CVaR or any other risk measure, then we have a generalization of the Markowitz portfolio model.

Gaivoronski and Pflug [18] pointed out that the portfolio optimization problem (5) associated with VaR is numerically very challenging due to its lack of convexity. In contrast, the CVaR-based portfolio optimization problem (5) is a convex program and hence is more tractable computationally. The CVaR-based portfolio model becomes even more popular and more practical when Rockafellar and Uryasev [30] shown that the convex program can in fact be formulated as a liner program. The key to the development of Rockafellar-Uryasev's

linear optimization scheme of CVaR-based portfolio selection problem is a characterization of $\phi_\alpha(\mathbf{x})$ and $\zeta_\alpha(\mathbf{x})$ in terms of a special function $F_\alpha(\mathbf{x}, \zeta)$ given by

$$F_\alpha(\mathbf{x}, \zeta) = \zeta + \frac{1}{1-\alpha} E[(\mathbf{l}(\mathbf{x}, \mathbf{y}) - \zeta)^+] = \zeta + \frac{1}{1-\alpha} \sum_{i=1}^m p_i (l_i - \zeta)^+. \quad (6)$$

As shown in Rockafellar and Uryasev [31], if $f(\mathbf{x}, \mathbf{y})$ is convex with respect to \mathbf{x} , then $\phi_\alpha(\mathbf{x})$ is convex with respect to \mathbf{x} . In this case, $F_\alpha(\mathbf{x}, \zeta)$ is also jointly convex in (\mathbf{x}, ζ) .

Armed with these findings, they derived the following key equivalence formulation (Rockafellar and Uryasev [31, Theorem 14]):

Theorem 2.1. *Minimizing $\phi_\alpha(\mathbf{x})$ with respect to $\mathbf{x} \in \mathbf{S}$ is equivalent to minimizing $F_\alpha(\mathbf{x}, \zeta)$ over all $(\mathbf{x}, \zeta) \in \mathbf{S} \times \mathbb{R}$, in the sense that*

$$\min_{\mathbf{x} \in \mathbf{S}} \phi_\alpha(\mathbf{x}) = \min_{(\mathbf{x}, \zeta) \in \mathbf{S} \times \mathbb{R}} F_\alpha(\mathbf{x}, \zeta) \quad (7)$$

where moreover

$$(\mathbf{x}^{*T}, \zeta^*) \in \underset{(\mathbf{x}, \zeta) \in \mathbf{S} \times \mathbb{R}}{\operatorname{argmin}} F_\alpha(\mathbf{x}, \zeta) \iff \mathbf{x}^* \in \underset{\mathbf{x} \in \mathbf{S}}{\operatorname{argmin}} \phi_\alpha(\mathbf{x}), \zeta^* \in \underset{\zeta \in \mathbb{R}}{\operatorname{argmin}} F_\alpha(\mathbf{x}^*, \zeta) \quad (8)$$

The above theorem links the representation (6) explicitly to both VaR and CVaR simultaneously. The theorem asserts that for the purpose of determining an optimal portfolio with respect to CVaR, we can replace $\phi_\alpha(\mathbf{x})$ by $F_\alpha(\mathbf{x}, \zeta)$ in portfolio selection problems. More importantly, by exploiting (6) the general convex programming of CVaR portfolio optimization problem can be linearized into a linear objective function with an additional linear auxiliary constraints. With such linear representation we can cast any portfolio selection problem with CVaR objective and linear constraint(s) as a linear program (LP). This significantly reduces the computation effort in obtaining the optimal portfolios.

3. Coherent Risk Measure (CRM) and Distortion Risk Measure (DRM)

In this section, we review two important families of risk measures known as coherent risk measure (CRM) and distortion risk measure (DRM).

The uncertainty for future value of an investment position is usually described by a function $X : \Omega \mapsto \mathbb{R}$, where Ω is a fixed set of scenarios with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let \mathcal{X} be a linear space of random variables on Ω , i.e., a set of functions $X : \Omega \mapsto \mathbb{R}$. It is assumed that X is bounded. In particular, $\mathcal{X} \subseteq L^\infty(\Omega, \mathcal{F}, \mathbb{P})$.⁴ For introduction, X can be thought of as a loss from an uncertain position. For $X, Y \in \mathcal{X}$, we denote the state-wise dominance by $X \geq Y$, i.e., $X \geq Y \Leftrightarrow X(\omega) \geq Y(\omega)$ for all $\omega \in \Omega$.

In Artzner et al. [3] CRM is defined through the following set of axioms:

⁴When we impose that $|\Omega|$ is finite and supported by finite elements, this is automatically satisfied.

Definition 3.1. A mapping $\rho : \mathcal{X} \mapsto \mathbb{R}$ is called a coherent risk measure if it satisfies, for all $X, Y \in \mathcal{X}$:

- C1** Monotonicity: $\rho(X) \geq \rho(Y)$ for all $X, Y \in \mathcal{X}$ and $X \geq Y$.
- C2** Translation invariance: $\rho(X + a) = \rho(X) + a$ for all $X \in \mathcal{X}$ and $a \in \mathbb{R}$.
- C3** Positive homogeneity: $\rho(\lambda X) = \lambda \rho(X)$ for all $X \in \mathcal{X}$ and $\lambda > 0$.
- C4** Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$ for all $X, Y \in \mathcal{X}$.

The financial meaning of monotonicity is clear: The risk of a portfolio is at least as much as another one if former incurs at least as much losses as the latter in very state of economy. Translation invariance is motivated by the interpretation of $\rho(X)$ as a reserve requirement, i.e., $\rho(X)$ is the amount which should be raised in order to make X acceptable from the point of view of a supervising agency. Thus, if there is a constant loss added to all future state of economy, then the reserve requirement is increased by the same amount. The positive homogeneity axiom states that risk scales linearly with the size of a position. Under positive homogeneity, the axiom of subadditivity is equivalent to convexity, which fosters the notion that diversification should not increase risk.

Similar to the axiomatic approach for constructing CRM, Wang et al. [37] proposed the following four axioms to characterize DRM.

Definition 3.2. A mapping $\rho : \mathcal{X} \mapsto \mathbb{R}$ is called a distortion risk measure if it satisfies, for all $X, Y \in \mathcal{X}$

- D1* Conditional state independence: $\rho(X) = \rho(Y)$ if X and Y have the same distribution. This means that the risk of a position is determined only by the loss distribution.
- D2* Monotonicity: $\rho(X) \leq \rho(Y)$ if $X \leq Y$.
- D3* Comonotonic additivity: $\rho(X + Y) = \rho(X) + \rho(Y)$ if X and Y are comonotonic, where random variables X and Y are comonotonic if and only

$$(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geq 0 \quad \text{a.s. for } \omega_1, \omega_2 \in \Omega$$

