Hedging Equity-Linked Products Under Stochastic Volatility Models

Anne MacKay, ASA

Department of Mathematics and Statistics
Concordia University, Montreal

August 13, 2011

Joint work with
Dr. Patrice Gaillardetz, Concordia University, Montreal
Dr. Etienne Marceau, Université Laval, Québec

Research funded by the Natural Sciences and Engineering Research Council of Canada (NSERC)
and by the Fonds québécois de la recherche sur la nature et technologie (FQRNT)

Outline of the Presentation:

Heston Model

- Introduced by Heston (1993)
- Better at describing the high peaks and heavy tails of the empirical distribution of log-returns
- Stock index price dynamics under the physical measure given by

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dZ_t^{(1)},$$
$$dv_t = \kappa'(\theta' - v_t)dt + \sigma\sqrt{v_t} dZ_t^{(2)},$$

  where μ, κ', θ', σ are constants and $\langle dZ_t^{(1)} dZ_t^{(2)} \rangle = \rho \, dt$.
- Market price of volatility risk is given by λ
- Risk-neutral parameters $\kappa = \kappa' + \lambda$ and $\theta = \frac{\kappa'\theta'}{\kappa'+\lambda}$

Price of a European call option under the Heston Model

Price of a European call option given by

$$C^H(x_t, v_t, \tau) = Ke^{-r\tau}(e^{x_t} P_1(x_t, v_t, \tau) - P_0(x_t, v_t, \tau))$$

where $x_t = \log(\frac{e^{r(T-t)} S_t}{K})$, $\tau = T - t$ and

$$P_j(x_t, v_t, \tau) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re \left(\frac{\exp(iux_t + C_j(u,\tau)\theta + D_j(u,\tau)v_t)}{iu} \right) du,$$

for $j = 0, 1$, with $C_j(u, \tau)$ and $D_j(u, \tau)$ functions of u, τ, κ, θ, σ and ρ.

Equity-Linked Products

- Insurance policies that offer participation in financial market while protecting the initial investment
- May offer other types of benefits
- Two main categories:
  - Variable Annuities
  - Equity-Indexed Annuities (EIAs)

Review of Literature

- First studied under the Black-Scholes model by Brennan and Schwartz (1976) and Boyle and Schwartz (1977)
- Hardy (2003) discusses product design and pricing techniques
- Tiong (2000) and Lee (2003) present closed-form expressions for the price of the financial guarantees embedded in EIAs
- Lin and Tan (2003) prices EIAs under stochastic interest rate models
- Lin et al. (2009) uses a regime-switching model to value EIAs

Equity-Indexed Annuities

- First sold in 1995 by Keyport Life
- Premium invested for 5 to 15 years
- Guaranteed return on initial investment
- Additional return based on the performance of a stock index
- Additional return may be reduced or capped
- Actual return of the EIA depends on its design (point-to-point, annual reset, ...)

Point-to-Point EIA

- Payoff based on the value of the index at inception and at maturity of the contract
- Participation rate α in additional return, $0 < \alpha \leq 1$
- Participation rate ϱ in guaranteed return g, $0 < \varrho \leq 1$
- Payoff:

$$B^{PTP}(S_T, T) = \max\left(1 + \alpha\left(\frac{S_T}{S_0} - 1\right), \varrho(1+g)^T\right) \quad (1)$$

Pricing Point-to-Point EIAs

Let $K = \varrho(1+g)^T$ and $L = S_0\left(\frac{K-1+\alpha}{\alpha}\right)$.
Re-write (1) as:

$$B^{PTP}(S_T, T) = K + \frac{\alpha}{S_0}\max(S_T - L, 0).$$

Under the no-arbitrage assumption, we have that

$$P_t(S_t, \tau) = Ke^{-r\tau} + \frac{\alpha}{S_0}C(S_t, L, \tau),$$

where $P_t(S_t, \tau)$ is the price at time t of the point-to-point EIA of maturity T and $C(S_t, L, T)$ is the price at time t of a European call option of strike L and maturity T.

