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Spending Retirement on Planet Vulcan: The Impact of Longevity Risk Aversion on Optimal Withdrawal Rates

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RECOMMENDATIONS FROM THE MEDIA AND FINANCIAL PLANNERS REGARDING RETIREMENT SPENDING RATES DEVIATE CONSIDERABLY FROM UTILITY MAXIMIZATION MODELS. This study argues that wealth managers should advocate dynamic spending in proportion to survival probabilities, adjusted up for exogenous pension income and down for longevity risk aversion.

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In our study, we attempted to derive, analyze, and explain the optimal retirement spending policy for a utility-maximizing consumer facing (only) a stochastic lifetime. We deliberately ignored financial market risk by assuming that all investment assets are allocated to risk-free bonds (e.g., Treasury Inflation-Protected Securities [TIPS]). We made this simplifying assumption in order to focus attention on the role of longevity risk aversion in determining optimal consumption and spending rates during a retirement period of stochastic length.

By longevity risk aversion, we mean that different people might have different attitudes toward the "fear" of living longer than anticipated and possibly depleting their financial resources. Some might respond to this economic risk by spending less early on in retirement, whereas others might be willing to take their chances and enjoy a higher standard of living while they are still able to do so.

Indeed, the impact of financial risk aversion on optimal asset allocation has been the subject of many studies and is intuitively well understood by practitioners. On the one hand, investors who are particularly concerned about losing money (i.e., risk averse) invest conservatively and thus sacrifice the potential upside, leading to a reduced lifetime standard of living. On the other hand, financially risk-tolerant investors accept more risk in their portfolios in exchange for the potential—never a guarantee—of a higher standard of living in retirement. But the impact of longevity risk aversion on retirement spending behavior has not received as much attention, and most practitioners are unfamiliar with the concept.

Although neither our framework nor our mathematical solution is original—they can be traced back almost 80 years—we believe that the insights from a normative life-cycle model (LCM) are worth emphasizing in the current environment, which has grown jaded by economic models and their prescriptions. Our pedagogical objective was to contrast the optimal (i.e., utility-maximizing) retirement spending policy with popular recommendations offered by the investment media and financial planners.

The main results of our investigation are as follows: Counseling retirees to set initial spending from investable wealth at a constant inflation-adjusted rate (e.g., the widely popular 4 percent rule) is consistent with life-cycle consumption smoothing only under a very limited set of implausible preference parameters—that is, there is no universally optimal or safe retirement spending rate. Rather, the optimal forward-looking behavior in the face of personal longevity risk is to consume in proportion to survival probabilities—adjusted upward for pension income and downward for longevity risk aversion—as opposed to blindly withdrawing constant income for life.

HISTORY OF THE PROBLEM

The first problem I propose to tackle is this: How much of its income should a nation save?

With those words, the 24-year-old Cambridge University economist Frank R. Ramsey began a celebrated paper published two years before his tragic death, in 1930. The so-called Ramsey (1928) model and the resultant Keynes–Ramsey rule, implicitly adopted by thousands of economists in the last 80 years (including Fisher 1930; Modigliani and Brumberg 1954; Phelps 1962; Yaari 1965; Modigliani 1986), form the foundation for life-cycle utility optimization. They are also the workhorse supporting the original asset allocation models of Samuelson (1969) and Merton (1971).

In its basic form, the normative LCM assumes a rational individual who seeks to maximize the discounted additive utility of consumption over his entire life. Despite its macroeconomic origins, the Ramsey model has been extended by scores of economists. Indeed, ask a first-year graduate student in economics how a consumer should be "spending" over some deterministic time horizon T, and she will most likely respond with a Ramsey-type model that spreads human capital and financial capital (i.e., total wealth) between time zero and the terminal time, T.

The pertinent finance literature has advanced since 1928 and now falls under the rubric "portfolio choice" or extensions of the Merton model. We counted more than 50 scholarly articles on this topic published in the top finance journals over the last decade alone. *Unfortunately, much of the financial planning community has ignored these dynamic optimization models, and nowhere is this ignorance more evident than in the world of "retirement income planning."*

Lamentably, the financial crisis, coupled with general skepticism toward financial models, has moved the practice of personal finance even further away from a dynamic optimization approach. In fact, many popular and widely advocated strategies are at odds with the prescriptions of financial economics. For examples of how economists "think about" problems in personal finance and how their thinking differs from conventional wisdom, see Bodie and Treussard (2007); Kotlikoff (2008); Bodie, McLeavey, and Siegel (2008); Ayres and Nalebuff (2010).

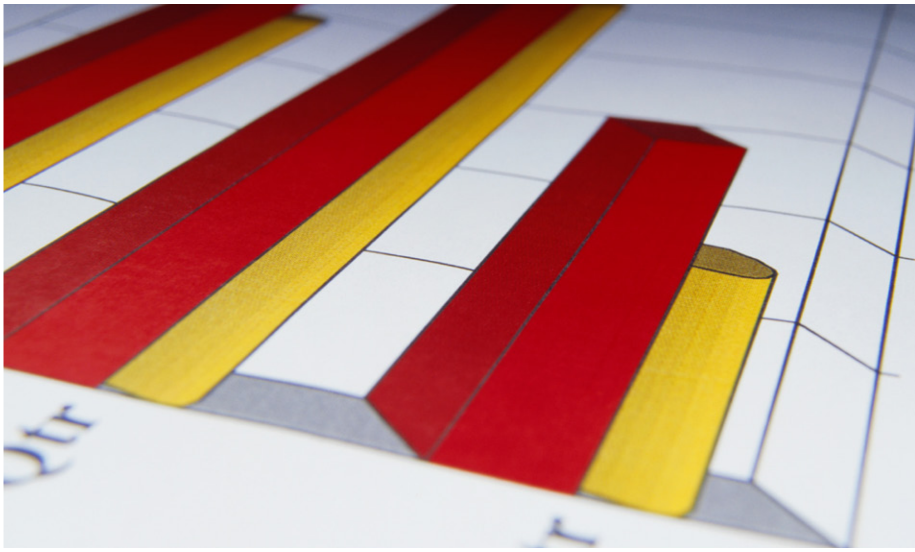
Along the same lines, we attempted to narrow the gap between the advice of the financial planning community regarding retirement spending policies and the "advice" of financial economists who use a rational utility-maximizing model of consumer choice.[1]

In particular, we focused exclusively on the impact of life span uncertainty—longevity risk—on the optimal consumption and retirement spending policy. To isolate the impact of longevity risk on optimal portfolio retirement withdrawal rates, we placed our deliberations on Planet Vulcan, where investment returns are known and unvarying, the inhabitants are rational and utility-maximizing consumption smoothers, and only life spans are random.

LITERATURE ON RETIREMENT SPENDING RATES

Within the community of retirement income planners, a frequently cited study is Bengen (1994), in which he used historical equity and bond returns to search for the highest allowable spending rate that would sustain a portfolio for 30 years of retirement. Using a 50/50 equity/bond mix, Bengen settled on a rate between 4 percent and 5 percent. In fact, this rate has become known as the Bengen or 4 percent rule among retirement income planners and has caught on like wildfire. The rule simply states that for every $100 in the retirement nest egg, the retiree should withdraw $4 adjusted for inflation each year—forever, or at least until the portfolio runs dry or the retiree dies, whichever occurs first.