- D4* Continuity:

$$\lim_{d \rightarrow 0} \rho((X - d)^+) = \rho(X^+), \lim_{d \rightarrow \infty} \rho(\min\{X, d\}) = \rho(X), \lim_{d \rightarrow -\infty} \rho(\max\{X, d\}) = \rho(X)$$

$$\text{where } (X - d)^+ = \max(X - d, 0)$$

If two random variable X and Y are comonotonic, then $X(\omega)$ and $Y(\omega)$ always move in the same direction as the state ω changes. The notion of comonotonicity is central in risk measures. See discussions on comonotonicity in Dhaene et al. [15] and Dhaene et al. [16]. Wang et al. [37] imposed axiom D3 based on the argument that the comonotonic random variables do not hedge against each other, leading to additivity of risks. They also proved (Wang et al. [37, Theorem 3]) that if \mathcal{X} contains all the *Bernoulli*(p) random variables,

$0 \leq p \leq 1$, then risk measure ρ satisfies axioms D1-D4 and $\rho(1) = 1$ if and only if ρ has a Choquet integral representation with respect to a distorted probability; i.e.,

$$\rho_g(X) = \int X d(g \circ \mathbb{P}) = \int_{-\infty}^0 [g(\mathbb{P}(X > x)) - 1] dx + \int_0^{\infty} g(\mathbb{P}(X > x)) dx \quad (9)$$

where $g(\cdot)$ is called the distortion function which is nondecreasing with $g(0) = 0$ and $g(1) = 1$, and $g \circ \mathbb{P}(A) := g(\mathbb{P}(A))$ is called the distorted probability.

The Choquet integral representation of DRM can be used to explore its mathematical properties. Furthermore, calculations of DRMs can be easily done by taking the expected value of X under probability measure $\mathbb{P}^* := g \circ \mathbb{P}$ (see Wang [35, Theorem 1] and Wang [36, Definition 4.2]). Here we list some commonly used distortion functions:

- CVaR distortion:

$$g_{CVaR}(x, \alpha) = \min\left\{\frac{x}{1 - \alpha}, 1\right\} \text{ with } \alpha \in [0, 1] \quad (10)$$

- Wang Transform(WT) distortion:

$$g_{WT}(x, \beta) = \Phi[\Phi^{-1}(x) - \Phi^{-1}(\beta)] \text{ with } \beta \in (0, 1) \quad (11)$$

- Proportional hazard(PH) distortion:

$$g_{PH}(x, \gamma) = x^\gamma \text{ with } \gamma \in (0, 1] \quad (12)$$

- Lookback(LB) distortion:

$$g_{LB}(x, \delta) = x^\delta(1 - \delta \ln x) \text{ with } \delta \in (0, 1]. \quad (13)$$

The connection between CVaR and the distortion function (10) was observed in Wirch and Hardy [39] while the remaining three distortion functions were proposed in Wang [36], Wang [34], and Hürlimann [22], respectively.

For discretely distributed portfolio losses random variable $\mathbf{l} = (l_1, \dots, l_m)$ and its probability masses $Pr[\mathbf{l} = l_i] = p_i$ for $i = 1, \dots, m$, we can obtain the cumulative distribution function $F_l(l) = \sum_{i=1}^m p_i \mathbf{1}_{\{l_i \leq l\}}$. Then the discrete survival function is given by $\mathbb{P}(X > x) = S_l(l) = 1 - F_l(l)$ and is applied in the distorted probability representation of (9). We have

$$\rho_g(\mathbf{l}) = \int_{-\infty}^0 [g(S_l(l)) - 1] dl + \int_0^{\infty} g(S_l(l)) dl = E^*[\mathbf{l}] = \sum_{i=1}^m p_i^* l_i \quad (14)$$

where

$$p_i^* = \begin{cases} 1 - g(S_l(l_{(1)})), & \text{for } i = 1 \\ p_{(i)}^* = g(S_l(l_{(i-1)})) - g(S_l(l_{(i)})), & \text{for } i = 2, \dots, m. \end{cases} \quad (15)$$

Since g is non-decreasing, $g(0) = 0$ and $g(1) = 1$, hence $p_i^* \geq 0$ for $i = 1, \dots, m$, and $\sum_{i=1}^m p_i^* = 1 - g(S_l(l_{(m)})) = 1$.

4. Coherent Distortion Risk Measure (CDRM)-based Portfolio Selection

There are two ways to derive and define CDRM: Bellini and Caperdoni [6] defined CDRM as a subclass of DRM, namely DRM with concave distortion function g ; Bertsimas and Brown [8] defined CDRM as a subclass of CRM, namely CRM that is also comonotonic and law invariant.⁵ These two definitions are indeed equivalence since it is shown in Bellini and Caperdoni [6] that the class of coherent distortion risk measures coincides with the class of comonotonic law invariant coherent risk measures.

Definition 4.1. We say ρ is a coherent distortion risk measure (CDRM) if:

- ρ_g is a distortion risk measure (DRM) with a concave distortion function g , or equivalently,
- ρ is a coherent risk measure (CRM) that is also comonotonic and law-invariant.

The following representation theorem for CDRM is the key result that enables us to develop a convex optimization framework for any CDRM portfolio selection problem.

Theorem 4.1. For any random variable X and a given concave distortion function g , risk measure ρ_g is a CDRM if and only if there exists a function $w : [0, 1] \mapsto [0, 1]$, satisfying $\int_{\alpha=0}^1 w(\alpha) d\alpha = 1$, such that:

$$\rho_g(X) = \int_{\alpha=0}^1 w(\alpha) \phi_{\alpha}(X) d\alpha \quad (16)$$

where $\phi_{\alpha}(X)$ is the α -CVaR of X .

This representation theorem says that any CDRM can be represented as a convex combination of $CVaR_{\alpha}(X)$, $\alpha \in [0, 1]$ and we can construct any CDRM based on a convex combination of $CVaR_{\alpha}(X)$, $\alpha \in [0, 1]$. Such result was proved by Kusuoka [25] for continuous portfolio loss distributions. Bertsimas and Brown [8] proved and strengthened the representation theorem that any CDRM can be represented as a convex combination of finite number of $CVaR_{\alpha}(X)$ s under the assumption that the portfolio loss has discrete uniform distribution. In addition to developing our convex programming formulation CDRM portfolio selection problem, we also generalize the finite generation theorem for CDRM to general discrete loss distributions. Before stating the main results of the paper, it is useful to state the following definition:

Definition 4.2. For a given loss observation $\mathbf{l} = (l_1, \dots, l_m)$ and the corresponding ordered losses $l_{(1)} < l_{(2)} < \dots < l_{(m)}$. Let $p_{(i)}$ be the probability of realizing $l_{(i)}$, $i = 1, \dots, m$ and let $S_{\mathbf{l}}(l_{(i)}) = 1 - \sum_{j=1}^i p_{(j)}$. Define a CVaR-matrix $\mathbf{Q} \in \mathbb{R}^m \times \mathbb{R}^m$ with columns $\mathbf{Q}_i \in \mathbb{R}^m$, $i = 1, \dots, m$ as

⁵Definition 4.5 in Bertsimas and Brown [8] should be defining CDRM as oppose to defining DRM.

