A Time-homogeneous Diffusion Model with Tax

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Abstract

We study the two-sided exit problem of a time-homogeneous diffusion process with tax payments of loss-carry-forward type and obtain explicit formulas for the exit probabilities. If the lower boundary is understood as the default threshold, then the non-default probability is solved as a special case. A sufficient and necessary condition for the tax identity is discovered.

Keywords: Diffusion; Hitting time; Markov property; Tax; Two-sided exit problem

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1 Introduction

We are interested in the default risk of a firm. Throughout the paper, denote by $a \geq 0$ the default threshold, so the firm is defaulted whenever its value is below a. In particular, the threshold a is set to 0 in ruin theory. Suppose that the value of the firm before taxation is modelled by a time-homogeneous diffusion process $X = \{X_t, t \geq 0\}$ satisfying

$$\mathrm{d}X_t = \mu\left(X_t\right)\mathrm{d}t + \sigma\left(X_t\right)\mathrm{d}B_t, \qquad t \geq 0, \tag{1.1}$$

where $X_0 = u > a$ is the initial wealth, $\{B_t, t \geq 0\}$ is a standard Brownian motion, and $\mu(\cdot)$ and $\sigma(\cdot) > 0$ are two measurable functions on $[a, \infty)$. As usual, assume that $\mu(\cdot)$ and $\sigma(\cdot)$ satisfy the conditions of the existence and uniqueness theorem for stochastic differential equations; namely, there exists a constant $K > 0$ such that, for all $x_1, x_2 \in [a, \infty)$,

$$|\mu(x_1) - \mu(x_2)| + |\sigma(x_1) - \sigma(x_2)| \leq K\left|x_1 - x_2\right|, \qquad \mu^2(x_1) + \sigma^2(x_1) \leq K^2\left(1 + x_1^2\right). \tag{1.2}$$

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Then the unique solution of (1.1) possesses strong Markov property. See Gīhman and Skorohod (1972, pages 40 and 107).

Denote by $T^X(x) = \inf\{t \geq 0 : X_t = x\}$ the first hitting time of X at level $x \geq a$. In particular, $T^X(a)$ stands for the time of default. Throughout the paper, let

$$0 \leq a < u < b. \tag{1.3}$$

The two-sided exit problem for the diffusion process X has been well studied in the literature. The exit probabilities from the interval $[a, b]$ can be expressed in terms of the function

$$G(y) = C \exp\left\{-\int_a^y \frac{2\mu(x)}{\sigma^2(x)} \mathrm{d}x\right\}, \qquad y \geq a,$$

where $C > 0$ is some constant. Note that $S(z) = \int_a^z G(y)\mathrm{d}y$ for $z \geq a$ is termed the scale function of X. More precisely, under (1.2), it is well known that

$$\mathrm{P}\left(T^X(a) > T^X(b)\right) = \frac{\int_a^u G(y)\mathrm{d}y}{\int_a^b G(y)\mathrm{d}y}, \qquad \mathrm{P}\left(T^X(a) < T^X(b)\right) = \frac{\int_u^b G(y)\mathrm{d}y}{\int_a^b G(y)\mathrm{d}y}; \tag{1.4}$$

see, e.g. Gīhman and Skorohod (1972, page 110) or Klebaner (2005, Section 6.4). The non-default probability follows immediately by letting $b \uparrow \infty$ in the first relation in (1.4), as

$$\mathrm{P}\left(T^X(a) = \infty\right) = \frac{\int_a^u G(y)\mathrm{d}y}{\int_a^\infty G(y)\mathrm{d}y}. \tag{1.5}$$

Recently, ruin problems with tax have become an attractive research topic. Albrecher and Hipp (2007) first introduced tax payments with constant rate at profitable times to the compound Poisson risk model and established a charming tax identity for the non-ruin probability. Later on, Albrecher et al. (2009) found a simple proof using downward excursions and extended the study to a wealth-dependent tax rate. Further extensions to the Lévy framework were done by Albrecher et al. (2008), Kyprianou and Zhou (2009) and Renaud (2009), among others. See also Hao and Tang (2009) for the study in the Lévy framework but under periodic taxation. So far there is little study beyond the Lévy framework with difficulty mainly in the two-sided exit problem.