Indeed, it is hard to overestimate the influence of the Bengen (1994) study and its embedded "rule" on the community of retirement income planners. Other studies in the same vein include Cooley, Hubbard, and Walz (1998), often referred to as the Trinity Study. In the last two decades, these and related studies have been quoted and cited thousands of times in the popular media (e.g., *Money Magazine, USA Today, Wall Street Journal*).[2] The 4 percent spending rule now seems destined for the same immortality enjoyed by other (unduly simplistic) rules of thumb, such as "buy term and invest the difference" and dollar cost averaging. And although numer-



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ous authors have extended, refined, and recalibrated these spending rules, the spirit of each rule remains intact across all versions.[3]

We are not the first to point out that this "start by spending *x* percent" strategy has no basis in economic theory. For example, Sharpe, Scott, and Watson (2007) and Scott, Sharpe, and Watson (2008) raised similar concerns and alluded to the need for a life-cycle approach, but they never actually solved or calibrated such a model. The goal of our study was to illustrate the solution to the lifecycle problem and demonstrate how longevity risk aversion—in contrast to financial risk aversion, so familiar to financial analysts—affects retirement spending rates.

Other researchers have recently teased out the implications of mortality and longevity risk for portfolio choice and asset allocation (see, e.g., Bodie, Detemple, Ortuba, and Walter 2004; Dybvig and Liu 2005; Babbel and Merrill 2006; Chen, Ibbotson, Milevsky, and Zhu 2006; Jiménez-Martín and Sánchez Martín 2007; Lachance, forthcoming 2011). Likewise, Milevsky and Robinson (2005) argued that retirement spending rates should be reduced because the embedded equity risk premium (ERP) assumption is too high. In our study, however, we used an economic LCM approach to retirement income planning.

NUMERICAL EXAMPLES AND CASES

The model we used is fully described in Appendix A so that readers can select their own parameter values and derive optimal values under any assumptions. Using our equations, readers can obtain values quite easily in Microsoft Excel. We selected one (plausible) set to illustrate the main qualitative insights, which are rather insensitive to assumed parameter values.

Note that we use the following terms (somewhat loosely and interchangeably, depending on the context) throughout the article: (1) The consumption rate is an annualized dollar amount that includes withdrawals from the portfolio, as well as pension income, and is scaled to reflect an initial portfolio value of $100. The retirement consumption rate is synonymous with the retirement spending rate. (2) The portfolio withdrawal rate (PWR) is the annualized ratio of the amount withdrawn from the portfolio divided by the value of the portfolio at that time. (3) The initial PWR is the annualized ratio of the initial amount withdrawn from the portfolio divided by the initial value of the portfolio.

Recall that we are spending our retirement on Vulcan, where only life spans are random. Our approach forced us to specify a real (inflation-adjusted) investment return. So, after carefully examining the real yield from U.S. TIPS over the last 10 years on the basis of data from the Fed, we found that the maximum real yield over the period was 3.15 percent for the 10-year bond and 4.24 percent for the 5-year bond. The average yield was 1.95 percent and 1.50 percent, respectively.

The longer-maturity TIPS exhibited higher yields but obviously entailed some duration risk. After much deliberation, we decided to assume a real interest rate of 2.5 percent for most of the numerical examples, even though current (fall 2010) TIPS rates were substantially lower. Readers can code the formulas in Appendix A—with their favorite riskfree-return estimate—to obtain their own rational spending rates. Our values are consistent with the view expressed by Arnott (2004) regarding the future of the lower ERP.

As for longevity risk, we exercised a great deal of modeling caution because it was the impetus for our investigation. We assumed that the retiree's remaining lifetime obeys a (unisex) biological law of mortality under which the hazard rate increases exponentially over time. This notion is known as the Gompertz assumption in the actuarial literature, and we calibrated this model to common pension annuitant mortality tables. (See Appendix A for a full description of the mortality law.)

In most of our numerical examples, therefore, we assumed an 86.6 percent probability that a 65-year-old will survive to the age of 75, a 57.3 percent probability of surviving to 85, a 36.9 percent probability of reaching 90, a 17.6 percent probability of reaching 95, and a 5 percent probability of attaining 100. Again, note that we do not plan for a life expectancy or an ad hoc 30-year retirement. Rather, we account for the entire term structure of mortality.

Our main objective was to focus attention on the impact of risk aversion on the optimal PWR and especially the initial PWR. Therefore, we display results for a range of values—for example, for a retiree with a very low (γ = 1) and a relatively high (γ = 8) coefficient of relative risk aversion (CRRA).

To aid a clear understanding of mortality risk aversion, we offer the following analogy to classical asset allocation models. An investor with a CRRA value of (γ = 4) would invest 40 percent of her assets in an equity portfolio and 60 percent in a bond portfolio, assuming an equity risk premium of 5 percent and volatility of 18 percent. This analogy comes from the famed Merton ratio. Our model does not have a risky asset and does not require an ERP, but the idea is that the CRRA can be mapped onto more easily understood risk attitudes. Along the same lines, the very low risk-aversion value of (γ = 1), which is often labeled the Bernoulli utility specification, would lead to an equity allocation of 150 percent, and a high risk-aversion value of (γ= 8) implies an equity allocation of 20 percent (all rounded to the nearest 5 percent).

Finally, to complete the parameter values required for our model, we assume that the subjective discount rate (ρ), which is a proxy for personal impatience, is equal to the risk-free rate (mostly 2.5 percent in our numerical examples). To those familiar with the basic LCM without lifetime uncertainty, this assumption suggests that the optimal consumption rates would be constant over time in the absence of longevity risk considerations. Again, our motivation for all these assumptions is to tease out the impact of pure longevity risk aversion.

In the language of economics, when the subjective discount rate (SDR) in an LCM is set equal to the constant and risk-free interest rate, a rational consumer will spend his total (human plus financial) capital evenly and in equal amounts over time. In other words, in a model with no horizon uncertainty, consumption rates and spending amounts are, in fact, constant, regardless of the consumer's elasticity of intertemporal substitution (EIS).[4]

The question is, what happens when lifetimes are stochastic?

Table 1. Optimal Rate (pre-$100) under Medium Risk Aversion (γ=4)

	Real Interest Rate			
	0.5%	1.5%	2.5%	3.5%
Retire at age 65	$3.330	$3.941	$4.605	$5.318
5 years later	3.286	3.888	4.544	5.247
10 years later	3.212	3.801	4.442	5.130
20 years later	2.898	3.429	4.007	4.627
30 years later	2.156	2.552	2.982	3.444

Notes: The initial portfolio (nest egg) is worth $100 and is invested at the indicated rates. There is a 5 percent probability of survival to age 100. The Gompertz mortality parameters are m = 89.335 and b = 9.5. No pension income is assumed. All consumption spending is from the investment portfolio.

Let us now take a look at some results. We will assume a 65-year-old with a (standardized) $100 nest egg. Initially, we allow for no pension annuity income, and therefore, all consumption must be sourced to the investment portfolio that is earning a deterministic interest rate. On Planet Vulcan, financial wealth must be depleted at the very end of the life cycle (say, age 120) and bequest motives are nonexistent. So, according to Equation A5 (see Appendix A), the optimal consumption rate at retirement age 65 is $4.605 when the risk-aversion parameter is set to (γ = 4) (see Table 1), and the optimal consumption rate is $4.121 when the risk-aversion parameter is set to (γ = 8).