$$\mathbf{Q} = [\mathbf{Q}_1, \mathbf{Q}_2, \dots, \mathbf{Q}_m] = \begin{bmatrix} p(1) & 0 & 0 & \cdots & 0 \\ p(2) & \frac{p(2)}{1-S_{\mathbf{l}}(l(1))} & 0 & \cdots & 0 \\ p(3) & \frac{p(3)}{1-S_{\mathbf{l}}(l(1))} & \frac{p(3)}{1-S_{\mathbf{l}}(l(2))} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p(m) & \frac{p(m)}{1-S_{\mathbf{l}}(l(1))} & \frac{p(m)}{1-S_{\mathbf{l}}(l(2))} & \cdots & \frac{p(m)}{1-S_{\mathbf{l}}(l(m-1))} = 1 \end{bmatrix}. \quad (17)$$

Since portfolio losses are discretely distributed at m points, there are m jumps in the cumulative function of \mathbf{l} . By defining

$$\alpha_i = \begin{cases} 0 & \text{for } i = 1 \\ \sum_{j=1}^{i-1} p(j) & \text{for } i = 2, \dots, m \end{cases} \quad (18)$$

at these m jumps, then the m CVaRs at these probability levels are then given by

$$\phi_{\alpha_i}(\mathbf{l}) = \frac{1}{1 - \alpha_i} \sum_{j=i}^m p(j)l(j) = \sum_{j=i}^m \frac{p(j)}{1 - S_{\mathbf{l}}(l(m-1))} l(j) = \sum_{j=i}^m Q_{ij}l(j), \quad (19)$$

for $i = 1, \dots, m$ and Q_{ij} is the (i, j) -th entry of \mathbf{Q} . Note that column \mathbf{Q}_i is essential to the calculation of $CVaR_{\frac{i-1}{m}}(\mathbf{l})$ and hence explains the name of the matrix.

We now give a finite generation result for the CDRM, which generalizes Theorem 4.2 of Bertsimas and Brown [8] to general discrete loss distributions.

Theorem 4.2. *For a give portfolio loss sample $\mathbf{l} = (l_1, \dots, l_m)$, the corresponding ordered losses $l_{(1)}, \dots, l_{(m)}$ and a given concave distortion function g , the resulting CDRM ρ_g is given by*

$$\rho_g(\mathbf{l}) = \sum_{i=1}^m q_i l_{(i)}. \quad (20)$$

where q_i , $i = 1, \dots, m$ are defined in Equation (15).

Moreover, every such \mathbf{q} can be written in the form

$$\mathbf{q} = \mathbf{Q}\mathbf{w} \quad (21)$$

where $\mathbf{w}^T = (w_1, \dots, w_m)$ denotes the convex weights satisfying $w_i \geq 0$, $i = 1, \dots, m$, and $\sum_{i=1}^m w_i = 1$, and \mathbf{Q} is to the CVaR-matrix (17). The convex weights \mathbf{w} are given by

$$w_i = \begin{cases} \frac{q_1}{p(1)} & \text{if } i = 1 \\ \left(q_i - \frac{p(i)}{p(i-1)} q_{i-1} \right) \frac{S_{\mathbf{l}}(l_{(i-1)})}{p(i)} & \text{if } i = 2, \dots, m. \end{cases} \quad (22)$$

We now make the following observations. First, it is easy to verify that the convex weights defined in (22) satisfy $w_i \geq 0$ for $i = 1, \dots, m$ and $\sum_{i=1}^m w_i = 1$. Theorem 4.2 implies that every CDRM can be defined as a convex combination of the ordered losses $l_{(1)}, \dots, l_{(m)}$ via (20) or equivalently as a convex combination of CVaRs via (21). The latter formulation is what we adopt in our CDRM portfolio optimization model.

Motivated by Theorem 2.1 and Theorem 4.2, we consider the following special function

$$M_g(\mathbf{x}, \boldsymbol{\zeta}) = \int_{\alpha=0}^1 w(\alpha) F_\alpha(\mathbf{x}, \zeta_\alpha) d\alpha \quad (23)$$

where $w(\alpha) \geq 0$ and $\int_{\alpha=0}^1 w(\alpha) d\alpha = 1$.

The representation Theorem 4.1 of CDRM ensures the existence of $w(\alpha)$, $\alpha \in [0, 1]$ and defines CDRM for a given set of weights. For each $\alpha \in [0, 1]$ there is a corresponding auxiliary variable ζ_α . Taking partial derivatives with respect to all ζ_α for $\alpha \in [0, 1]$ and setting them equal to zeros give the extremal properties of $M_g(\mathbf{x}, \boldsymbol{\zeta})$. This provides more insights about the connection between a particular CDRM, $\rho_g(\mathbf{x})$, and its convex representation $M_g(\mathbf{x}, \boldsymbol{\zeta})$. Yet $\boldsymbol{\zeta}$ may have infinite many entries ζ_α . Taking partial derivative with respect to all ζ_α for $\alpha \in [0, 1]$ requires calculus of variations, which is outside the scope of this thesis. We alleviate such difficulty by applying properties of Choquet integrals because CDRM is a subclass of DRM.

We conclude the section by presenting the following key result of the paper. This generalizes the CVaR-based portfolio model of Rockafellar and Uryasev [30] to the more general class of CDRM-based portfolio model:

Theorem 4.3. *Let $\rho_g(\mathbf{x})$ be a CDRM with a corresponding distortion function g . Minimizing $\rho_g(\mathbf{x})$ with respect to $\mathbf{x} \in \mathbf{D}$ is equivalent to minimizing $M_g(\mathbf{x}, \boldsymbol{\zeta})$ over all $(\mathbf{x}, \boldsymbol{\zeta}) \in \mathbf{D} \times \mathbb{R}^{|\zeta|}$, in the sense that*

$$\min_{\mathbf{x} \in \mathbf{D}} \rho_g(\mathbf{x}) = \min_{(\mathbf{x}, \boldsymbol{\zeta}) \in \mathbf{D} \times \mathbb{R}^{|\zeta|}} M_g(\mathbf{x}, \boldsymbol{\zeta}) \quad (24)$$

where moreover

$$(\mathbf{x}^{*T}, \boldsymbol{\zeta}^{*T}) \in \underset{(\mathbf{x}, \boldsymbol{\zeta}) \in \mathbf{D} \times \mathbb{R}}{\operatorname{argmin}} M_g(\mathbf{x}, \boldsymbol{\zeta}) \iff \mathbf{x}^* \in \underset{\mathbf{x} \in \mathbf{D}}{\operatorname{argmin}} \rho_g(\mathbf{x}), \boldsymbol{\zeta}^* \in \underset{\boldsymbol{\zeta} \in \mathbb{R}}{\operatorname{argmin}} M_g(\mathbf{x}^*, \boldsymbol{\zeta}) \quad (25)$$

Proof. Since CDRM is a subclass of DRM, all results of DRM and of Choquet integrals can be applied. In particular, one of the properties of Choquet integral states that if a random variable X_n has a finite number of values and converges to X , i.e., $X_n \xrightarrow{W} X$, then $\rho_g(X_n) \xrightarrow{W} \rho(X)$ provided that $\rho_g(X)$ exists. This property implies that it is sufficient to prove the statement for the discrete random variables, and then carry over the result to the general continuous case.