Following this new trend of ruin theory, we introduce a wealth-dependent tax rate to the time-homogeneous diffusion model (1.1). More precisely, whenever the process X coincides with its running maximum M^X defined by $M_t^X = \sup_{0 \leq \tau \leq t} X_\tau$, $t \geq 0$, the firm pays tax at rate $\gamma(M_t^X)$, where $\gamma(\cdot) : [u, \infty) \to [0, 1)$ is a measurable function satisfying

$$\int_u^\infty (1 - \gamma(z))\, \mathrm{d}z = \infty. \tag{1.6}$$

This is the so-called loss-carry-forward taxation. It is easy to understand that the wealth process after taxation satisfies

$$U_t = X_t - \int_0^t \gamma\left(M_\tau^X\right) \mathrm{d}M_\tau^X, \qquad t \geq 0, \tag{1.7}$$

with $U_0 = X_0 = u$. Our goal is to solve the two-sided exit problem of U and, hence, to obtain the non-default probability of U as a corollary.

The rest of this paper consists of two sections. In Section 2 we present our main result and its corollaries and in Section 3 we prove these results.

2 Main Result and Related Discussions

Recall the initial wealth u, the lower boundary a and the upper boundary b as specified by (1.3). Following Kyprianou and Zhou (2009), we define

$$\bar{\gamma}(x) = x - \int_u^x \gamma(z)\mathrm{d}z = u + \int_u^x (1-\gamma(z))\,\mathrm{d}z, \qquad x \geq u,$$

which is strictly increasing and continuous in x, with $\bar{\gamma}(u) = u$ and $\bar{\gamma}(\infty) = \infty$ by (1.6). Thus, its inverse function $\bar{\gamma}^{-1}(\cdot)$ is well defined on $[u, \infty)$. Note that both $x - \bar{\gamma}(x)$ and $\bar{\gamma}^{-1}(s) - s$ are non-decreasing and continuous functions.

As before, denote by $T^U(x) = \inf\{t \geq 0 : U_t = x\}$ the first hitting time of U at level $x \geq a$. Our main result is the following:

Theorem 2.1 *Under (1.2) and (1.6), it holds that*

$$\mathrm{P}\left(T^U(a) > T^U(b)\right) = \exp\left\{-\int_u^{\bar{\gamma}^{-1}(b)} \frac{G(x)}{\int_{x-\bar{\gamma}(x)+a}^x G(y)\mathrm{d}y}\mathrm{d}x\right\} \tag{2.1}$$

and that $\mathrm{P}\left(T^U(a) < T^U(b)\right) = 1 - \mathrm{P}\left(T^U(a) > T^U(b)\right)$.

The proof of Theorem 2.1 is deferred to Section 3. Clearly, relation (2.1) agrees with the first relation in (1.4) provided $\gamma(\cdot) \equiv 0$. Letting $b \uparrow \infty$ in (2.1) yields the non-default probability of U as follows:

Corollary 2.1 *Under (1.2) and (1.6), it holds that*

$$\mathrm{P}\left(T^U(a) = \infty\right) = \exp\left\{-\int_u^{\infty} \frac{G(x)}{\int_{x-\bar{\gamma}(x)+a}^x G(y)\mathrm{d}y}\mathrm{d}x\right\}. \tag{2.2}$$

Tax payments increase default risk, of course. The non-default probability with tax given by (2.2) is always smaller than the non-default probability without tax given by (1.5) unless $x - \bar{\gamma}(x) \equiv 0$ (or, equivalently, $\gamma(\cdot) = 0$ almost everywhere). Thus, relation (2.2) provides us with a quantitative understanding of the impact of the tax payments on default risk. In particular, the following example shows that the standard Black-Scholes model without tax has a positive probability to survive forever while any constant tax rate, no matter how small it is, will drive the firm to default.

