Coherent Distortion Risk Measures in Portfolio Selection
(Joint work with Dr Ken Seng Tan)

Ming Bin Feng
University of Waterloo

The 46th Actuarial Research Conference

August 11, 2011
Abstract

The theme of this presentation relates to solving portfolio selection problems using linear and fractional programming. Two key contributions:

- Generalization of the CVaR linear optimization framework (see Rockafellar and Uryasev [3, 4]).
- Equivalences among four formulations of CDRM optimization problems.
Outline

1. Introduction
   - Motivations
   - Goals

2. CDRM Optimization

3. Case Studies

4. Conclusions and Future Directions
Motivations

- Practical portfolio selection problems
- Good risk measures
- Well-studied programming models

Question

Can we connect this together? We want to solve practical portfolio optimization problems with sophisticated risk measures using a programming model that can be solved efficiently.
We wish to..

- Incorporate a general class of risk measure into a well-studied programming model
- Study equivalences among different formulations of portfolio selection problems
- Solve portfolio selection problems of interest efficiently
Outline

1. Introduction

2. CDRM Optimization
   - CVaR Optimization
   - CDRM Representation Theorem
   - CDRM Optimization
   - Formulation Equivalences

3. Case Studies

4. Conclusions and Future Directions
Scenario Generation

Loss Matrix

\[
\begin{bmatrix}
L_{11} & L_{12} & \cdots & L_{1n} \\
L_{21} & L_{22} & \cdots & L_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
L_{m1} & L_{m2} & \cdots & L_{mn}
\end{bmatrix}
\Rightarrow
\begin{align*}
l_1 &= l(x, p_1) \\
l_2 &= l(x, p_2) \\
&\vdots \\
l_m &= l(x, p_m)
\end{align*}
\]

Let \( l_{(1)} \leq \cdots \leq l_{(m)} \) be the ordered losses, \( p_{(i)}, i = 1, \cdots, m \) be the corresponding probability masses.

Return/Price/Premium/Profit Vector

\[
c = [c_1, \cdots, c_m]'
\]
Background

Consider the special function

\[ F(x, \zeta) = \zeta + \frac{1}{1 - \alpha} \sum_{j=1}^{m} p_j (l_j - \zeta)^+ \]

Rockafellar and Uryasev [3, 4] showed that

1. \( CVaR_\alpha(x) = \min_{\zeta \in \mathbb{R}} F(x, \zeta) \)
2. \( \min_{x \in X} CVaR_\alpha(x) = \min_{(x, \zeta) \in X \times \mathbb{R}} F(x, \zeta) \)
CVaR portfolio selection problems can be formulated as LPs. Suppose \( X \) is the set of all feasible portfolios.

**CVaR minimization subject to a return constraint**

\[
\begin{align*}
\text{minimize} & \quad \zeta + \frac{1}{1-\alpha} \sum_{j=1}^{m} p_j z_j \\
\text{subject to} & \quad c' x \geq \mu \\
& \quad l(x, p_j) - \zeta \leq z_j \quad j = 1, \ldots, m \\
& \quad 0 \leq z_j \quad j = 1, \ldots, m \\
& \quad (x, \zeta) \in X \times \mathbb{R}
\end{align*}
\]
Return maximization subject to CVaR constraint(s)

\[
\begin{align*}
\text{maximize} & \quad c'x \\
\text{subject to} & \quad \zeta_i + \frac{1}{1-\alpha} \sum_{j=1}^{m} p_j z_{ij} \leq \eta_i \quad i = 1, \ldots, k \\
& \quad l(x, p_j) - \zeta_i \leq z_{ij} \quad \forall i, j \\
& \quad 0 \leq z_{ij} \quad \forall i, j \\
& \quad (x, \zeta) \in X \times \mathbb{R}^k
\end{align*}
\]
Definition and Representation Theorem

Two Equivalent Definitions

A risk measure $\rho(x)$ is a CDRM if it is

- A comonotone law-invariant coherent risk measure
- A distortion risk measure with a concave distortion function