Consider a discrete portfolio loss random variable $\mathbf{l} = (l_1, \dots, l_m)$ induced by the choice of portfolio $\mathbf{x} \in \mathbb{R}^n$ and the random vector $\mathbf{y} \in \mathbb{R}^m$; i.e. $l_i = l(\mathbf{x}, y_i)$. It follows from Theorem 4.2 that

$$\rho_g(\mathbf{x}) = \sum_{i=1}^m q_i l_{(i)} = \sum_{i=1}^m w_i \phi_{\alpha_i}(\mathbf{x}).$$

Consider now the discrete analog of (23); i.e.

$$M_g(\mathbf{x}, \boldsymbol{\zeta}) = \sum_{i=1}^m w_i F_{\alpha_i}(\mathbf{x}, \zeta_{\alpha_i})$$

where $F_{\alpha_i}(\mathbf{x}, \zeta_{\alpha_i})$ and α_i , $i = 1, \dots, m$ are defined by (6) and (18), respectively. Since $F_{\alpha_i}(\mathbf{x}, \zeta_{\alpha_i})$, $i = 1, \dots, m$ are all joint convex functions of \mathbf{x} and ζ_{α_i} and $M_g(\mathbf{x}, \boldsymbol{\zeta})$ is a convex combination of $F_{\alpha_i}(\mathbf{x}, \zeta_{\alpha_i})$ for $i = 1, \dots, m$, then $M_g(\mathbf{x}, \boldsymbol{\zeta})$ is a joint convex function of \mathbf{x} and $\boldsymbol{\zeta}$.

For a given portfolio \mathbf{x} , we want to find $\boldsymbol{\zeta}^*$ that minimizes $M_g(\mathbf{x}, \boldsymbol{\zeta})$. Since $M_g(\mathbf{x}, \boldsymbol{\zeta})$ is a convex function of $\boldsymbol{\zeta}$, we can simply set the gradient of $M_g(\mathbf{x}, \boldsymbol{\zeta})$ with respect to $\boldsymbol{\zeta}$ equal to zero. This leads to

$$\begin{aligned} 0 &= \frac{\partial M_g(\mathbf{x}, \boldsymbol{\zeta})}{\partial \zeta} \\ 0 &= \frac{\partial}{\partial \zeta_{\alpha_i}} w_j [\zeta_{\alpha_i} + \frac{1}{1-\alpha_i} \sum_{i=1}^m p_i (l_i - \zeta_{\alpha_i})^+], \quad i = 1, \dots, m \\ 0 &= w_j [1 - \frac{1}{1-\alpha_i} \sum_{i=1}^m p_i \mathbf{1}_{(l_i - \zeta_{\alpha_i})}], \quad i = 1, \dots, m \\ &\Leftrightarrow \begin{cases} \zeta_{\alpha_i}^* \in [l_i, l_{i+1}) & \text{if } w_i \neq 0 \\ \zeta_{\alpha_i}^* \text{ unconstrained} & \text{if } w_i = 0. \end{cases} \end{aligned}$$

Substituting these extremal conditions into $M_g(\mathbf{x}, \boldsymbol{\zeta})$, we have

$$\begin{aligned} \min_{\boldsymbol{\zeta} \in \mathbb{R}^m} M_g(\mathbf{x}, \boldsymbol{\zeta}) &= \sum_{i=1}^m w_i \left[\zeta_{\alpha_i}^* + \frac{1}{1-\alpha_i} \sum_{j=1}^m p_j (l_j - \zeta_{\alpha_i}^*)^+ \right] \\ &= \sum_{i=1}^m w_i \left[\zeta_{\alpha_i}^* + \frac{1}{1-\alpha_i} \sum_{j=i}^m p_j (l_j - \zeta_{\alpha_i}^*)^+ \right] \\ &= \sum_{i=1}^m w_i \left[\zeta_{\alpha_i}^* + \frac{1}{1-\alpha_i} \sum_{j=i}^m p^{(j)} l^{(j)} - \frac{\sum_{j=i}^m p^{(j)}}{1-\alpha_i} \zeta_{\alpha_i}^* \right] \\ &= \sum_{i=1}^m w_i \left[\frac{1}{1-\alpha_i} \sum_{j=i}^m p^{(j)} l^{(j)} \right] \\ &= \sum_{i=1}^m w_i \phi_{\alpha_i}(\mathbf{x}) \\ &= \rho_g(\mathbf{x}). \end{aligned}$$

The minimum value of $M_g(\mathbf{x}, \boldsymbol{\zeta})$ is precisely $\rho_g(\mathbf{x})$ and such result holds for any portfolio \mathbf{x} . Therefore the equivalences in Theorem 4.3 hold.

According to Theorem 4.3, we can replace $\rho_g(\mathbf{x})$ with $M_g(\mathbf{x}, \boldsymbol{\zeta})$ in portfolio selection problems. Since $M_g(\mathbf{x}, \boldsymbol{\zeta})$ is a joint convex function w.r.t $(\mathbf{x}, \boldsymbol{\zeta})$, therefore a portfolio selection

problem induces a convex programming problem if the feasible set \mathbf{D} is convex. Since Theorem 4.3 relates closely to the Rockafellar-Uryasev CVaR optimization approach, we can cast portfolio selection problems with CDRM objective/constraint(s) similar to those with CVaR objective/constraint(s).

5. Case Study: Optimal Investments with CDRMs

Recall that the key result derived in the last section allows us to reformulate a CDRM-based portfolio optimization model as a linear programming. This facilitates us in obtaining the optimal portfolios over a much wider class of risk measure and hence this generalizes the Rockafellar-Uryasev CVaR portfolio model. To demonstrate the flexibility and the applicability of our portfolio optimization model, we provide some empirical studies. Subsection 5.1 first addresses the efficiency of $\frac{1}{n}$ -portfolio strategy, a portfolio strategy that is commonly used in practice. Subsection 5.2 then compares and contrasts the optimal portfolios arising from various specification of CDRMs. All programming problems are solved with AMPL using the Gurobi 4.5.1 solver.

We begin our empirical analysis by first defining our “universe” of stocks, which consists of 20 stocks from S&P 500 (see Table 1). Moreover, 2 stocks from each of the 10 sectors defined in Global Industry Classification Standard (GICS) are chosen so that this universe of stocks can be a proxy of the real market. Weekly closing prices (adjusted for dividends and splits) from 02/01/2001 to 31/05/2011 (a total of 543 weeks, hence 542 weekly returns) for these stocks were obtained from finance.yahoo.com.⁶ Since we have confined our portfolio to be constructed from these stocks, Figure 1 shows the time series for the sum of these 20 stock prices. Clearly there had been market declines from 2001 to 2003 as the aftershock from 9 – 11 terrorist attack in 2001. We also observe that market declines from 2007 to 2009 resulted from the so-called “sub-prime mortgage financial crisis”. Moreover, the portfolio increases gradually from mid-2003 to 2005, from mid-2005 to 2007, and after 2009. We replace scenario generation by historical data of stock returns and assume equal probability for each scenario.