Example 2.1 *Consider the geometric Brownian motion*

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}B_t,$$

where $X_0 = u > 0$ *is the initial wealth and* μ, σ *are positive constants satisfying* $c = 2\mu/\sigma^2 > 1$. *In addition, we assume the default threshold* $a > 0$. *Then by relation (1.5) with* $G(y) = C\,(a/y)^c$ *for* $y \geq a$, *the non-default probability without tax is*

$$\mathrm{P}\left(T^X(a) = \infty\right) = 1 - \left(\frac{a}{u}\right)^{c-1} > 0.$$

However, in the presence of a constant tax rate $0 < \gamma < 1$, *by (2.2) we have*

$$\mathrm{P}\left(T^U(a) = \infty\right) = \exp\left\{-\int_u^\infty \frac{x^{-c}}{\int_{\gamma x - \gamma u + a}^x y^{-c} \mathrm{d}y} \mathrm{d}x\right\} = 0.$$

In order to compare our Theorem 2.1 with Theorem 1.1 of Kyprianou and Zhou (2009), we can use change of variables $x = \bar{\gamma}^{-1}(s)$ to rewrite equation (2.1). In particular, if $\gamma(\cdot) \equiv \gamma \in [0, 1)$ is constant, then $\bar{\gamma}(x) = x - \gamma x + \gamma u$ and relation (2.1) is reduced to

$$\mathrm{P}\left(T^U(a) > T^U(b)\right) = \exp\left\{-\int_u^b \frac{G\left(\frac{s-\gamma u}{1-\gamma}\right)}{\int_{\frac{\gamma s - \gamma u}{1-\gamma}+a}^{\frac{s-\gamma u}{1-\gamma}} G(y) \mathrm{d}y} \mathrm{d}s\right\}^{\frac{1}{1-\gamma}}. \tag{2.3}$$

As mentioned in Section 1, for the case of a constant tax rate γ, the tax identity

$$\mathrm{P}\left(T^U(0) = \infty\right) = \left(\mathrm{P}\left(T^X(0) = \infty\right)\right)^{\frac{1}{1-\gamma}} \tag{2.4}$$

has been established by Albrecher and Hipp (2007), Albrecher et al. (2008), Albrecher et al. (2009) and Kyprianou and Zhou (2009) in various situations within the Lévy framework. However, relation (2.3) indicates that such an identity does not hold in general within the diffusion framework.

Motivated by these cited works, we now consider under what condition the identity

$$\mathrm{P}\left(T^U(a) > T^U(b)\right) = \left(\mathrm{P}\left(T^X(a) > T^X(b)\right)\right)^{\frac{1}{1-\gamma}} \tag{2.5}$$

holds. Interestingly, the answer is that $\mu(\cdot)/\sigma^2(\cdot)$ has to be constant.

Corollary 2.2 *Consider constant tax rates and assume (1.2) and (1.6).*

(1) For arbitrarily fixed u *and* a *with* $0 \leq a < u$, *relation (2.5) holds for all* $b > u$ *and* $0 \leq \gamma < 1$ *if and only if* $\mu(x)/\sigma^2(x)$ *is constant for* $x \geq a$.

(2) For arbitrarily fixed a *and* b *with* $0 \leq a < b$, *relation (2.5) holds for all* $a < u < b$ *and* $0 \leq \gamma < 1$ *if and only if* $\mu(x)/\sigma^2(x)$ *is constant for* $a \leq x \leq b$.

The proof of Corollary 2.2 is deferred to Section 3. By letting $b \uparrow \infty$ and $a = 0$ in part (2) of Corollary 2.2 and going along the same lines of its proof, we obtain the following:

Corollary 2.3 *Consider constant tax rates and assume (1.2), (1.6) and* $\int_0^\infty G(y)\mathrm{d}y < \infty$. *Then relation (2.4) holds for all* $0 < u < \infty$ *and* $0 \leq \gamma < 1$ *if and only if* $\mu(x)/\sigma^2(x)$ *is constant for* $x \geq 0$.