Note that these rates—perhaps surprisingly—are within the range of numbers quoted by the popular press for optimal portfolio withdrawal (spending) rates. Thus, at first glance, these numbers seem to suggest that simple 4 percent rules of thumb are consistent with an LCM. Unfortunately, the euphoria is short-lived. The numbers (may) coincide only in the first year of withdrawals (at age 65) and for a limited range of risk-aversion coefficients (most importantly, no pension income). As retirees age, they rationally consume less each year—in proportion to their survival probability adjusted for risk aversion. For example, at our baseline intermediate level of risk aversion (γ = 4), the optimal consumption rate drops from $4.605 at age 65 to $4.544 at age 70,

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and then to \$4.442 at age 75, \$3.591 at 90, and \$2.177 at 100, assuming the retiree is still alive. All these values are derived from Equation A5.

Note how a lower real interest rate (e.g., 0.5 percent in Table 1) leads to a reduced optimal retirement consumption/spending rate. Indeed, in the yield curve and TIPS environment of fall 2010, our model offered an important message for Baby Boomers: Your parents' retirement plans might not be sustainable anymore.

The first insight in our model is that a fully rational plan is for retirees to spend less as they progress through retirement. Life-cycle optimizers (i.e., "consumption smoothers" on Vulcan) spend more at earlier ages and reduce spending as they age, even if their SDR is equal to the real interest rate in the economy.

Intuitively, they deal with longevity risk by setting aside a financial reserve and by planning to reduce consumption (if that risk materializes) in proportion to their survival probability adjusted for risk aversion—all without any pension annuity income.

As Irving Fisher (1930) observed in The Theory of Interest,

> *The shortness of life thus tends powerfully to increase the degree of impatience or rate of time preference beyond what it would otherwise be. . . . Everyone at some time in his life doubtless changes his degree of impatience for income. . . . When he gets a little older . . . he expects to die and he thinks: instead of piling up for the remote future, why shouldn't I enjoy myself during the few years that remain?* (pp. 85, 90)

Table 2. Initial PWR at age 65 with Pension Income, as a Function of Risk Aversion

Pension Income	$\gamma = 1$	$\gamma = 2$	$\gamma = 4$	$\gamma = 8$
$\pi_0 = \$0$	6.330%	5.301%	4.605%	4.121%
$\pi_0 = \$1$	6.798	5.653	4.873	4.324
$\pi_0 = \$2$	7.162	5.924	5.078	4.480
$\pi_0 = \$5$	8.015	6.553	5.551	4.839

Notes: The mortality assumption is that there is a 5 percent probability of survival to age 100. The Gompertz mortality parameters are $m = 89.335$ and $b = 9.5$. The interest rate is 2.5 percent.

Including Pension Annuities.

Let us now use the same model to examine what happens when the retiree has access to a defined benefit (DB) pension income annuity, which provides a guaranteed lifetime cash flow. In the United States, the maximum benefit from Social Security, which is the ultimate real pension annuity, is approximately \$25,000 per annuitant. Let us examine the behavior of a retiree with 100, 50, and 20 times this amount in her nest egg—that is, \$2,500,000, \$1,250,000, and \$500,000 in investable retirement assets.

Alternatively, one can interpret Table 2 as displaying the optimal policy for four different retirees with varying degrees of longevity risk aversion, each with \$1,000,000 in investable retirement assets. The first retiree has no pension ($\pi_0 = \$0$), the second has an annual pension of \$10,000 ($\pi_0 = \$1$), the third has an annual pension of \$20,000 ($\pi_0 = \$2$), and the fourth has a pension of \$50,000 ($\pi_0 = \$5$).

Table 2 shows the net initial PWR (i.e., the optimal amount withdrawn from the investment portfolio) as a function of the risk-aversion values and pre-existing pension income. Thus, for example, when the ($\gamma = 4$) retiree (medium risk aversion) has \$1,000,000 in investable assets and is entitled to a real lifetime pension of \$50,000—which, in our language, is a scaled nest egg of \$100 and a pension ($\pi_0 = \$5$)—the optimal total consumption rate is \$10.551 in the first year. Of that amount, \$5.00 obviously comes from the pension and \$5.551 is withdrawn from the portfolio. The net initial PWR is thus 5.551 percent.

In contrast, if the retiree has the same \$1,000,000 in assets but is entitled to only \$10,000 in lifetime pension income, the optimal total consumption rate is \$5.873 per \$100 of assets at age 65, of which \$1.00 comes from the pension and \$4.873 is withdrawn from the portfolio. Hence, the initial PWR is 4.873 percent. All these numbers are derived directly from Equation A5.

The main point of our study can be summarized in one sentence: The optimal portfolio withdrawal rate depends on longevity risk aversion and the level of pre-existing pension income. The larger the amount of the pre-existing pension income, the greater the optimal consumption rate and the greater the PWR.

The pension acts primarily as a buffer and allows the retiree to consume more from discretionary wealth. Even at high levels of longevity risk aversion, the risk of living a long life does not "worry" retirees too much because they have pension income to fall back on should that chance (i.e., a long life) materialize. We believe that this insight is absent from most of the popular media discussion (and practitioner implementation) of optimal spending rates. If a potential client has substantial income from a DB pension or Social Security, she can afford to withdraw more—percentagewise—than her neighbor, who is relying entirely on his investment portfolio to finance his retirement income needs.

Table 2 confirms a number of other important results. Note that the optimal PWR—for a range of risk-aversion and pension income levels—is between 8 percent and 4 percent, but only when the inflation-adjusted interest rate is assumed to be a rather generous 2.5 percent. Adding another 100 bps to the investment return assumption raises the initial PWR by 60–80 bps. Reducing interest assumptions, however, will have the opposite effect. Readers can input their own assumptions into Equation A5 to obtain suitable consumption/spending rates.

Table 3. Impact of Pensionization on Retirement Consumption Rates

Percent: Initial Portfolio and Pension	Lower Longevity Risk Aversion $\gamma = 2.0$		Medium Longevity Risk Aversion $\gamma = 4.0$		Higher Longevity Risk Aversion $\gamma = 8.0$	
	Consume at Age 65	Consume at Age 80	Consume at Age 65	Consume at Age 80	Consume at Age 65	Consume at Age 80
0%: F_0=\$100, π_0=\$0.000	\$5.3014	\$4.5696	\$4.6051	\$4.2755	\$4.1187	\$3.9684
20%: F_0=\$80, π_0=\$1.2661	\$5.9193	\$5.1021	\$5.2637	\$4.8869	\$4.8013	\$4.6263
40%: F_0=\$60, π_0=\$2.5321	\$6.3760	\$5.4958	\$5.7963	\$5.3815	\$5.3858	\$5.1893
60%: F_0=\$40, π_0=\$3.7982	\$6.7040	\$5.7784	\$6.2292	\$5.7833	\$5.8921	\$5.6774
80%: F_0=\$20, π_0=\$5.0643	\$6.8631	\$5.9156	\$6.5328	\$6.0651	\$6.2983	\$6.0687
100%: F_0=\$0, π_0=\$6.3303	\$6.3303	\$6.3303	\$6.3303	\$6.3303	\$6.3303	\$6.3303

***Notes**: Assumes a subjective discount rate (ρ) equal to the interest rate (r) of 2.5%, and annuity factor of \$15.7971 at age 65, under Gompertz mortality with parameters (m=89.335, b=9.5), truncated at age 122. All numbers rounded to four decimals.*

The impact of longevity risk aversion can be described in another way. If the remaining future lifetime has a modal value of (m = 89.335) and a dispersion (volatility) value of (b = 9.5), then a consumer averse to longevity risk behaves (consumes) as if the modal value were [$m^* = m + b\ln(\gamma)$] but with the same dispersion parameter, b.