5.1. Efficiency of $\frac{1}{n}$ -Portfolio Strategy

In this subsection we examine the performance of a simple yet common investment portfolio, the equally weighted portfolio, also known as the $\frac{1}{n}$ -portfolio. In an $\frac{1}{n}$ -portfolio, initial wealth is invested equally, in monetary amount, in all available stocks. Benartzi and Thaler [7] observed that many participants in defined contribution plans used this simple strategy. Windcliff and Boyle [38] explored this simple investment strategy in classical Markowitz framework and gave merits to this diversification rule when parameter estimation risks and parameter estimation errors are considered. DeMiguel et al. [13] preformed extensive empirical study across 14 portfolio selection models and found that none of them consistently outperforms the $\frac{1}{n}$ -portfolio in terms of the Sharpe Ratio.

⁶Last access on 25/07/2011.

GICS Sector	Company	Ticker Symbol
Consumer Discretionary	Ford Motor	F
	McDonald's Corp.	MCD
Consumer Staples	Coca Cola Co.	KO
	Wal-Mart Stores	WMT
Energy	Marathon Oil Corp.	MRO
	Exxon Mobil Corp.	XOM
Financials	American Intl Group Inc	AIG
	Citigroup Inc.	C
Health Care	CIGNA Corp.	CI
	Humana Inc.	HUM
Industrials	General Dynamics	GD
	General Electric	GE
Information Technology	Microsoft Corp.	MSFT
	National Semiconductor	NSM
Materials	FMC Corporation	FMC
	International Paper	IP
Telecommunication Services	AT&T Inc	T
	Verizon Communications	VZ
Utilities	American Electric Power	AEP
	Entergy Corp.	ETR

Table 1: Companies Selected for Investment Portfolio Construction Case Study

For our selected universe of assets, we derive the efficient frontiers at beginning of years 2003, 2005, 2007, and 2009 (defined as return vs 95%-CVaR) using the linear programming discussed in Section 2. The results are plotted in Figure 2. To compare and contrast the efficiency of the $\frac{1}{n}$ -portfolios, the panel also depicts the corresponding risk-reward trade off for the constructed $\frac{1}{n}$ -portfolio (shown as solid diamonds). It is clear from these comparisons that the $\frac{1}{n}$ -portfolio is far from being efficient. In fact, three out of four times the $\frac{1}{n}$ -portfolio lies below the minimum-risk portfolio. Moreover, the $\frac{1}{n}$ -portfolio lies significantly farther from the efficient frontier in periods of market declines (beginning of years 2003 and 2009) than in periods of market increases (beginning of years 2005 and 2007). One possible explanation is that, assets are more correlated when the market performs poorly hence the benefit of risk diversification for $\frac{1}{n}$ -portfolio becomes the disadvantage of risk aggregation.

5.2. Comparisons among Different CDRMs

In this subsection, we consider the optimal portfolios under various CDRM. The initial portfolio consists of \$100 cash and the portfolio is rebalanced weekly according to the optimal CDRM optimal portfolios. For each chosen CDRM (and subject to various constraints), we determine the optimal portfolios on 442 overlapping 100-week periods. We impose a budget constraint $\sum_{i=1}^{20} x_i = 1$, no-short selling constraints $\mathbf{x} \geq \mathbf{0}$, upper-limit constraints $\mathbf{x} \leq \mathbf{0.2}$

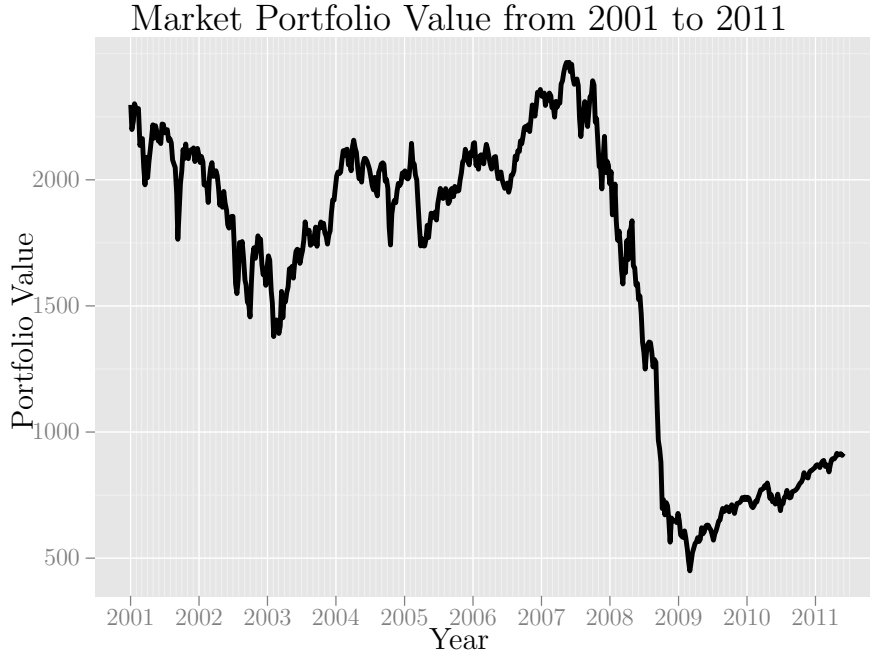


Figure 1: Time Series Plot of Market Portfolio

so that no more than 20% of the total portfolio value should be invested in one single stock, and a return constraint $R(\mathbf{x}) \geq \mu$ where μ is the expected return of the $\frac{1}{n}$ -portfolio.

The linear programming implementation of CDRM-based portfolio model in Theorem 4.3 enables to easily obtain the optimal portfolios over a wider class of risk measures. In particular, we examine the following four members of CDRMs:

1. the CVaR measure (4) with $\alpha \in \{0.9, 0.95, 0.99\}$.
2. the Wang Transform (WT) measure (11) with $\beta \in \{0.75, 0.85, 0.95\}$.
3. the Proportional Hazard (PH) transform measure (12) with $\gamma \in \{0.1, 0.5, 0.9\}$.
4. the lookback (LB) distortion measure (13) with $\delta \in \{0.1, 0.5, 0.9\}$.

We implement the linear programming of CVaR portfolio model using the approach of Rockafellar and Uryasev [30]. For the latter three portfolio models, we use the results in Theorem 4.3 to determine the optimal portfolios. In these cases, we need to determine the weight vectors \mathbf{q} and \mathbf{w} which are plotted in Figure 3 to Figure 5. Summary statistics for the expected returns and realized returns of the optimal portfolios under each CDRM risk measure are listed in Table 2.