The condition $\int_0^\infty G(y)\mathrm{d}y < \infty$ in Corollary 2.3 is necessary; otherwise, the probability $\mathrm{P}\left(T^X(0) = \infty\right)$ is equal to 0 and relation (2.4) becomes trivial.

3 Proofs

As before, denote by $M_t^U = \sup_{0\le\tau\le t} U_\tau$, $t \ge 0$, the running maximum of U. In terms of the function $\bar\gamma(\cdot)$, we can rewrite the process U in (1.7) as

$$U_t = X_t - M_t^X + \bar\gamma\left(M_t^X\right), \qquad t \ge 0. \tag{3.1}$$

As shown in Lemma 2.1 of Kyprianou and Zhou (2009), we have

$$M_t^U = M_t^X - \int_0^t \gamma\left(M_\tau^X\right) \mathrm{d}M_\tau^X = \bar\gamma\left(M_t^X\right), \qquad t \ge 0, \tag{3.2}$$

and, hence, $T^U(s) = T^X\left(\bar\gamma^{-1}(s)\right)$ for $s \ge u$.

Trivially, in order for U to hit b before a, for every $s \in [u, b)$, after $T^U(s)$ the process U must enter (s, ∞) before it hits a. By equation (3.1), this fact can be restated in terms of X as follows. After $T^X\left(\bar\gamma^{-1}(s)\right)$, the process X must enter $(\bar\gamma^{-1}(s), \infty)$ before it hits $\bar\gamma^{-1}(s) - s + a$. Thus, the event $\left(T^U(a) > T^U(b)\right)$ necessitates a two-sided exit problem of X for every $s \in [u, b)$. Based on this intuition, we establish lower and upper discrete approximations for the event $\left(T^U(a) > T^U(b)\right)$ in Lemma 3.1 and a precise discrete approximation in Lemma 3.2. Our idea stems from the work of Lehoczky (1977). Nonetheless, we consider a much more complicated situation due to taxation.

Lemma 3.1 *Let $u = s_0 < s_1 < \cdots < s_n = b$ form a partition of the interval $[u, b]$, $n \in \mathbb{N}$. Then, almost surely,*

$$\bigcap_{i=1}^n A_i \subset \left(T^U(a) > T^U(b)\right) \subset \bigcap_{i=1}^n B_i, \tag{3.3}$$

where each A_i denotes the event that after $T^X\left(\bar\gamma^{-1}(s_{i-1})\right)$, the process X hits $\bar\gamma^{-1}(s_i)$ before $\bar\gamma^{-1}(s_i) - s_i + a$ while each B_i denotes the event that after $T^X\left(\bar\gamma^{-1}(s_{i-1})\right)$, the process X hits $\bar\gamma^{-1}(s_i)$ before $\bar\gamma^{-1}(s_{i-1}) - s_{i-1} + a$.

Proof. To prove the first inclusion in (3.3), assume that the path of X is continuous such that $\bigcap_{i=1}^n A_i$ holds. Arbitrarily choose $t \in [0, T^X\left(\bar\gamma^{-1}(b)\right)]$ and suppose that t falls into the interval $[T^X\left(\bar\gamma^{-1}(s_{i-1})\right), T^X\left(\bar\gamma^{-1}(s_i)\right)]$ for some $i = 1, \ldots, n$. Then $M_t^X \le \bar\gamma^{-1}(s_i)$ and, by relation (3.1), the monotonicity of $s - \bar\gamma(s)$ and the decription of A_i, we have

$$U_t = X_t - (M_t^X - \bar\gamma(M_t^X)) \ge X_t - \left(\bar\gamma^{-1}(s_i) - \bar\gamma\left(\bar\gamma^{-1}(s_i)\right)\right) > a.$$

In sum, $U_t > a$ for all $t \in [0, T^X\left(\bar\gamma^{-1}(b)\right)]$. Hence, $T^U(a) > T^X\left(\bar\gamma^{-1}(b)\right) = T^U(b)$.