Longevity risk aversion manifests itself by (essentially) assuming that retirees will live longer than the biological/medical estimate. Only extremely risk-tolerant retirees (γ = 1) behave as if their modal life spans were the true (biological) modal value. Note that this behavior is not risk neutrality, which would ignore longevity risk altogether.

In the asset allocation literature, the closest analogy to these risk-adjusted mortality rates is the concept of risk-adjusted investment returns. A risk-averse investor observes a 10 percent expected portfolio return and adjusts it downward on the basis of the volatility of the return and her risk aversion. If the (subjectively) adjusted investment return is less than the risk-free rate, the investor shuns the risky asset. Of course, this analogy is not quite correct because retirees cannot shun longevity risk, but the spirit is the same. The longevity probability they see is not the longevity probability they feel.

Again, an important take-away is the impact of pension annuities on retirement consumption. Although the point of our study was not to advocate for pension annuities or examine the market for longevity protection—already well achieved in a recent book by Sheshinski (2008), as well as the excellent collection of studies by Brown, Mitchell, Poterba, and Warshawsky (2001)—we present yet another way to use Equations A5 and A6.

Table 3 reports the optimal consumption rate at various ages, assuming that a fixed percentage of the retirement nest egg is used to purchase a pension annuity ("pensionized"). The cost of each lifetime dollar of income

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Figure 1. Optimal Consumption: $5 Pension Income with Investment Rate = 2.5%

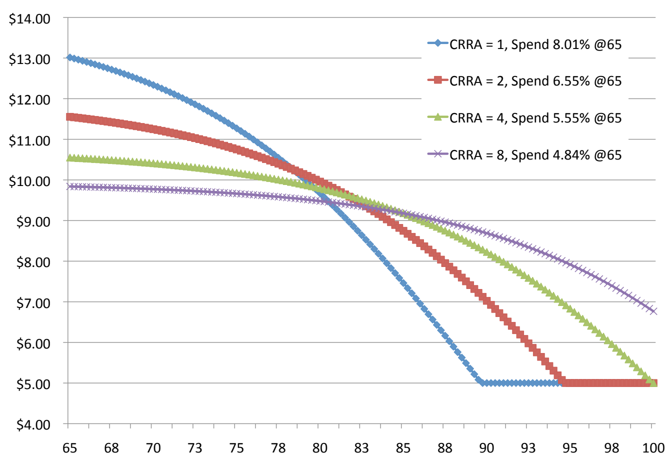
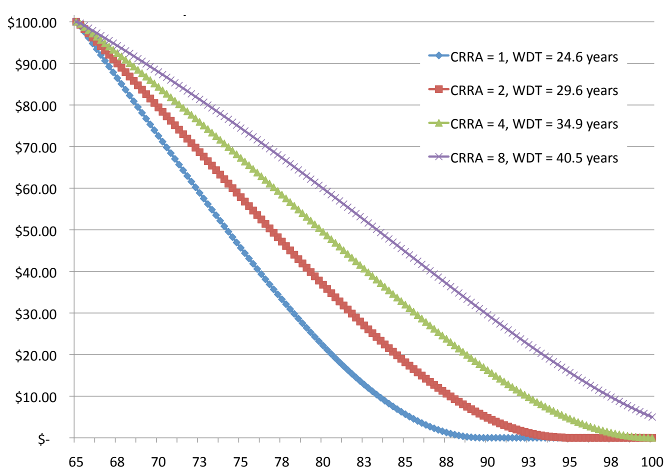


Figure 2. Financial Capital: $5 Pension Income with Investment Rate = 2.5%



is displayed in Equation A2, which is the expression for the pension annuity factor. So, if 30 percent of $100 is pensionized, the corresponding value of F_0 is $70 and the resulting pension annuity income is

$$\$30 / a_{65}^{55}\left(0.025,\ 89.335,\ 9.5\right) = \$1.899.$$

We note that the pricing of pension (income) annuities by private sector insurance companies usually involves mortality rates that differ from population rates owing to anti-selection concerns. This factor could be easily incorporated by using different mortality parameters, but we will keep things simple to illustrate the impact of lifetime income on optimal total spending rates.

Results are reported for a retirement age of 65 and planned consumption 15 years later (assuming the retiree is still alive), at age 80. We illustrate with a variety of scenarios in which 0 percent, 20 percent 40 percent, 60 percent, or 100 percent of initial wealth is pensionized—that is, a nonreversible pension annuity (priced by Equation A2) is purchased on the basis of the going market rate.[5]

Table 3 shows total dollar consumption rates, including the corresponding pension annuity income. These rates are not (only) the PWRs that are reported in percentages in Table 2. For example, if the retiree with medium risk aversion allocates $20 (from the $100 available) to purchase a pension annuity that pays $1.261 for life, optimal consumption will be $1.261 + $3.997 = $5.258 at age 65. Note that the $3.997 withdrawn from the remaining portfolio of $80 is equivalent to an initial PWR of 4.996 percent.

In contrast, the retiree with a high degree of longevity risk aversion (γ = 8) will receive the same $1.261 from the $20 that has been pensionized but will optimally spend only $3.535 from the portfolio (a withdrawal rate of 4.419 percent), for a total consumption rate of $4.801 at age 65.

If the entire nest egg is pensionized at 65, leading to $6.3303 of lifetime income, the consumption rate is constant for life—and independent of risk aversion—because there is no financial capital from which to draw down any income. This example is yet another way to illustrate the benefit of converting financial wealth into a pension income flow. The $6.3303 of annual consumption is the largest of all the consumption plans. Thus, most financial economists are strong advocates of pensionizing (or at least annuitizing) a portion of one's retirement nest egg.

Visualizing the Results. Figure 1 depicts the optimal consumption path from retirement to the maximum length of life as a function of the retiree's level of longevity risk aversion (γ in our model). This figure provides yet another perspective on the rational approach and attitude toward longevity risk management. It uses Equation A5 to trace the entire consumption path, from retirement at age 65 to age 100.

"Using our methodology, one can examine the optimal reaction to financial shocks over the retirement horizon."

Note that the optimal consumption rate declines with age and in relation to the retiree's attitude toward longevity risk as measured by the CRRA.

Figure 1 plots four cases that correspond to various levels of the CRRA. Note that the consumption rate eventually hits $5, which is the pension income flow. For example, the consumer with a CRRA of 2 (i.e., very low aversion to longevity risk) will start retirement by withdrawing 6.55 percent from his nest egg plus his pension income of $5. The withdrawals from the portfolio will continue until the retiree rationally exhausts his wealth at age 95. From the wealth depletion time (WDT) onward, he consumes only his pension.[6]

Figure 2 shows the corresponding trajectory for financial capital. At all levels of longevity risk aversion, the curve begins at $100 and then declines. The rate of decline is higher and faster for lower levels of longevity risk aversion because the retiree is "unafraid" of living to an advanced age. She will deplete her wealth after 24.6 years (at age 90), after which she will live on her pension ($5).