Representation Theorem for CDRM

A risk measure $\rho(x)$ is a CDRM if and only if there exists a function $w : [0, 1] \mapsto [0, 1]$, satisfying $\int_{\alpha=0}^{1} w_{\alpha} d\alpha = 1$, such that

$$\rho(x) = \int_{\alpha=0}^{1} CVaR_{\alpha}(x) w_{\alpha} d\alpha$$
Representation Theorem in Discrete Case

Finite Generation Theorem for CDRM

Given a concave distortion function $g$, $\rho(x) = \sum_{i=1}^{m} q_i l(i)$, moreover

$$\rho(x) = \sum_{i=1}^{m} w_i CVaR_{\frac{i-1}{m}}(x),$$

where

$$w_i = \begin{cases} \frac{q_1}{p(1)} & \text{if } i = 1 \\ (q_i - \frac{p(i)}{p(i-1)} q_{i-1}) \frac{\sum_{j=i}^{m} p(j)}{p(i)} & \text{if } i = 2, \cdots, m \end{cases}$$
CDRM minimization subject to a return constraint

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{m} w_i (\zeta_i + \frac{1}{1-\alpha} \sum_{j=1}^{m} p_j z_{ij}) \\
\text{subject to} & \quad c' x \geq \mu \\
& \quad l(x, p_j) - \zeta_i \leq z_{ij} \\
& \quad 0 \leq z_{ij} \\
& \quad (x, \zeta) \in X \times \mathbb{R}^m
\end{align*}
\]
Return maximization subject to one CDRM constraint

\[
\begin{align*}
\text{maximize} & \quad c' x \\
\text{subject to} & \quad \sum_{i=1}^{m} w_i (\zeta_i + \frac{1}{1-\alpha} \sum_{j=1}^{m} p_j z_{ij}) \leq \eta \\
& \quad l(x, p_j) - \zeta_i \leq z_{ij} \quad \forall i, j \\
& \quad 0 \leq z_{ij} \quad \forall i, j \\
& \quad (x, \zeta) \in X \times \mathbb{R}^m
\end{align*}
\]
Return-CDRM utility maximization

\[
\begin{align*}
\text{maximize} & \quad \mathbf{c}' \mathbf{x} - \tau \sum_{i=1}^{m} w_i (\zeta_i + \frac{1}{1-\alpha} \sum_{j=1}^{m} p_j z_{ij}) \\
\text{subject to} & \quad l(\mathbf{x}, p_j) - \zeta_i \leq z_{ij} \quad \forall i, j \\
& \quad 0 \leq z_{ij} \quad \forall i, j \\
& \quad (\mathbf{x}, \zeta) \in \mathbf{X} \times \mathbb{R}^m
\end{align*}
\]

This formulation is very similar to a return maximization problem with \(m\) CVaR constraints. Yet we converted \(m\) CVaR constraints into the objective function.
CDRM-based Sharpe ratio maximization

\[
\begin{align*}
\text{maximize} & \quad \frac{c'x - \nu}{\sum_{i=1}^{m} w_i (\zeta_i + \frac{1}{1-\alpha} \sum_{j=1}^{m} p_j z_{ij})} \\
\text{subject to} & \quad l(x, p_j) - \zeta_i \leq z_{ij} \quad \forall i, j \\
& \quad 0 \leq z_{ij} \quad \forall i, j \\
& \quad (x, \zeta) \in X \times \mathbb{R}^m
\end{align*}
\]

This is an LFP, but we can solve it by solving at most two related LPs using a variable transformation method studied by Charnes and Cooper [1].
### Equivalences among four formulations, part 1

<table>
<thead>
<tr>
<th>Problem</th>
<th>Max-Return</th>
<th>Min-CDRM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preset Parameter</td>
<td>$\eta$</td>
<td>$\mu$</td>
</tr>
<tr>
<td>Implied Parameters</td>
<td>$\eta = \text{N/A}$</td>
<td>$\mu = \rho(x^*)$</td>
</tr>
<tr>
<td></td>
<td>$\mu = \mathbf{c}' \mathbf{x}^*$</td>
<td>$\mu = \text{N/A}$</td>
</tr>
<tr>
<td></td>
<td>$\tau = \mathbf{u}^1$</td>
<td>$\tau = \frac{1}{\mathbf{u}^2}$</td>
</tr>
<tr>
<td></td>
<td>$\nu = \mathbf{c}' \mathbf{x}^* - \mathbf{u}^1 \rho(\mathbf{x}^*)$</td>
<td>$\nu = R(x^<em>) - \frac{1}{\mathbf{u}^2} \rho(\mathbf{x}^</em>)$</td>
</tr>
</tbody>
</table>