To prove the second inclusion in (3.3). Assume by contradiction that there exists some $i = 1, \ldots, n$ such that after $T^X\left(\bar\gamma^{-1}(s_{i-1})\right)$ the path of X hits $\bar\gamma^{-1}(s_{i-1}) - s_{i-1} + a$ before $\bar\gamma^{-1}(s_i)$. Then at the moment of hitting $\bar\gamma^{-1}(s_{i-1}) - s_{i-1} + a$, by relation (3.1), the monotonicity of $s - \bar\gamma(s)$ and $M_t^X \ge \bar\gamma^{-1}(s_{i-1})$, we have

$$U_t = X_t - (M_t^X - \bar\gamma(M_t^X)) \le \bar\gamma^{-1}(s_{i-1}) - s_{i-1} + a - \left(\bar\gamma^{-1}(s_{i-1}) - \bar\gamma\left(\bar\gamma^{-1}(s_{i-1})\right)\right) = a,$$

which contradicts to $T^U(a) > T^U(b)$. ■

The two bounds given by (3.3) can actually be made arbitrarily close to each other, as shown in the following:

Lemma 3.2 *Let $\{s_{n,i}, i = 0, \ldots, m_n\}$, $n \in \mathbb{N}$, constitute a sequence of increasing partitions of the interval $[u, b]$ with $u = s_{n,0} < s_{n,1} < \cdots < s_{n,m_n} = b$ and the maximum length of subintervals $\Delta_n = \max_{1 \le i \le n}(s_{n,i} - s_{n,i-1}) \downarrow 0$ as $n \to \infty$. Then, almost surely,*

$$\left(T^U(a) > T^U(b)\right) = \bigcup_{n=1}^{\infty} \bigcap_{i=1}^{m_n} A_{n,i}, \tag{3.4}$$

where each $A_{n,i}$ denotes the event that after $T^X\left(\bar{\gamma}^{-1}(s_{n,i-1})\right)$, the process X hits $\bar{\gamma}^{-1}(s_{n,i})$ before $\bar{\gamma}^{-1}(s_{n,i}) - s_{n,i} + a$.

Proof. For every $\delta > 0$, there exists some $n_\delta \in \mathbb{N}$ such that, for all $n \ge n_\delta$,

$$\bar{\gamma}^{-1}(s_{n,i}) - s_{n,i} + a \le \bar{\gamma}^{-1}(s_{n,i-1}) - s_{n,i-1} + a + \delta, \qquad i = 1, \ldots, m_n.$$

Denote by $B_{n,i}^\delta$ the event that after $T^X\left(\bar{\gamma}^{-1}(s_{n,i-1})\right)$, the process X hits $\bar{\gamma}^{-1}(s_{n,i})$ before $\bar{\gamma}^{-1}(s_{n,i-1}) - s_{n,i-1} + a + \delta$. Applying Lemma 3.1 twice, we obtain, for all $n \ge n_\delta$,

$$\left(T^U(a) > T^U(b)\right) \supset \bigcap_{i=1}^{m_n} A_{n,i} \supset \bigcap_{i=1}^{m_n} B_{n,i}^\delta \supset \left(T^U(a+\delta) > T^U(b)\right).$$

Note that $\left(T^U(a) > T^U(b)\right) = \bigcup_{\delta > 0}\left(T^U(a+\delta) > T^U(b)\right)$. Since $\bigcap_{i=1}^{m_n} A_{n,i}$ is increasing in n, we have

$$\left(T^U(a) > T^U(b)\right) = \bigcup_{\delta>0} \bigcap_{i=1}^{m_n} A_{n,i} = \bigcup_{n=1}^{\infty} \bigcap_{i=1}^{m_n} A_{n,i}.$$

This proves relation (3.4). ■

The proof of Theorem 2.1 below is based on Lemma 3.2 but we point out that Lemma 3.1 can play the same role here.

Proof of Theorem 2.1. By relation (3.2) and the fact $U \le X$, the event that U always stays in (a, b) implies the event that X always stays in $(a, \bar{\gamma}^{-1}(b))$. By this and (1.4),

$$\mathrm{P}\left(T^U(a) = T^U(b) = \infty\right) \le \mathrm{P}\left(T^X(a) = T^X\left(\bar{\gamma}^{-1}(b)\right) = \infty\right) = 0.$$

Thus, $\mathrm{P}\left(T^U(a) < T^U(b)\right) = 1 - \mathrm{P}\left(T^U(a) > T^U(b)\right)$.