In contrast, the retiree with a longevity risk aversion of CRRA = 8 does not (plan to) deplete wealth until age 105 and draws down wealth at a much slower rate. When there is no pension annuity income at all, the WDT is exactly at the end of the terminal horizon, which is the last possible age on the mortality table. In other words, wealth is never completely exhausted. This result can also be seen from Equation A8, in which the only way to obtain zero (on the right-hand side) is for the survival probability to be zero, which can happen only when τ equals the maximum length of life.

Reacting to Financial Shocks. Using our methodology, one can examine the optimal reaction to financial shocks over the retirement horizon. Take someone who experiences a 30 percent loss in his investment portfolio and wants to rationally reduce spending to account for the depleted nest egg. The rule of thumb suggesting that retirees spend 4–5 percent says nothing about how to update the rule in response to a shock to wealth.

The rational reaction to a financial shock at time s, which results in a new (reduced) portfolio value, would be to follow these steps:

1. Using Equation A8, recalibrate the model from time zero but with the shocked level of wealth and compute the new WDT.
2. Use Equation A7 to compute the new level of initial consumption, which will be different from the old consumption level because of the financial shock.
3. Continue retirement consumption from time s onward on the basis of Equation A5.

To understand how this approach would work in practice, let us begin with a (CRRA = 4) retiree who has $100 in investable assets and is entitled to $2 of lifetime pension income. With a real interest rate of r = 2.5 percent, the optimal policy is to consume a total of $7.078 at age 65 ($2 from the pension and $5.078 from the portfolio) and adjust withdrawals downward over time in proportion to the survival probability to the power of the risk-aversion coefficient. The WDT is at age 105.

Under this dynamic policy, the expectation is that at age 70, the financial capital trajectory will be $86.668 and total consumption will be $6.984 if the retiree follows the optimal consumption path for the next five years.

Now let us assume that the retiree survives the next five years and experiences a financial shock that reduces the portfolio value from the expected $86.668 to $60 at age 70, which is 31 percent less than planned. In this case, the optimal plan is to reduce consumption to $5.583, which is obtained by solving the problem from the beginning but with a starting age of 70. This result is a reduction of approximately 20 percent compared with the original plan.

Of course, this scenario is a bit of an apples-to-oranges comparison because (1) a shock is not allowed in our model and (2) the time zero consumption plan is based on a conditional probability of survival that could change on the basis of realized health status. The problem of stochastic versus hazard rates obviously takes us far beyond the simple agenda of our study.

In sum, a rational response to an *x* percent drop in one's retirement portfolio is not to reduce consumption and spending by the same *x* percent. Consumption smooth-

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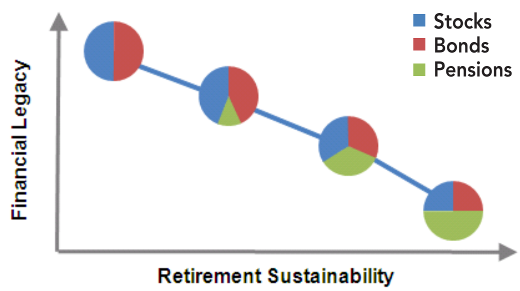
Figure 3. Economic Tradeoffs at Retirement:



ing in the LCM is about amortizing unexpected losses and gains over the remaining lifetime horizon, adjusted for survival probabilities.

SUMMARY

To a financial economist, the optimal retirement consumption rate, asset allocation (investments), and product allocation (insurance) are a complicated function of mortality expectations, economic forecasts, and the trade-off between the preference for retirement sustainability and the desire to leave a financial legacy (bequest motive). Although it is not an easy problem to solve even under some very simplifying assumptions, the qualitative trade-off can be illustrated (see Figure 3).

Retirees can afford to spend more if they are willing to leave a smaller financial legacy and risk early depletion times. They should spend less if they desire a larger legacy and greater sustainability. Optimization of investments and insurance products occurs on this retirement income frontier. Ergo, a simple rule that advises all retirees to spend *x* percent of their nest egg adjusted up or down in some *ad hoc* manner is akin to the broken clock that tells time correctly only twice a day.

We are not the first authors—and will certainly not be the last—to criticize the "spend *x* percent" approach to retirement income planning. For example, as noted by Scott, Sharpe, and Watson (2008),

> *The 4 percent rule and its variants finance a constant, non-volatile spending plan using a risky, volatile investment strategy. Two of the rule's inefficiencies—the price paid for funding its unspent surpluses and the overpayments for its spending distribution—apply to all retirees, independent of their preferences. (p. 18)*

Although we obviously concur, the focus of our study was to illustrate what exactly a life-cycle model says about optimal consumption rates. Our intention was to contrast *ad hoc* recommendations with "advice" that a financial economist might give to a utility-maximizing consumer and see whether the two approaches have any *overlap* and how much they differ. In particular, we shined a light on aversion to longevity risk—uncertainty about the human life span—and examined how this aversion affects optimal spending rates.[7]

Computationally, we solved an analytic LCM that was calibrated to actuarial mortality rates (see Appendix A). Our model can easily be used by anyone with access to an Excel spreadsheet. Our main insights are as follows:

1. The optimal initial PWR, which the "planning literature" says should be an exogenous percentage of (only) one's retirement nest egg, critically depends on both the consumer's risk aversion—where risk concerns longevity and not just financial markets—and any preexisting pension annuity income. For example, if the portfolio's assumed annual real investment return is 2.5 percent, the optimal initial PWR can be as low as 3 percent for highly risk-averse retirees and as high as 7 percent for those who are less risk averse. The same approach applies to any pension annuity income. The greater the amount of pre-existing pension income, the larger the initial PWR, all else being equal. Of course, if one assumes a healthier retiree and/or lower inflation-adjusted returns, the optimal initial PWR is lower.
2. The optimal consumption rate (c_t^*)—which is the total amount of money consumed by the retiree in any given year, including all pension income—is a declining function of age. In other words, retirees (on Vulcan) should consume less at older ages. The consumption rate for discretionary wealth is proportional to the survival probability $({}_tp_x)$ and is a func-

"You might live a very long time, so you better make sure to own a lot of stocks and equity."

tion of risk aversion, even when the subjective rate of time preferences (ρ) is equal to the interest rate. The rational consumer—planning at age 65—is willing to sacrifice some income at 100 in exchange the age of 100 the same preference weight as the age of 80 can be explained within an LCM only if the SDR (ρ_t) is a time-dependent function that exactly offsets the declining survival probability. That people might have such preferences is highly unrealistic.

3. The interaction between (longevity) risk aversion and survival probability is quite important. In particular, risk aversion tends to increase the effective probability of survival. So, imagine two retirees with the same amount of initial retirement wealth and pension income (and the same SDR) but with different levels of risk aversion (γ). The retiree with greater risk aversion behaves as if her modal value of life were higher. Specifically, she behaves as if it were increased by an amount proportional to $\ln(\gamma)$ and spends less in anticipation of a longer life. Observers will never know whether such retirees are averse to longevity risk or simply believe they are much healthier than the population.
4. The optimal trajectory of financial capital also declines with age. Moreover, for retirees with pre-existing pension income, spending down wealth by some advanced age, and thereafter living exclusively on pension income, is rational. The WDT can be at age 90—or even 80 if the pension income is sufficiently large. Greater (longevity) risk aversion, which is associated with lower consumption, induces greater financial capital at all ages. Planning to deplete wealth by some advanced age is neither wrong nor irrational.[8]
5. The rational reaction to portfolio shocks (i.e., losses) is nonlinear and dependent on when the shock occurs and the amount of pre-existing pension income. One does not reduce portfolio withdrawals by the exact amount of a financial shock unless the risk aversion is ($\gamma = 1$), known as the Bernoulli utility. For example, if the portfolio suffers an unexpected loss of 30 percent, the retiree might reduce consumption by only 30 percentage points.
6. Converting some of the initial nest egg into a stream of lifetime income increases consumption at all ages regardless of the cost of the pension annuity. Even when interest rates are low and the cost of \$1 of lifetime income is (relatively) high, the net effect is that pensionization increases consumption. Note that we are careful to distinguish between real-world pension annuities—in which the buyer hands over a nonrefundable sum in exchange for a constant real stream—and tontine annuities, which are the foundation of most economic models but are completely unavailable in the marketplace.

