Hedging Equity-Linked Products Under Stochastic Volatility Models

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Heston Model

- Introduced by Heston (1993)
- Better at describing the high peaks and heavy tails of the empirical distribution of log-returns
- Stock index price dynamics under the physical measure given by

\[ dS_t = \mu S_t dt + \sqrt{v_t} S_t dZ_t^{(1)}, \]
\[ dv_t = \kappa' (\theta' - v_t) dt + \sigma \sqrt{v_t} dZ_t^{(2)}, \]

where \( \mu, \kappa', \theta', \sigma \) are constants and \( \langle dZ_t^{(1)} dZ_t^{(2)} \rangle = \rho \ dt \).

- Market price of volatility risk is given by \( \lambda \)
- Risk-neutral parameters \( \kappa = \kappa' + \lambda \) and \( \theta = \frac{\kappa' \theta'}{\kappa' + \lambda} \)
Price of a European call option given by

\[ C^H(x_t, v_t, \tau) = K e^{-r\tau} (e^{x_t} P_1(x_t, v_t, \tau) - P_0(x_t, v_t, \tau)) \]

where \( x_t = \log\left(\frac{e^{r(T-t)}S_t}{K}\right) \), \( \tau = T - t \) and

\[ P_j(x_t, v_t, \tau) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left( \frac{\exp(izu_x + C_j(u, \tau)\theta + D_j(u, \tau)v_t)}{iu} \right) du, \]

for \( j = 0, 1 \), with \( C_j(u, \tau) \) and \( D_j(u, \tau) \) functions of \( u, \tau, \kappa, \theta, \sigma \) and \( \rho \).
Equity-Linked Products

- Insurance policies that offer participation in financial market while protecting the initial investment
- May offer other types of benefits
- Two main categories:
  - Variable Annuities
  - Equity-Indexed Annuities (EIAs)
Review of Literature

- First studied under the Black-Scholes model by Brennan and Schwartz (1976) and Boyle and Schwartz (1977)
- Hardy (2003) discusses product design and pricing techniques
- Tiong (2000) and Lee (2003) present closed-form expressions for the price of the financial guarantees embedded in EIAs
- Lin and Tan (2003) prices EIAs under stochastic interest rate models
- Lin et al. (2009) uses a regime-switching model to value EIAs
Equity-Indexed Annuities

- First sold in 1995 by Keyport Life
- Premium invested for 5 to 15 years
- Guaranteed return on initial investment
- Additional return based on the performance of a stock index
- Additional return may be reduced or capped
- Actual return of the EIA depends on its design (point-to-point, annual reset, ...)

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Hedging Equity-Linked Products Under Stochastic Volatility
Point-to-Point EIA

- Payoff based on the value of the index at inception and at maturity of the contract
- Participation rate $\alpha$ in additional return, $0 < \alpha \leq 1$
- Participation rate $\varrho$ in guaranteed return $g$, $0 < \varrho \leq 1$
- Payoff:

$$B^{PTP}(S_T, T) = \max \left( 1 + \alpha \left( \frac{S_T}{S_0} - 1 \right), \varrho(1 + g)^T \right) \quad (1)$$
Pricing Point-to-Point EIAs

Let $K = \varrho (1 + g)^T$ and $L = S_0 \left( \frac{K - 1 + \alpha}{\alpha} \right)$.

Re-write (1) as:

$$B^{PTP}(S_T, T) = K + \frac{\alpha}{S_0} \max(S_T - L, 0).$$

Under the no-arbitrage assumption, we have that

$$P_t(S_t, \tau) = Ke^{-r\tau} + \frac{\alpha}{S_0} C(S_t, L, \tau),$$

where $P_t(S_t, \tau)$ is the price at time $t$ of the point-to-point EIA of maturity $T$ and $C(S_t, L, T)$ is the price at time $t$ of a European call option of strike $L$ and maturity $T$. 
The Greeks

- \( \Delta \): Sensitivity to changes in the price of the underlying
- \( \Gamma \): Sensitivity of the delta to changes in the price of the underlying
- \( \mathcal{V} \): Sensitivity to changes in the volatility

For the European call option in the Heston model:

\[
\Delta^H_{C,t} = P_1 + \frac{\partial P_1}{\partial x_t} - e^{-x_t} \frac{\partial P_0}{\partial x_t} \\
\Gamma^H_{C,t} = \frac{1}{S_t} \left[ \left( \frac{\partial P_1}{\partial x_t} - \frac{\partial^2 P_1}{\partial x_t^2} \right) - e^{-x_t} \left( \frac{\partial^2 P_0}{\partial x_t^2} - \frac{\partial P_0}{\partial x_t} \right) \right] \\
\mathcal{V}^H_{C,t} = Ke^{-r\tau} \left( e^{x_t} \frac{\partial P_1}{\partial v_t} - \frac{\partial P_0}{\partial v_t} \right)
\]
Delta Hedging