Next we focus on the proof of relation (2.1). As mentioned before, the intersection $\bigcap_{i=1}^{m_n} A_{n,i}$ is increasing in n. Thus, by Lemma 3.2 we have

$$\mathrm{P}\left(T^U(a) > T^U(b)\right) = \lim_{n\to\infty} \mathrm{P}\left(\bigcap_{i=1}^{m_n} A_{n,i}\right) = \lim_{n\to\infty} P_n.$$

By virtue of the strong Markov property and time homogeneity of the diffusion X, it is easy to see that, for all large n,

$$\begin{aligned} P_n &= \prod_{i=1}^{m_n} \mathrm{P}\left(T^X\left(\bar{\gamma}^{-1}(s_{n,i})\right) < T^X\left(\bar{\gamma}^{-1}(s_{n,i}) - s_{n,i} + a\right) \Big| X_0 = \bar{\gamma}^{-1}(s_{n,i-1})\right) \\ &= \exp\left\{\sum_{i=1}^{m_n} \log\left(1 - q\left(\bar{\gamma}^{-1}(s_{n,i}) - s_{n,i} + a, \bar{\gamma}^{-1}(s_{n,i}) \Big| \bar{\gamma}^{-1}(s_{n,i-1})\right)\right)\right\}, \end{aligned}$$

where $q(c_1, c_2|c_0)$, $0 \le c_1 < c_0 < c_2$, denotes the probability of $\left(T^X(c_1) < T^X(c_2)\right)$ conditional on $X_0 = c_0$. By the second relation in (1.4),

$$q\left(\bar\gamma^{-1}(s_{n,i}) - s_{n,i} + a, \bar\gamma^{-1}(s_{n,i}) \big| \bar\gamma^{-1}(s_{n,i-1})\right) = \frac{\int_{\bar\gamma^{-1}(s_{n,i-1})}^{\bar\gamma^{-1}(s_{n,i})} G(y)\mathrm{d}y}{\int_{\bar\gamma^{-1}(s_{n,i})-s_{n,i}+a}^{\bar\gamma^{-1}(s_{n,i})} G(y)\mathrm{d}y}$$
$$\le C \max_{1\le i\le m_n} \left(\bar\gamma^{-1}(s_{n,i}) - \bar\gamma^{-1}(s_{n,i-1})\right),$$

where the constant C is defined as

$$C = \frac{\sup_{u\le y\le \bar\gamma^{-1}(b)} G(y)}{\inf_{u\le w\le \bar\gamma^{-1}(b)} \int_{w-u+a}^{w} G(y)\mathrm{d}y} < \infty.$$

This means that the conditional probabilities $q(c_1, c_2|c_0)$ appearing in P_n are uniformly small for all large n. Therefore, by the elementary relation $\log(1-q) \sim -q$ as $q \downarrow 0$, it holds for arbitrarily fixed $0 < \varepsilon < 1$ and all large n that

$$P_n \le \exp\left\{-(1-\varepsilon)\sum_{i=1}^{m_n} q\left(\bar\gamma^{-1}(s_{n,i}) - s_{n,i} + a, \bar\gamma^{-1}(s_{n,i}) \big| \bar\gamma^{-1}(s_{n,i-1})\right)\right\}$$
$$= \exp\left\{-(1-\varepsilon)\sum_{i=1}^{m_n} \frac{\int_{\bar\gamma^{-1}(s_{n,i-1})}^{\bar\gamma^{-1}(s_{n,i})} G(y)\mathrm{d}y}{\int_{\bar\gamma^{-1}(s_{n,i})-s_{n,i}+a}^{\bar\gamma^{-1}(s_{n,i})} G(y)\mathrm{d}y}\right\}.$$