7. Although not pursued in the numerical examples, one result that follows from our analysis is counter-intuitive and perhaps even controversial: Borrowing against pension income might be optimal at advanced ages. For retirees with relatively large pre-existing (DB) pension income, preconsuming and enjoying their pensions while they are still able to do so might make sense. The lower the longevity risk aversion, the more optimal this path becomes.

The "cost" of our deriving a simple analytic expression—described by Equations A1–A8—is that we had to assume a deterministic investment return. Although we assumed a safe and conservative return for most of

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our numerical examples, we essentially ignored the last 50 years of portfolio modeling theory. Recall, however, that our goal was to shed light on the oft-quoted rules of thumb and how they relate to longevity risk, as opposed to developing a full-scale dynamic optimization model.

CONCLUSION: BACK TO PLANET EARTH

How might a full stochastic model—with possible shocks to health and their related expenses— change optimal consumption policies? Assuming agreement on a reasonable model and parameters for long-term portfolio returns, the risk-averse retiree would be exposed to the risk of a negative (early) shock and would plan for this risk by consuming less. With a full menu of investment assets and products available, however, the retiree would be free to optimize around pension annuities and other downside-protected products, in addition to long-term-care insurance and other retirement products. In other words, even the formulation of the problem itself becomes much more complex.

More importantly, the optimal allocation depends on the retiree's preference for personal consumption versus bequest, as illustrated in Figure 3. A product and asset allocation suitable for a consumer with no bequest or legacy motives— those in the lower left-hand corner of the figure—is quite different from the optimal portfolio for someone with strong legacy preferences. In our study, we assumed that the retiree's objective is to maximize utility of lifetime consumption without any consideration for the value of bequest or legacy.

Although some have argued that a behavioral explanation is needed to rationalize the desire for a constant consumption pattern in retirement, we note that very high longevity risk aversion leads to relatively constant spending rates and might "explain" these fixed rules. In other words, we do not need a behavioral model to justify constant 4 percent spending. Extreme risk aversion does that for us.

That said, we believe that another important take-away from our study is that offering the following advice to retirees is internally inconsistent: "You might live a very long time, so you better make sure to own a lot of stocks and equity." The first part of the sentence implies longevity risk aversion, while the second part is suitable only for risk-tolerant retirees. Risk is risk.

To make this sort of statement more precise, we are working on a follow-up study in which we derive the optimal portfolio withdrawal rate for both pension and tontine annuities in a robust capital market environment *à la* Richard (1975) and Merton (1971) but with a model that breaks the reciprocal link between the elasticity of intertemporal substitution and general risk aversion. Another fruitful line of research would be to explore the optimal time to retire in the context of a mortalityonly LCM, which would take us far beyond the current literature.[9]

One thing seems clear: Longevity risk aversion and pension annuities remain very important factors to consider when giving advice regarding optimal portfolio withdrawal rates. That is the main message of our study, a message that does not change here on Planet Earth.

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APPENDIX A. LIFE-CYCLE MODEL IN RETIREMENT

The value function in the LCM during retirement years when labor income is zero, assuming no bequest motive, can be written as follows:

$$\max_c V(c) = \int_0^D e^{-\rho t} \left({}_t p_x \right) u(c_t) dt, \qquad \text{(A1)}$$

where

x = the age of the retiree when the consumption/spending plan is formulated (e.g., 60 or 65)

D = the maximum possible life span years in retirement (the upper bound of the utility integration, which is currently 122 on the basis of the world's longest-lived person, Jeanne Calment, who died in France in 1997)

ρ = the SDR, or personal time preference (which ranges in value from 0 percent to as high as 20 percent in some empirical studies)

tPx = the conditional probability of survival from retirement age x to age $x + t$, which is based on an actuarial mortality table

We parameterize (tPx) on the basis of the Gompertz law of mortality, under which the biological hazard rate is $\lambda_t = (1/b)e^{(x-m+t)/b}$, which grows exponentially with age—m denotes the modal value of life (e.g., 80 years), and b denotes the dispersion coefficient (e.g., 10 years) of the future lifetime random variable. Both numbers are calibrated to U.S. mortality tables to fit advanced-age survival rates.

In our study, we assumed that the utility function of consumption exhibits constant elasticity of intertemporal substitution, which is synonymous with (and the reciprocal of) constant relative risk aversion (RRA) under conditions of perfect certainty and time-separable utility. The exact specification is $u(c) = c^{1-\gamma}/(1-\gamma)$, where γ is the coefficient of relative (longevity) risk aversion, which can take on values from Bernoulli ($\gamma = 1$) up to infinity.

The actuarial present value function, denoted by $a_x^T(v,m,b)$ depends implicitly on the survival probability curve (tpx) via the parameters (m,b). It is defined and computed by using the following:

$$a_x^T(v,m,b) = \int_0^T e^{-vs}\left({}_s p_x\right)ds, \qquad \text{(A2)}$$

which is the retirement age "price"—under a real, constant discount rate v—of a life-contingent pension annuity that pays a real $1 a year until death or time T, whichever comes first. Although we do not include a mortality risk premium from the perspective of the insurance company in this valuation model, one could include it by tilting the survival rate toward a longer life.

A closed-form representation of Equation A2 is possible in terms of the incomplete gamma function $\Gamma(A,B)$, which is available analytically:

$$a_x^T(v,m,b) = \frac{b\Gamma\left[-vb, \exp\left(\frac{x-m}{b}\right)\right]}{\exp\left[(m-x)v - \exp\left(\frac{x-m}{b}\right)\right]} - \frac{b\Gamma\left[-vb, \exp\left(\frac{x-m+T}{b}\right)\right]}{\exp\left[(m-x)v - \exp\left(\frac{x-m}{b}\right)\right]}. \qquad \text{(A2a)}$$

See Milevsky (2006, p. 61) for instructions on how to code the gamma function in Microsoft Excel.

The wealth trajectory (financial capital during retirement) is denoted by F_t, and the dynamic constraint in our model—linked to the objective function in Equation A1—can now be expressed as follows:

$$\dot{F}_t = v(t,F_t)F_t - c_t + \pi_0, \qquad \text{(A3)}$$

where the dot is shorthand notation for a derivative of wealth (financial capital) with respect to time, π_0 denotes the income (in real dollars) from any preexisting pension annuities, and the function multiplying wealth itself is defined by

$$v(t,F_t) = \begin{cases} r, & F_t \geq 0 \\ R+\lambda_t, & F_t < 0 \end{cases}, \qquad \text{(A3a)}$$

where $R \geq r$. The discontinuous function $v(t,F_t)$ denotes the interest rate on financial capital and allows F_t to be negative. For credit cards and other unsecured lines of credit, $v(t,F_t) = R + \lambda_t$. The borrower pays R plus the insurance (to protect the lender in the event of the borrower's death).