If the return and CDRM constraints are binding at respective optimal solutions, the preset parameter for Max-Return equals to the implied parameter for Min-CDRM and vice versa.
### Formulation Equivalences

#### Equivalences among four formulations, part 1

<table>
<thead>
<tr>
<th>Problem</th>
<th>Max-Utility</th>
<th>Max-Sharpe</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preset Parameter</td>
<td>$\tau$</td>
<td>$\nu$</td>
</tr>
<tr>
<td>Implied Parameters</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\eta = \rho(x^*)$</td>
<td></td>
<td>$\rho(x^*)$</td>
</tr>
<tr>
<td>$\mu = c'x^*$</td>
<td>$c'x^*$</td>
<td>$c'x^*$</td>
</tr>
<tr>
<td>$\tau = \frac{c'x^<em>-\nu}{\rho(x^</em>)}$</td>
<td>N/A</td>
<td>$\frac{c'x^<em>-\nu}{\rho(x^</em>)}$</td>
</tr>
<tr>
<td>$\nu = c'x^* - \tau \rho(x^*)$</td>
<td>N/A</td>
<td>N/A</td>
</tr>
</tbody>
</table>

We will see that the preset parameter for Max-Return equals to the implied parameter for Min-CDRM and vice versa.
Outline

1. Introduction
2. CDRM Optimization
3. Case Studies
   - Case 1: Reinsurance portfolio selection with simulated data
   - Case 2: Investment portfolio selection with historical data
4. Conclusions and Future Directions
### Case Study 1: Constructing Reinsurance Portfolios

We wish to construct profit-$CVaR_{0.95}(L)$ efficient portfolios from the following 10 risk contracts. Simulations are done for 10,000 scenarios.

<table>
<thead>
<tr>
<th>Contract</th>
<th>Premium</th>
<th>Premium Mean</th>
<th>Premium STD</th>
<th>Losses Mean</th>
<th>Losses STD</th>
<th>95%VaR</th>
<th>95%CVaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>554271</td>
<td>311388</td>
<td>1377843</td>
<td>2613161</td>
<td>5885442</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>364272</td>
<td>222117</td>
<td>1172497</td>
<td>588329</td>
<td>4338214</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>91763</td>
<td>55953</td>
<td>739026</td>
<td>0</td>
<td>1119065</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>867176</td>
<td>437968</td>
<td>1806626</td>
<td>3845685</td>
<td>7937610</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>798005</td>
<td>438464</td>
<td>2913258</td>
<td>0</td>
<td>8769284</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>107585</td>
<td>43381</td>
<td>263019</td>
<td>0</td>
<td>867624</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>878525</td>
<td>375438</td>
<td>1375166</td>
<td>3160679</td>
<td>5974087</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>3081188</td>
<td>1283828</td>
<td>2199151</td>
<td>5661191</td>
<td>8442634</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>65162</td>
<td>29352</td>
<td>324061</td>
<td>0</td>
<td>587044</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>885897</td>
<td>385173</td>
<td>1047454</td>
<td>1506500</td>
<td>3693435</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Case Study 1: Constructing Reinsurance Portfolios

Balanced portfolio consisting of 0.1 unit of each risk.

Summary of balanced portfolio

<table>
<thead>
<tr>
<th>Premium</th>
<th>Losses</th>
<th>Expected Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>STD</td>
</tr>
<tr>
<td>769384</td>
<td>358306</td>
<td>667647</td>
</tr>
<tr>
<td></td>
<td>769384</td>
<td>358306</td>
</tr>
</tbody>
</table>

Profit-95%CVaR utility maximization with $\tau = 0.2$

Summary of target portfolio

<table>
<thead>
<tr>
<th>Premium</th>
<th>Losses</th>
<th>Expected Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>STD</td>
</tr>
<tr>
<td>769384</td>
<td>305689</td>
<td>492425</td>
</tr>
<tr>
<td></td>
<td>769384</td>
<td>305689</td>
</tr>
</tbody>
</table>
Case 1: Reinsurance portfolio selection with simulated data