Since the function $G(\cdot)$ is continuous and away from 0 over the interval $[u, \bar\gamma^{-1}(b)]$, it holds for all large n and $i = 1, \ldots, m_n$ that

$$\int_{\bar\gamma^{-1}(s_{n,i-1})}^{\bar\gamma^{-1}(s_{n,i})} G(y)\mathrm{d}y \ge (1-\varepsilon) G\left(\bar\gamma^{-1}(s_{n,i})\right)\left(\bar\gamma^{-1}(s_{n,i}) - \bar\gamma^{-1}(s_{n,i-1})\right).$$

It follows that

$$\limsup_{n\to\infty} P_n \le \lim_{n\to\infty} \exp\left\{-(1-\varepsilon)^2 \sum_{i=1}^{m_n} \frac{G\left(\bar\gamma^{-1}(s_{n,i})\right)}{\int_{\bar\gamma^{-1}(s_{n,i})-s_{n,i}+a}^{\bar\gamma^{-1}(s_{n,i})} G(y)\mathrm{d}y}\left(\bar\gamma^{-1}(s_{n,i}) - \bar\gamma^{-1}(s_{n,i-1})\right)\right\}$$
$$= \exp\left\{-(1-\varepsilon)^2 \int_u^{\bar\gamma^{-1}(b)} \frac{G(x)}{\int_{x-\bar\gamma(x)+a}^{x} G(y)\mathrm{d}y}\mathrm{d}x\right\}.$$

By the arbitrariness of ε, we have

$$\limsup_{n\to\infty} P_n \le \exp\left\{-\int_u^{\bar\gamma^{-1}(b)} \frac{G(x)}{\int_{x-\bar\gamma(x)+a}^{x} G(y)\mathrm{d}y}\mathrm{d}x\right\}.$$

The inequality for $\liminf_{n\to\infty} P_n$ can be established symmetrically. This ends the proof of Theorem 2.1. ■

Proof of Corollary 2.2. Clearly, $\mu(x)/\sigma^2(x)$ is a constant for $x \ge a$ if and only if

$$G(x) = c_1 \mathrm{e}^{c_2 x}, \qquad x \ge a,$$

for some constants $c_1 > 0$ and c_2. The sufficiency of both parts can be checked directly. We now prove the necessity separately for both parts.

(1) For arbitrarily fixed u and a with $0 \le a < u$, we assume that relation (2.5) holds for all $b > u$ and $0 \le \gamma < 1$. By (2.3), (2.5) and (1.4),

$$\int_u^b \frac{G\left(\frac{s-\gamma u}{1-\gamma}\right)}{\int_{\frac{\gamma s-\gamma u}{1-\gamma}+a}^{\frac{s-\gamma u}{1-\gamma}} G(y)\mathrm{d}y}\mathrm{d}s = \int_u^b \frac{G(s)}{\int_a^s G(y)\mathrm{d}y}\mathrm{d}s, \qquad b > u, 0 \le \gamma < 1.$$

It follows that

$$\frac{G\left(\frac{\gamma s-\gamma u}{1-\gamma}+s\right)}{\int_{\frac{\gamma s-\gamma u}{1-\gamma}+a}^{\frac{\gamma s-\gamma u}{1-\gamma}+s} G(y)\mathrm{d}y} = \frac{G(s)}{\int_a^s G(y)\mathrm{d}y}, \qquad s > u, 0 \le \gamma < 1.$$

Using change of variables $x = (\gamma s - \gamma u)/(1-\gamma)$ on the left-hand side of above, upon some simple rearrangement we obtain

$$\frac{\int_a^s G(y)\mathrm{d}y}{G(s)} G(x+s) = \int_{x+a}^{x+s} G(y)\mathrm{d}y, \qquad s > u, x \ge 0.$$

By the continuity of $G(\cdot)$, it follows that

$$\frac{\int_a^s G(y)\mathrm{d}y}{G(s)} G(x+s) = \int_{x+a}^{x+s} G(y)\mathrm{d}y, \qquad s \ge u, x \ge 0. \tag{3.5}$$