Note that we do not assume a complete liquidity constraint that prohibits borrowing in the sense of Deaton (1991), Leung (1994), or Bütler (2001). We do not allow stochastic returns. Equations A1, A2, and A3 are

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essentially the Yaari (1965) setup, under which pension annuities, but not tontine annuities, are available.

The initial condition is $F_0 = W$, where W denotes the investable assets at retirement. The terminal condition is $F_\tau = 0$, where τ denotes the wealth depletion time, at which point only the pension annuity income is consumed. Leung (1994, 2007) explored the existence of a WDT in a series of theoretical papers. In theory, the WDT can be at the final horizon time ($\tau = D$) if the pension income is minimal (or zero) and/or the borrowing rate is relatively low. To be very precise, it is possible for $F_t < 0$ for some time $t < D$. We are not talking about the zero values of the function. Rather, the definition of our WDT is $F_t = 0$; $\forall t > \tau$ permanently. One can show that when $R > \rho$, borrowing is not optimal and $\tau < D$ under certain conditions. For our numerical results, we assume a high-enough value of R.

The Euler–Lagrange theorem from the calculus of variations leads to the following. The optimal trajectory, F_t, in the region over which it is positive, assuming that $v(t,F_t) = r$, can be expressed as the solution to the following second-order nonhomogeneous differential equation:

$$\ddot{F}_t - (k_t + r)\dot{F}_t + rk_t F_t = -\pi_0 k_t, \tag{A4}$$

where the double dots denote the second derivative with respect to time and the time-dependent function $k_t = (r - \rho - \lambda_t)/\gamma$ is introduced to simplify notation. The real interest rate, r, is a positive constant and a pivotal input to the model. We reiterate that Equation A4 is valid only until the wealth depletion time, τ. But one can always force a wealth depletion time $\tau < D$ by assuming a minimal pension annuity, as well as a large-enough (arbitrary) interest rate $v(t,F_t)$ on borrowing when $F_t < 0$. For a more detailed discussion, including the impact of a stochastic mortality rate, see Huang, Milevsky, and Salisbury (2010).

The solution to the differential Equation A4 is obtained in two stages. First, the optimal consumption rate while $F_t > 0$ can be shown to satisfy the equation

$$c_t^* = c_0^* e^{kt} \left({}_t p_x\right)^{1/\gamma}, \tag{A5}$$

where $k = (r - \rho)/\gamma$ and the unknown initial consumption rate, c_0^*, can be solved for. The optimal consumption rate declines when the SDR, ρ, is equal to the interest rate, r, and hence, $k = 0$. This outcome is a very important implication (and observable result) from the LCM. Planning to even if ($\rho = r$).

Note also that consumption as defined earlier includes the pension annuity income, π_0. Therefore, the portfolio withdrawal rate (PWR), which is the main item of interest in our study, is $(c_t^* - \pi_0)/F_t$, and the initial PWR (i.e., the retirement spending rate) is $(c_0^* - \pi_0)/F_0$. The optimal financial capital trajectory (also defined as only until time $t < \tau$), which is the solution to Equation A4, can be expressed as a function of c_0^* as follows:

$$\begin{aligned} F_t = {} & \left(W + \frac{\pi_0}{r}\right) e^{rt} \\ & - a_x^t\left(r - k, m^*, b\right) c_0^* e^{rt} - \frac{\pi_0}{r}, \end{aligned} \tag{A6}$$

where the modified modal value in the annuity factor is $m^* = m + b\ln(\gamma)$. The actuarial present value term multiplying time zero consumption values a life-contingent pension annuity under a shifted modal value of $m + b\ln(\gamma)$ and a shifted valuation rate of $r - (r - \rho)/\gamma$ instead of r. Plugging Equation A6 into the differential Equation A4, however, confirms that the solution is correct and valid over the domain $t \in (0,\tau)$.

In other words, the value function in Equation A1—and thus life-cycle utility—is maximized when the consumption rate and the wealth trajectory satisfy Equations A5 and A6, respectively. Of course, these two equations are functions of two unknowns—c_0^* and τ—and we must now solve for them, which we will do sequentially.

First, from Equation A6 and the definition of the WDT ($F_\tau = 0$), we can solve for the initial consumption rate:

$$c_0^* = \frac{\left(W + \pi_0 / r\right) e^{r\tau} - \pi_0 / r}{a_x^\tau\left(r - k, m^*, b\right) e^{r\tau}}. \tag{A7}$$

Note that when $\gamma = 1$, $\pi_0 = 0$, and $\rho = r$, Equation A7 collapses to W/a_x^τ.

Finally, the WDT, τ, is obtained by substituting Equation A7 into Equation A5 and searching the resulting nonlinear equation over the range $(0,D)$ for the value of τ that solves $c_t^* - \pi_0 = 0$. In other words, if a WDT exists, then for consumption to remain smooth at that point—which is part of the foundation of life-cycle theory—it must converge to π_0.

Mathematically, the WDT, τ, satisfies the equation

$$\frac{(W + \pi_0 / r)e^{r\tau} - \pi_0 / r}{a_x^\tau(r-k, m^*, b)e^{r\tau}} e^{k\tau} \left({}_\tau p_x\right)^{1/\gamma} = \pi_0. \qquad \text{(A8)}$$

Put another way,

$$\tau = f\left(\gamma, \pi_0 \middle| W, \rho, r, x, m, b\right). \qquad \text{(A8a)}$$

The optimal consumption policy (described by Equation A5) and the optimal trajectory of wealth (described by Equation A6) are now available explicitly. Practically speaking, the WDT ($\tau \leq D$) is extracted from Equation A8, and the initial consumption rate is then obtained from Equation A7. Everything else follows. These expressions can be coded in Excel in a few minutes.

NOTES

1. Thus, our use of "Planet Vulcan" in the title of our study, inspired by Thaler and Sunstein (2008), who distinguished "humans" from perfectly rational "econs," much like the Star Trek character Spock, who is from Vulcan.
2. See, for example, Walter Updegrave, "Retirement: The 4 Percent Solution," Money Magazine (16 August 2007): http:/ /money.cnn.com/2007/08/13/pf/expert/expert.moneymag/ index.htm.
3. In fact, to some extent, Milevsky and Robinson (2005) encouraged this approach by deriving and publishing an analytic expression for the lifetime ruin probability that assumes a constant consumption spending rate.
4. For detailed information on possible parameter estimates for the EIS and how they affect consumption under deterministic life-cycle models in which the SDR is not equal to the interest rate, see Hanna, Fan, and Chang (1995) and Andersen, Harrison, Lau, and Rutstrom (2008).
5. This annuity is quite different from the Yaari (1965) tontine annuity, in which mortality credits are paid out instantaneously by adding the mortality hazard rate, λ_t, to the investment return, r. Thus, we use the term pensionization to distinguish it from economists' use of the term annuitization. The latter assumes a pool in which survivors inherit the assets of the deceased, whereas the former requires an insurance company or pension fund to guarantee the lifetime payments. See Huang, Milevsky, and Salisbury (2010) for a discussion of the distinction between the two and their impact on optimal retirement planning in a stochastic versus deterministic mortality model.
6. The consumption function is concave until the WDT, at which point it is nondifferentiable and set equal to the pension annuity income.
7. A (tongue-in-cheek) rule of thumb that could be substituted for the static 4 percent algorithm is to counsel retirees to pick any initial spending rate between 2 percent and 5 percent but to reduce the actual spending amount each year by the proportion of their friends and acquaintances who have died. This approach would roughly approximate the optimal decline based on anticipated survival rates.
8. Thus, one could say that there are bag ladies on Vulcan. 9. See Stock and Wise (1990) for an example of this burgeoning literature.