Case 2: Investment portfolio selection with historical data

Profit-CVaR Efficient Frontier (Enlarged)

- $x$-intercept $= 1815641$
- $y$-intercept $= 463695$
- $\tau = 0.2$
Case 1: Reinsurance portfolio selection with simulated data

Case 2: Investment portfolio selection with historical data

Profit-CVaR Efficient Frontier

Profit

95% -CVaR

Intercept = 100567, Slope = 0.2
Data descriptions

- 2 stocks from each of the 10 sectors defined in Global Industry Classification Standard (GICS).
- Weekly prices from Jan-02-2001 to May-31-2011
- Adjusted closing prices obtained from finance.yahoo.com
Sum of these 20 stocks’ prices can be viewed as the “market”
Optimization Settings

- Replace scenario generation by historical data
- Constant “sample” size of 100.
- $\mathbf{c} =$ expected sample returns, $\mathbf{L} =$ negative returns matrix.
- Weekly rebalancing via CDRM-minimization.
- $\mathbf{x} \geq 0, \mathbf{x} \leq 0.2$, budget constraint, and return constraint.
Case 1: Reinsurance portfolio selection with simulated data

Case 2: Investment portfolio selection with historical data

Ming Bin Feng
Case 1: Reinsurance portfolio selection with simulated data

Case 2: Investment portfolio selection with historical data
Portfolio Selection over Different CDRMs

**Well-known CDRMs**

- **CVaR\(_\alpha\)** distortion:
  \[ g_{CVaR}(x, \alpha) = \min\{\frac{x}{1-\alpha}, 1\} \]

- **Wang Transform (WT)** distortion:
  \[ g_{WT}(x, \beta) = \Phi[\Phi^{-1}(x) - \Phi^{-1}(\beta)] \]

- **Proportional hazard (PH)** distortion:
  \[ g_{PH}(x, \gamma) = x^\gamma \text{ with } \gamma \in (0, 1] \]

- **Lookback (LB)** distortion:
  \[ g_{LB}(x, \delta) = x^\delta (1 - \delta \ln x) \text{ with } \delta \in (0, 1] \]
Case 1: Reinsurance portfolio selection with simulated data

Case 2: Investment portfolio selection with historical data
### Summary statistics of optimal out-of-sample returns

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>STD</th>
<th>Skew</th>
<th>Kurt</th>
<th>Sharpe</th>
</tr>
</thead>
<tbody>
<tr>
<td>CVaR&lt;sub&gt;0.9&lt;/sub&gt;</td>
<td>0.00148</td>
<td>0.01891</td>
<td>-0.93697</td>
<td>6.08202</td>
<td>0.07833</td>
</tr>
<tr>
<td>CVaR&lt;sub&gt;0.95&lt;/sub&gt;</td>
<td>0.00117</td>
<td>0.02050</td>
<td>-0.56738</td>
<td>4.96513</td>
<td>0.05718</td>
</tr>
<tr>
<td>CVaR&lt;sub&gt;0.99&lt;/sub&gt;</td>
<td>0.00139</td>
<td>0.02243</td>
<td>-0.20805</td>
<td>4.47107</td>
<td>0.06219</td>
</tr>
<tr>
<td>WT&lt;sub&gt;0.75&lt;/sub&gt;</td>
<td>0.00164</td>
<td>0.01919</td>
<td>-1.00243</td>
<td>7.06069</td>
<td>0.08560</td>
</tr>
<tr>
<td>WT&lt;sub&gt;0.85&lt;/sub&gt;</td>
<td>0.00261</td>
<td>0.01915</td>
<td>-0.77534</td>
<td>5.88635</td>
<td>0.07477</td>
</tr>
<tr>
<td>WT&lt;sub&gt;0.95&lt;/sub&gt;</td>
<td>0.00232</td>
<td>0.02107</td>
<td>-0.30517</td>
<td>5.46812</td>
<td>0.06628</td>
</tr>
</tbody>
</table>
Case 1: Reinsurance portfolio selection with simulated data
Case 2: Investment portfolio selection with historical data
### Summary statistics of optimal out-of-sample returns