- Protects the insurer against small changes in index prices.
- Based on following replicating portfolio

\[ H_t^\Delta = \Delta P_t S_t + \xi_t, \]

with \( \xi_t \) is an amount invested in a risk-free asset.
- \( \xi_t \) chosen so that \( H_t^\Delta = P_t(S_t, \tau) \).
- Strategy is self-financing when applied in continuous time.
Gamma Hedging

- Improves the delta hedging strategy when it is applied in discrete time
- Based on following replicating portfolio

\[ H_t^\Gamma = \alpha^\Gamma_{1,t} C(S_t, L, \bar{\tau}) + \alpha^\Gamma_{2,t} S_t + \xi_t, \]

with \( \xi_t \) is an amount invested in a risk-free asset and

\[ \alpha^\Gamma_{1,t} = \frac{\Gamma_{P,t}}{\Gamma_{C,t}} \]

\[ \alpha^\Gamma_{2,t} = \Delta_{P,t} - \alpha^\Gamma_{1,t} \Delta_{C,t}. \]

- To hedge EIAs, use calls with the longest maturity possible.
### Vega Hedging

- Protects the insurer against small changes in both index prices and volatility.
- Based on following replicating portfolio

\[
H_t^\gamma = \alpha_{1,t}^\gamma C(S_t, L, \bar{\tau}) + \alpha_{2,t}^\gamma S_t + \xi_t,
\]

with \(\xi_t\) is an amount invested in a risk-free asset and

\[
\alpha_{1,t}^\gamma = \frac{\nu_{P,t}}{\nu_{C,t}},
\]

\[
\alpha_{2,t}^\gamma = \Delta_{P,t} - \alpha_{1,t}^\gamma \Delta_{C,t}.
\]

- To hedge EIAs, use calls with the longest maturity possible.
Hedging Errors

- Due to the discretization of the hedging process
- Occur when rebalancing the replicating portfolio
- Hedging error at time $t$ defined by

$$HE_t = P_t(S_t, \tau) - H_t$$

- Total discounted hedging error given by

$$PV(HE) = \sum_{i=1}^{mT} e^{-ir/m} HE_i$$

if rebalancing occurs $m$ times a year at equal time intervals.

- Used to assess the performance of the hedging strategy.
Assumptions

- 10-year maturity point-to-point EIA with $g = 0$ and $\varrho = 1$.
- Participation rate $\alpha$ chosen so that the price of the EIA is 1.
- Risk-free rate $r = 0.02$.
- Black-Scholes parameters: $\mu_{BS} = 0.0636$ and $\sigma_{BS} = 0.19$.
- Heston parameters: $\kappa = 5.1793$, $\theta = 0.0178$, $\sigma = 0.1309$, $\nu_0 = 0.0286$, $\rho = -0.7025$.
- Index prices follow Heston model with different volatility risk premia $\lambda$. 
Black-Scholes Delta Hedging

Figure: Present values of hedging errors resulting from a Black-Scholes delta hedging strategy for different values of $\lambda, \alpha = 0.5723$
Black-Scholes Gamma Hedging

(a) $\lambda = -1$

(b) $\lambda = 0$

(c) $\lambda = 2.62$

Figure: Present values of hedging errors resulting from a Black-Scholes gamma hedging strategy for different values of $\lambda$, $\alpha = 0.5723$
Heston Delta Hedging

(a) $\lambda = -1$

(b) $\lambda = 0$

(c) $\lambda = 2.62$

**Figure:** Present values of hedging errors resulting from a Heston delta hedging strategy for different values of $\lambda$, $\alpha = 0.6961$
Heston Gamma Hedging

Figure: Present values of hedging errors resulting from a Heston gamma hedging strategy for different values of $\lambda$, $\alpha = 0.6961$
Heston Vega Hedging

(a) $\lambda = 0$

(b) $\lambda = 2.62$

**Figure:** Present values of hedging errors resulting from a Heston gamma hedging strategy for different values of $\lambda$, $\alpha = 0.6961$
Conclusion

- Stochastic volatility and volatility risk premium should be considered when hedging EIAs
- Future work:
  - Modify constant risk-free rate assumption
  - Analyze the effect of stochastic volatility on other designs
  - Consider transaction costs
Thank you for your attention