REFERENCES

Andersen, S., G.W. Harrison, M.I. Lau, and E.E. Rutstrom. 2008. "Eliciting Risk and Time Preferences." Econometrica, vol. 76, no. 3 (May):583–618.

Arnott, R.D. 2004. "Sustainable Spending in a Lower-Return World." Financial Analysts Journal, vol. 60, no. 5 (September/ October):6–9.

Ayres, I., and B. Nalebuff. 2010. Lifecycle Investing: A New, Safe, and Audacious Way to Improve the Performance of Your Retirement Portfolio. New York: Basic Books.

Babbel, D.F., and C.B. Merrill. 2006. "Rational Decumulation." Working paper, Wharton Financial Institutions Center.

Bengen, W.P. 1994. "Determining Withdrawal Rates Using Historical Data." Journal of Financial Planning, vol. 7, no. 4 (October):171–181.

CONTINUED ON PAGE 38

Bodie, Z., and J. Treussard. 2007. "Making Investment Choices as Simple as Possible, but Not Simpler." Financial Analysts Journal, vol. 63, no. 3 (May/June):42–47.

Bodie, Z., D. McLeavey, and L.B. Siegel, eds. 2008. The Future of Life-Cycle Saving and Investing. 2nd ed. Charlottesville, VA: Research Foundation of CFA Institute.

Bodie, Z., J.B. Detemple, S. Ortuba, and S. Walter. 2004. "Optimal Consumption-Portfolio Choice and Retirement Planning." Journal of Economic Dynamics & Control, vol. 28, no. 6 (March):1115–1148

Brown, J.R., O.S. Mitchell, J.M. Poterba, and M.J. Warshawsky. 2001. The Role of Annuity Markets in Financing Retirement. Cambridge, MA: MIT Press.

Bütler, M. 2001. "Neoclassical Life-Cycle Consumption: A Textbook Example." Economic Theory, vol. 17, no. 1 (January): 209–221.

Chen, P., R.G. Ibbotson, M.A. Milevsky, and K.X. Zhu. 2006. "Human Capital, Asset Allocation, and Life Insurance." Financial Analysts Journal, vol. 62, no. 1 (January/February):97–109.

Cooley, P.L., C.M. Hubbard, and D.T. Walz. 1998. "Retirement Savings: Choosing a Withdrawal Rate That Is Sustainable." AAII Journal, vol. 20, no. 2 (February):16–21.

Deaton, A. 1991. "Saving and Liquidity Constraints." Econometrica, vol. 59, no. 5 (September):1221–1248.

Dybvig, Philip H., and Hong Liu. 2005. "Lifetime Consumption and Investment: Retirement and Constrained Borrowing." Working paper, Washington University in St. Louis (September).

Fisher, I. 1930. The Theory of Interest: As Determined by

Impatience to Spend Income and Opportunity to Invest It. New York: Macmillan.

Hanna, S., J.X. Fan, and Y.R. Chang. 1995. "Optimal Life Cycle Savings." Financial Counseling and Planning, vol. 6:1–14.

Huang, H., M.A. Milevsky, and T.S. Salisbury. 2010. "Calibrating Yaari in the 21st Century." Working paper, IFID Centre at York University.

Jiménez-Martín, S., and A.R. Sánchez Martín. 2007. "An Evaluation of the Life Cycle Effects of Minimum Pensions on Retirement Behavior." Journal of Applied Econometrics, vol. 22, no. 5 (August):923–950.

Kotlikoff, L. 2008. "Economics' Approach to Financial Planning." Working paper (www.esplanner.com/learn/ economics-approach-financial-planning).

Lachance, M. Forthcoming 2011. "Optimal Onset and Exhaustion of Retirement Savings in a Life-Cycle Model." Journal of Pension Economics and Finance.

Leung, S.F. 1994. "Uncertain Lifetime, the Theory of the Consumer, and the Life Cycle Hypothesis." Econometrica, vol. 62, no. 5 (September):1233–1239. ———. 2007. "The Existence, Uniqueness, and Optimality of the Terminal Wealth Depletion Time in Life-Cycle Models of Saving under Uncertain Lifetime and Borrowing Constraint." Journal of Economic Theory, vol. 134, no. 1 (May):470–493.

Merton, R.C. 1971. "Optimum Consumption and Portfolio Rules in a Continuous-Time Model." Journal of Economic Theory, vol. 3, no. 4 (December):373–413.

Milevsky, M.A. 2006. The Calculus of Retirement Income: Financial Models for Pensions and Insurance. Cambridge, U.K.: Cambridge University Press.

Milevsky, M.A., and C. Robinson. 2005. "A Sustainable Spending Rate without Simulation." Financial Analysts Journal, vol. 61, no. 6 (November/December):89–100.

Modigliani, F. 1986. "Life Cycle, Individual Thrift, and the Wealth of Nations." American Economic Review, vol. 76, no. 3 (June):297–313.

Modigliani, F., and R. Brumberg. 1954. "Utility Analysis and the Consumption Function: An Interpretation of Cross-Section Data." In Post Keynesian Economics. Edited by K.K. Kurihara. New Brunswick, NJ: Rutgers University Press.

Phelps, E.S. 1962. "The Accumulation of Risky Capital: A Sequential Utility Analysis." Econometrica, vol. 30, no. 4 (October): 729–743.

Ramsey, F.P. 1928. "A Mathematical Theory of Saving." Economic Journal, vol. 38, no. 152 (December):543–559.

Richard, S.F. 1975. "Optimal Consumption, Portfolio and Life Insurance Rules for an Uncertain Lived Individual in a Continuous Time Model." Journal of Financial Economics, vol. 2, no. 2 (June):187–203.

Samuelson, P.A. 1969. "Lifetime Portfolio Selection by Dynamic Stochastic Programming." Review of Economics and Statistics, vol. 51, no. 3 (August):239–246.

Scott, J.S., W.F. Sharpe, and J.G. Watson. 2008. "The 4% Rule— At What Price?" Working paper (April): www.stanford.edu/ ~wfsharpe/retecon/4percent.pdf.

Sharpe, W.F., J.S. Scott, and J.G. Watson. 2007. "Efficient Retirement Financial Strategies." Pension Research Council Working Paper (July).

Sheshinski, E. 2008. The Economic Theory of Annuities. Princeton, NJ: Princeton University Press.

Stock, J.H., and D.A. Wise. 1990. "Pensions, the Option Value of Work, and Retirement." Econometrica, vol. 58, no. 5 (September): 1151–1180.

Thaler, R.H., and C.R. Sunstein. 2008. Nudge: Improving Decisions about Health, Wealth, and Happiness. New Haven, CT: Yale University Press.

Yaari, M.E. 1965. "Uncertain Lifetime, Life Insurance, and the Theory of the Consumer." Review of Economic Studies, vol. 32, no. 2 (April):137–150.