Taking derivative with respect to s, upon some simple rearrangement we obtain

$$\frac{G'(x+s)}{G(x+s)} = \frac{G'(s)}{G(s)}, \qquad s > u, x \ge 0.$$

This means that $G'(\cdot)/G(\cdot)$ is constant over the interval (u, ∞). Hence, by the positivity and continuity of $G(\cdot)$, it must hold that

$$G(x) = c_1 \mathrm{e}^{c_2 x}, \qquad x \ge u, \tag{3.6}$$

for some constants $c_1 > 0$ and c_2. Substituting (3.6) into (3.5) with $s = u$ yields

$$\mathrm{e}^{c_2 x} \int_a^u G(y)\mathrm{d}y = \int_{x+a}^{x+u} G(y)\mathrm{d}y, \qquad x \ge 0.$$

Taking derivative with respect to x and using (3.6), we have

$$G(x) = \mathrm{e}^{-c_2 a}\left(c_1 \mathrm{e}^{c_2 u} - c_2 \int_a^u G(y)\mathrm{d}y\right)\mathrm{e}^{c_2 x}, \qquad x \ge a.$$

Comparing this with (3.6), we must have $\mathrm{e}^{-c_2 a}\left(c_1 \mathrm{e}^{c_2 u} - c_2 \int_a^u G(y)\mathrm{d}y\right) = c_1$ since $G(\cdot)$ is continuous at u. One can also easily check this by substitution. Hence, $G(x) = c_1 \mathrm{e}^{c_2 x}$ is valid over $[a, \infty)$.

(2) For arbitrarily fixed a and b with $0 \le a < b$, we assume that relation (2.5) holds for all $u \in (a, b)$ and $\gamma \in [0, 1)$. Similarly as in the proof of part (1), by (2.1), (2.5) and (1.4) one sees that

$$\int_u^{\frac{b-\gamma u}{1-\gamma}} \frac{G(x)}{\int_{\gamma x-\gamma u+a}^x G(y)\mathrm{d}y}\mathrm{d}x = \frac{1}{1-\gamma}\int_u^b \frac{G(x)}{\int_a^x G(y)\mathrm{d}y}\mathrm{d}x, \qquad u \in (a,b), \gamma \in [0,1).$$

Taking derivative with respect to u and cancelling γ, we obtain that, over the range $u \in (a, b)$ and $\gamma \in (0, 1)$,

$$\frac{1}{1-\gamma}\frac{G(u)}{\int_a^u G(y)\mathrm{d}y} - \frac{1}{1-\gamma}\frac{G(\frac{b-\gamma u}{1-\gamma})}{\int_{\frac{\gamma b-\gamma u}{1-\gamma}+a}^{\frac{b-\gamma u}{1-\gamma}} G(y)\mathrm{d}y} = \int_u^{\frac{b-\gamma u}{1-\gamma}} \frac{G(x)\,G(\gamma x-\gamma u+a)}{\left(\int_{\gamma x-\gamma u+a}^x G(y)\mathrm{d}y\right)^2}\mathrm{d}x.$$

Letting $\gamma \to 0$ yields

$$\frac{G(u)}{\int_a^u G(y)\mathrm{d}y} - \frac{G(b)}{\int_a^b G(y)\mathrm{d}y} = \int_u^b \frac{G(x)\,G(a)}{\left(\int_a^x G(y)\mathrm{d}y\right)^2}\mathrm{d}x, \qquad u \in (a,b).$$

Upon some rearrangement we obtain

$$\frac{G(b)-G(a)}{\int_a^b G(y)\mathrm{d}y}\int_a^u G(y)\mathrm{d}y = \int_a^u G'(y)\mathrm{d}y, \qquad u \in (a,b),$$

which implies that

$$\frac{G(b)-G(a)}{\int_a^b G(y)\mathrm{d}y}G(x) = G'(x), \qquad x \in (a,b).$$

Therefore, it must hold that

$$G(x) = c_1\mathrm{e}^{c_2 x}, \qquad x \in [a,b],$$

for some constants $c_1 > 0$ and c_2 by the positivity and continuity of $G(\cdot)$. ■

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