- It is of interest to note that the PH portfolio optimization is almost equivalent to optimizing over two extreme CVaR-based portfolios: one with $\alpha = 0.99$ and the other with $\alpha = 0$. PH_γ . Recall that minimizing CVaR with high value of α implies that you are someone who is very risk averse and hence is interested in risk minimization. In contrast, minimizing CVaR with as low α as 0 implies an investor is risk seeker and

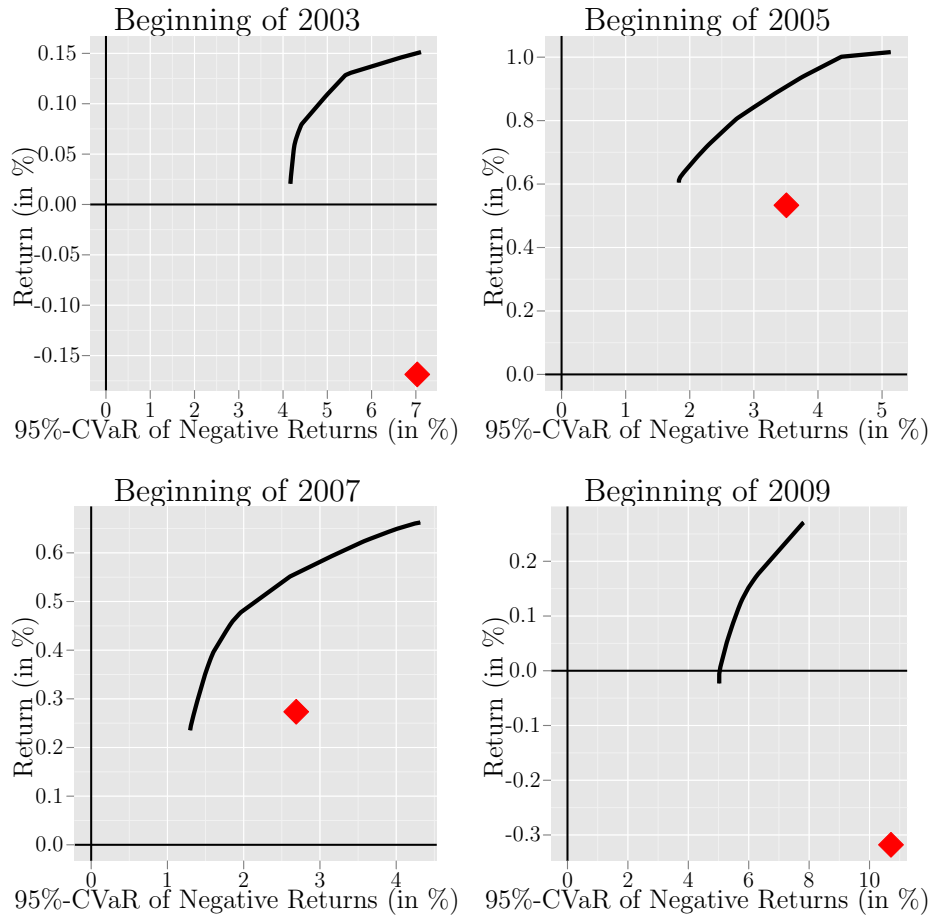


Figure 2: Efficiency of $\frac{1}{n}$ -Portfolio Strategy at Different Times

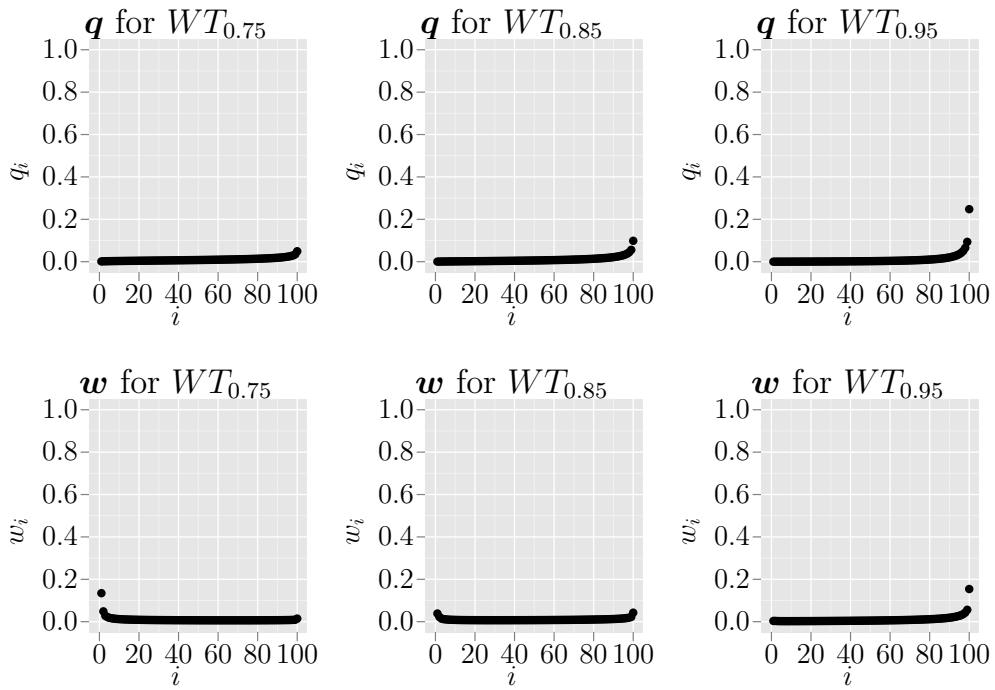


Figure 3: Weights for WT_β

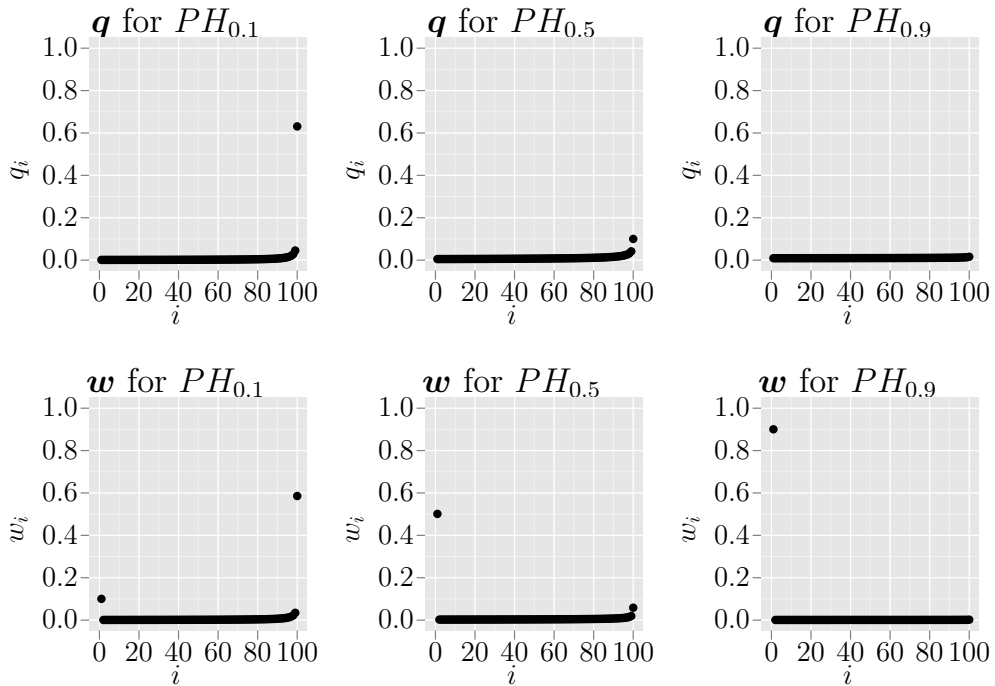


Figure 4: Weights for PH_γ

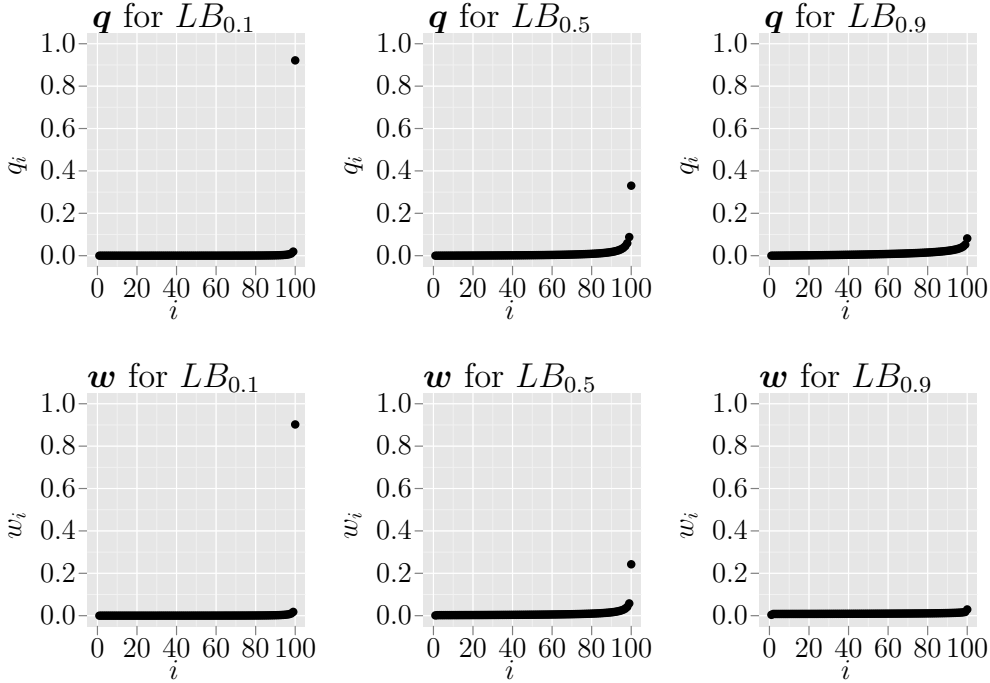


Figure 5: Weights for LB_δ

is only interested in maximizing expected return. Consequently seeking an optimal PH-based portfolio attempts to balance between two extreme portfolios.

- Consistent with the classical tradeoff theory on risk and reward, a more risk averse investor seeks an optimal portfolio with lower risk (as measured by the respective CDRM) but at the expense of lower expected return. Hence the expected return of the optimal portfolio decreases with α for CVaR, decreases with