The Greeks

- Δ: Sensitivity to changes in the price of the underlying
- Γ: Sensitivity of the delta to changes in the price of the underlying
- $\mathcal{V}$: Sensitivity to changes in the volatility
- For the European call option in the Heston model:

$$
\begin{aligned}
\Delta^H_{C,t} &= P_1 + \frac{\partial P_1}{\partial x_t} - e^{-x_t}\frac{\partial P_0}{\partial x_t} \\
\Gamma^H_{C,t} &= \frac{1}{S_t}\left[\left(\frac{\partial P_1}{\partial x_t} - \frac{\partial^2 P_1}{\partial x_t^2}\right) - e^{-x_t}\left(\frac{\partial^2 P_0}{\partial x_t^2} - \frac{\partial P_0}{\partial x_t}\right)\right] \\
\mathcal{V}^H_{C,t} &= Ke^{-r\tau}\left(e^{x_t}\frac{\partial P_1}{\partial v_t} - \frac{\partial P_0}{\partial v_t}\right)
\end{aligned}
$$

Delta Hedging

- Protects the insurer against small changes in index prices.
- Based on following replicating portfolio

$$H_t^{\Delta} = \Delta_{P,t} S_t + \xi_t,$$

with ξ_t is an amount invested in a risk-free asset.

- ξ_t chosen so that $H_t^{\Delta} = P_t(S_t, \tau)$.
- Strategy is self-financing when applied in continuous time.

Gamma Hedging

- Improves the delta hedging strategy when it is applied in discrete time
- Based on following replicating portfolio

$$H_t^{\Gamma} = \alpha_{1,t}^{\Gamma} C(S_t, L, \bar{\tau}) + \alpha_{2,t}^{\Gamma} S_t + \xi_t,$$

with ξ_t is an amount invested in a risk-free asset and

$$\begin{aligned} \alpha_{1,t}^{\Gamma} &= \frac{\Gamma_{P,t}}{\Gamma_{C,t}} \\ \alpha_{2,t}^{\Gamma} &= \Delta_{P,t} - \alpha_{1,t}^{\Gamma} \Delta_{C,t}. \end{aligned}$$

- To hedge EIAs, use calls with the longest maturity possible.

Vega Hedging

- Protects the insurer against small changes in both index prices and volatility.
- Based on following replicating portfolio

$$H_t^{\mathcal{V}} = \alpha_{1,t}^{\mathcal{V}} C(S_t, L, \bar{\tau}) + \alpha_{2,t}^{\mathcal{V}} S_t + \xi_t,$$

with ξ_t is an amount invested in a risk-free asset and

$$\begin{aligned} \alpha_{1,t}^{\mathcal{V}} &= \frac{\mathcal{V}_{P,t}}{\mathcal{V}_{C,t}} \\ \alpha_{2,t}^{\mathcal{V}} &= \Delta_{P,t} - \alpha_{1,t}^{\mathcal{V}} \Delta_{C,t}. \end{aligned}$$

- To hedge EIAs, use calls with the longest maturity possible.

Hedging Errors

- Due to the discretization of the hedging process
- Occur when rebalancing the replicating portfolio
- Hedging error at time t defined by

$$HE_t = P_t(S_t, \tau) - H_{t^-}$$

- Total discounted hedging error given by

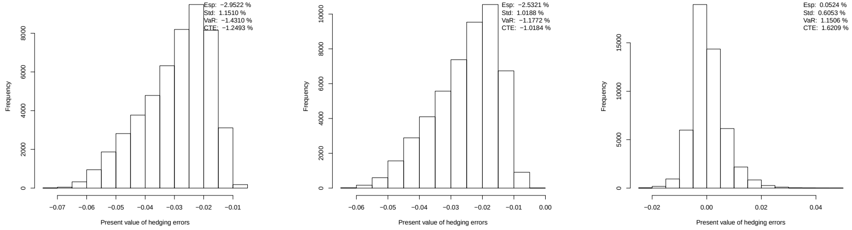
$$PV(HE) = \sum_{i=1}^{mT} e^{\frac{-ir}{m}} HE_i$$

if rebalancing occurs m times a year at equal time intervals.
- Used to assess the performance of the hedging strategy.

Assumptions

- 10-year maturity point-to-point EIA with $g = 0$ and $\varrho = 1$.
- Participation rate α chosen so that the price of the EIA is 1.
- Risk-free rate $r = 0.02$.
- Black-Scholes parameters: $\mu_{BS} = 0.0636$ and $\sigma_{BS} = 0.19$.
- Heston parameters: $\kappa = 5.1793$, $\theta = 0.0178$, $\sigma = 0.1309$, $v_0 = 0.0286$, $\rho = -0.7025$.
- Index prices follow Heston model with different volatility risk premia λ.