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>STD</th>
<th>Skew</th>
<th>Kurt</th>
<th>Sharpe</th>
</tr>
</thead>
<tbody>
<tr>
<td>PH0.1</td>
<td>0.00130</td>
<td>0.02218</td>
<td>-0.26156</td>
<td>5.14293</td>
<td>0.05844</td>
</tr>
<tr>
<td>PH0.5</td>
<td>0.00148</td>
<td>0.02091</td>
<td>-0.83421</td>
<td>8.50931</td>
<td>0.07091</td>
</tr>
<tr>
<td>PH0.9</td>
<td>0.00277</td>
<td>0.02622</td>
<td>-0.95739</td>
<td>6.78000</td>
<td>0.10574</td>
</tr>
<tr>
<td>LB0.1</td>
<td>0.00134</td>
<td>0.02230</td>
<td>-0.22880</td>
<td>4.59996</td>
<td>0.05995</td>
</tr>
<tr>
<td>LB0.5</td>
<td>0.00137</td>
<td>0.02130</td>
<td>-0.34008</td>
<td>5.15387</td>
<td>0.06439</td>
</tr>
<tr>
<td>LB0.9</td>
<td>0.00145</td>
<td>0.01893</td>
<td>-0.80400</td>
<td>6.04230</td>
<td>0.07645</td>
</tr>
</tbody>
</table>
Case 1: Reinsurance portfolio selection with simulated data
Case 2: Investment portfolio selection with historical data

Portfolio Values with Out-of-Sample Returns

Legend
- CVaR(0.9)
- WT(0.75)
- PH(0.9)
- LB(0.9)
- 1/n Portfolio
## Summary statistics of optimal out-of-sample returns

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>STD</th>
<th>Skew</th>
<th>Kurt</th>
<th>Sharpe</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{n}$-portfolio</td>
<td>0.00208</td>
<td>0.03038</td>
<td>0.25175</td>
<td>13.73943</td>
<td>0.06845</td>
</tr>
<tr>
<td>CVaR$_{0.9}$</td>
<td>0.00148</td>
<td>0.01891</td>
<td>-0.93697</td>
<td>6.08202</td>
<td>0.07833</td>
</tr>
<tr>
<td>WT$_{0.75}$</td>
<td>0.00164</td>
<td>0.01919</td>
<td>-1.00243</td>
<td>7.06069</td>
<td>0.08560</td>
</tr>
<tr>
<td>PH$_{0.9}$</td>
<td>0.00277</td>
<td>0.02622</td>
<td>-0.95739</td>
<td>6.78000</td>
<td>0.10574</td>
</tr>
<tr>
<td>LB$_{0.9}$</td>
<td>0.00145</td>
<td>0.01893</td>
<td>-0.80400</td>
<td>6.04230</td>
<td>0.07645</td>
</tr>
</tbody>
</table>
Outline

1. Introduction
2. CDRM Optimization
3. Case Studies
4. Conclusions and Future Directions
   - Concluding remarks
   - Future Directions
Linear optimization for CDRM portfolio selection

- CDRM portfolio optimization with LPS and LFPs
- CDRM includes CVaR, WT, PH, and LB
- Choose CDRM that suits specific risk appetites
- Four different CDRM formulations are equivalent
- Equivalences are helpful for interpretation of parameters, verification of consistencies, and estimation of implied information
Empirical results

- Simple portfolio construction rules can be very inefficient, active management is important.
- Despite the inefficiency of the $\frac{1}{n}$-portfolio, its terminal wealth (based on out-of-sample returns) can be high.
- We have found CDRM efficient portfolios with higher Sharpe ratio than the $\frac{1}{n}$-portfolio’s.
Future Directions

- Apply various decomposition methods to solve CDRM problems more efficiently
- Apply stochastic programming techniques to solve CDRM problems
- Apply CDRM approach in multi-period models
- Explore/identify other members of CDRM (Higher moment coherent risk measure)


R.T. Rockafellar and S. Uryasev.
Conditional value-at-risk for general loss distributions.