β for WT, increases with γ for PH, and increases with δ for LB. The reported Sharpe ratios (assuming zero risk-free interest rate) in Table 2 are also consistent with these observations.
- Figure 6 produces the out-of-sample realized returns (on the constructed optimal portfolios) over 442 overlapping 100-week periods. It is also of interest to note that while an optimal portfolio with a higher risk is compensated with a higher expected return, the portfolio does not necessary lead to higher realized out-of-sample return, as confirmed in some of these graphs.

To conclude our analysis, we perform a final comparison among the best performing portfolios in the aforementioned four members of CDRMs to the $\frac{1}{n}$ -portfolio, and the return maximization portfolio. Note that we can also view the return maximization problem as a CDRM minimization problem by minimizing $CVaR_0$ of the negative returns. The resulting time series are plotted in Figure 7 the summary statistics for $\frac{1}{n}$ -portfolio and profit maximization portfolio is given in Table 3.

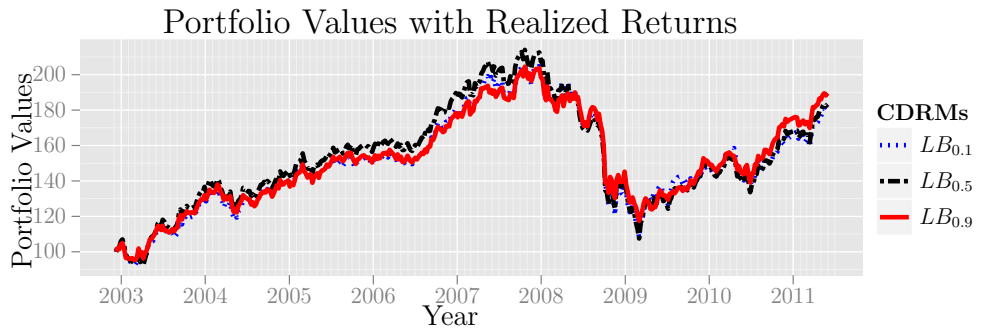
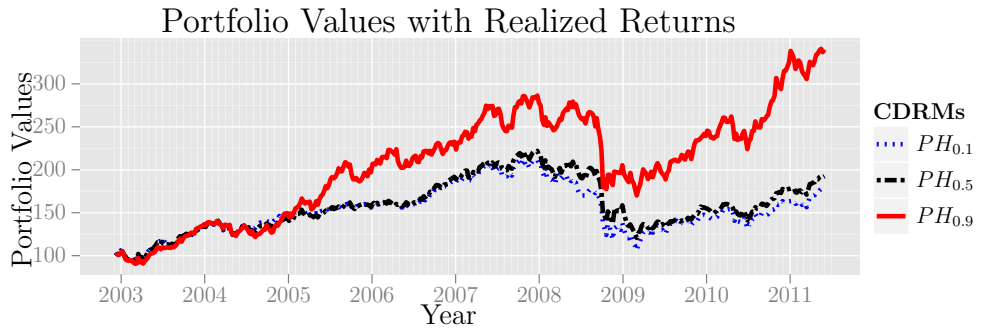
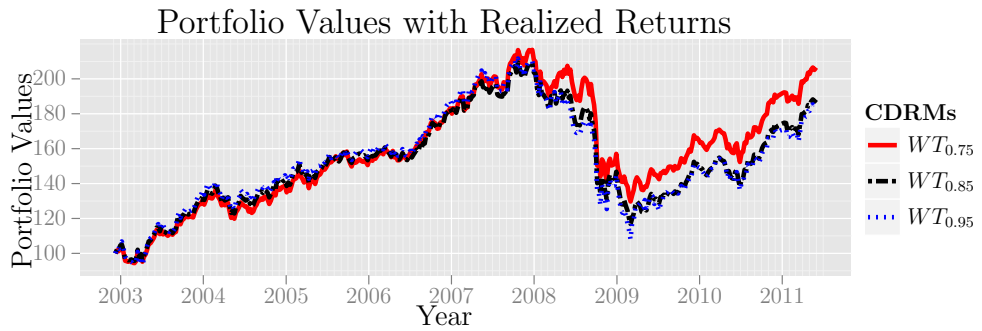
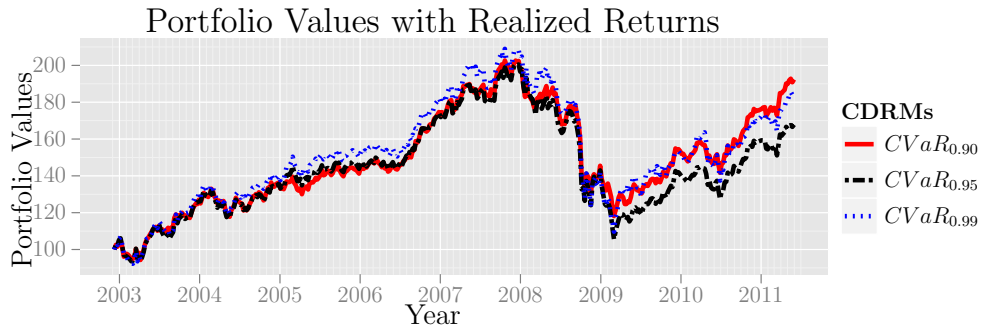