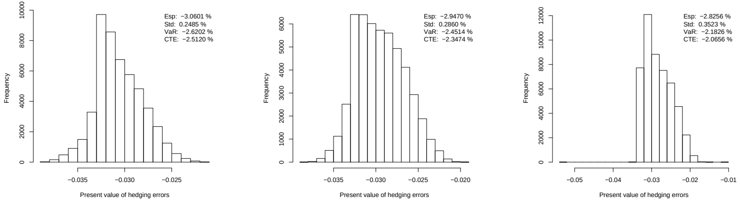
Black-Scholes Delta Hedging



(a) $\lambda = -1$ (b) $\lambda = 0$ (c) $\lambda = 2.62$

Figure: Present values of hedging errors resulting from a Black-Scholes delta hedging strategy for different values of λ, $\alpha = 0.5723$

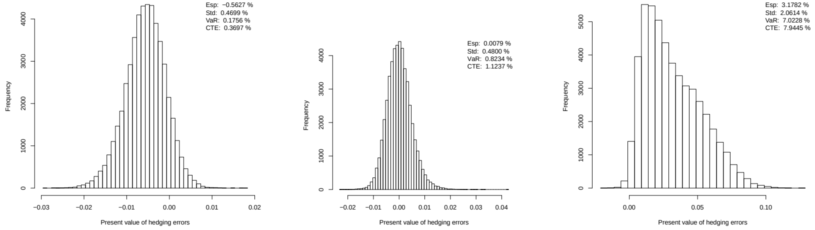
Black-Scholes Gamma Hedging



(a) $\lambda = -1$ (b) $\lambda = 0$ (c) $\lambda = 2.62$

Figure: Present values of hedging errors resulting from a Black-Scholes gamma hedging strategy for different values of λ, $\alpha = 0.5723$

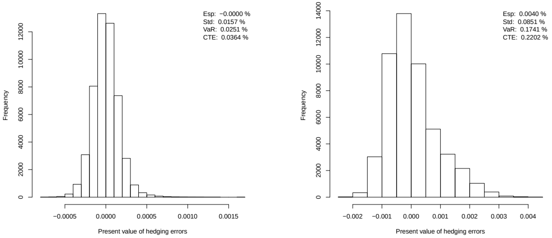
Heston Delta Hedging



(a) $\lambda = -1$ (b) $\lambda = 0$ (c) $\lambda = 2.62$

Figure: Present values of hedging errors resulting from a Heston delta hedging strategy for different values of λ, $\alpha = 0.6961$

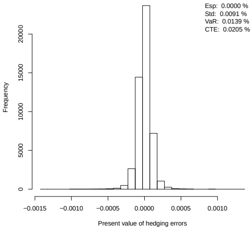
Heston Gamma Hedging



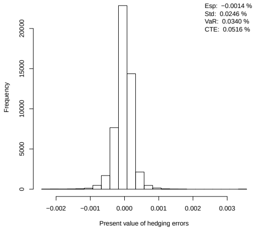
(a) $\lambda = 0$ (b) $\lambda = 2.62$

Figure: Present values of hedging errors resulting from a Heston gamma hedging strategy for different values of λ, $\alpha = 0.6961$

Heston Vega Hedging



(a) $\lambda = 0$



(b) $\lambda = 2.62$

Figure: Present values of hedging errors resulting from a Heston gamma hedging strategy for different values of λ, $\alpha = 0.6961$

Conclusion

- Stochastic volatility and volatility risk premium should be considered when hedging EIAs
- Future work:
  - Modify constant risk-free rate assumption
  - Analyze the effect of stochastic volatility on other designs
  - Consider transaction costs

Thank you for your attention

Boyle, Phelim P. and Eduardo S. Schwartz (1977), "Equilibrium prices of guarantees under equity-linked contracts." *The Journal of Risk and Insurance*, 44, 639–660.

Brennan, Michael J. and Eduardo S. Schwartz (1976), "The pricing of equity-linked life insurance policies with an asset value guarantee." *Journal of Financial Economics*, 3, 195–213.

Hardy, Mary (2003), *Investment Guarantees: Modeling and Risk Management for Equity-Linked Life Insurance*. John Wiley & Sons, Inc., Hoboken, New Jersey.

Heston, Steven L. (1993), "A closed-form solution for options with stochastic volatility with applications to bond and currency options." *The Review of Financial Studies*, 6, 327–343.

Lee, Hangsuck (2003), "Pricing equity-indexed annuities with path-dependent options." *Insurance: Mathematics and Economics*, 33, 667–690.

Lin, X. Sheldon and Ken Seng Tan (2003), "Valuation of equity-indexed annuities under stochastic interest rates." *North American Actuarial Journal*, 7, 7291.

Lin, X. Sheldon Lin, Ken Seng Tan, and Hailiang Yang (2009), "Pricing annuity guarantees under a regime-switching model." *North American Actuarial Journal*, 13, 316–338.

Tiong, Serena (2000), "Valuing equity-indexed annuities." *North American Actuarial Journal*, 4, 149–170.