Figure 6: Time Series Plots for Various CDRM Optimal Portfolios

	Expected	Realized	Expected	Realized	Expected	Realized
CVaR	$\alpha = 0.90$		$\alpha = 0.95$		$\alpha = 0.99$	
Mean	0.0025	0.0015	0.0023	0.0012	0.0021	0.0014
Std. Dev.	0.0021	0.0189	0.0023	0.0205	0.0026	0.0224
Skewness	0.3975	-0.9370	0.1961	-0.5674	-0.1758	-0.2081
Kurtosis	0.5856	6.0820	0.1382	4.9651	0.2113	4.4711
Sharpe	1.1710	0.0783	0.9738	0.0572	0.7853	0.0622
WT	$\beta = 0.75$		$\beta = 0.85$		$\beta = 0.95$	
Mean	0.0031	0.0016	0.0026	0.0014	0.0023	0.0014
Std. Dev.	0.0022	0.0192	0.0021	0.0191	0.0023	0.0211
Skewness	-0.0019	-1.0024	0.3737	-0.7753	0.1095	-0.3052
Kurtosis	-0.2304	7.0607	0.3865	5.8864	0.3137	5.4681
Sharpe	1.4267	0.0856	1.2493	0.0748	0.9964	0.0663
PH	$\gamma = 0.1$		$\gamma = 0.5$		$\gamma = 0.9$	
Mean	0.0021	0.0013	0.0030	0.0015	0.0061	0.0028
Std. Dev.	0.0026	0.0222	0.0023	0.0209	0.0035	0.0262
Skewness	-0.1909	-0.2616	-0.1962	-0.8342	1.5106	-0.9574
Kurtosis	0.3613	5.1429	-0.1318	8.5093	3.9670	6.7800
Sharpe	0.8243	0.0584	1.2824	0.0709	1.7294	0.1057
LB	$\delta = 0.1$		$\delta = 0.5$		$\delta = 0.9$	
Mean	0.0021	0.0013	0.0023	0.0014	0.0026	0.0014
Std. Dev.	0.0026	0.0223	0.0024	0.0213	0.0021	0.0189
Skewness	-0.1683	-0.2288	0.0358	-0.3401	0.4153	-0.8040
Kurtosis	0.2944	4.6000	0.2819	5.1539	0.4736	6.0423
Sharpe	0.7970	0.0600	0.9559	0.0644	1.2609	0.0764

Table 2: Summary statistics for the returns of optimal portfolios w.r.t various CDRMs

	$\frac{1}{n}$ -Portfolio		Max Return Portfolio	
	Expected	Realized	Expected	Realized
Mean	0.00168	0.00208	0.00639	0.00279
Std. Dev.	0.00279	0.03038	0.00352	0.02947
Skewness	-0.33372	0.25175	1.52513	-0.73268
Kurtosis	0.64192	13.73943	4.02045	4.95543
Sharpe Ratio	0.60129	0.06854	1.81415	0.09480

Table 3: Summary Statistics for $\frac{1}{n}$ -Portfolio and Return Maximization Portfolio

We see that for our selection of stocks, the return maximization produces the best portfolio in terms of its terminal wealth and the mean portfolio returns. However, this should not be a practical recommendation for portfolio manager because it might bear unacceptably high risks. For instance, the standard deviation of the expected returns for return maximiza-

Optimal Portfolio Values with Realized Returns

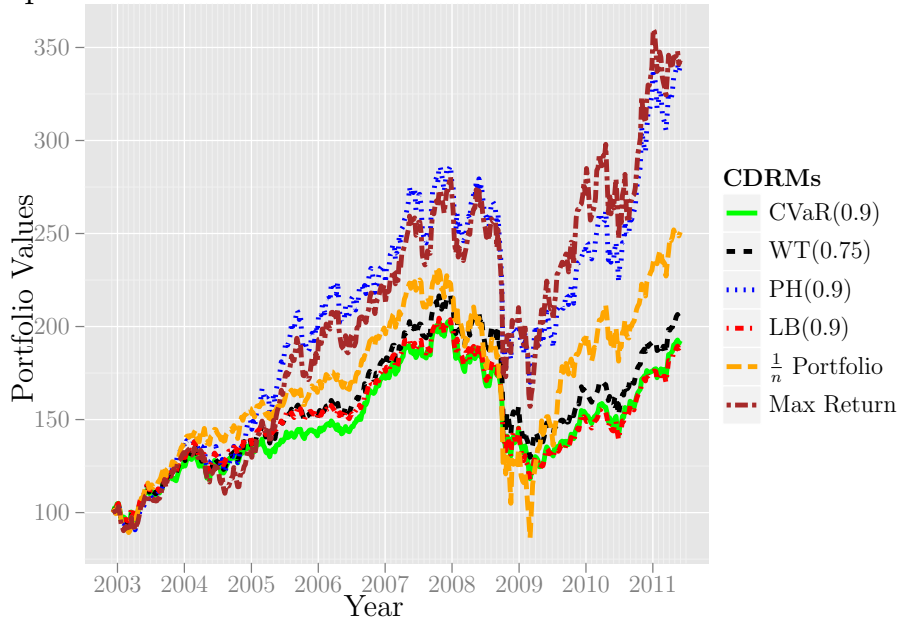


Figure 7: Summary Time Series Plots

tion portfolios is the highest among all portfolios we have discussed in this section. Investors should consider their own risk appetites and choose an appropriate risk measure in every investment decision. It is also of interest to note that the performance of PH with $\gamma = 0.9$ is very similar to the return maximization strategy. This should not be surprising since with such a high value of γ , the PH-based portfolio model is similar to return maximization.

6. Concluding Remarks

This paper extended the well-known linear optimization framework for CVaR (see Rockafellar and Uryasev [30]) to a general class of risk measure known as the CDRM. We first generalized the finite generation theorem for CDRM in Bertsimas and Brown [8] and shown that any CDRM can be defined as a convex combination of ordered portfolio losses and equivalently a convex combination of CVaRs. We make use of the latter to develop a CDRM-based portfolio optimization framework. We solved CDRM-based portfolio optimization via linear programming, which could handle problems with large number of variables and/or constraints. A case study was conducted on constructing a portfolio consisting of 20 S&P 500 stocks. Our empirical analysis suggested the importance of active risk management since naive portfolio construction, such as the $\frac{1}{n}$ -portfolio strategy, can be very inefficient. We also compared and contrasted optimizations over four different members of CDRM. Our numerical shows that different CDRMs reflected different risk appetites and hence different optimization focuses. Choosing risk measures wisely enables risk managers to achieve higher Sharpe ratios, or risk adjusted returns on their investment.

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