On the Haezendonck-Goovaerts Risk Measure for Extreme Risks

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Abstract

In this paper we are concerned with the calculation of the Haezendonck-Goovaerts risk measure, which is defined via a convex Young function and a parameter $q \in (0,1)$ representing the confidence level. We mainly focus on the case in which the risk variable follows a distribution function from a max-domain of attraction. For this case, we restrict the Young function to be a power function and we derive exact asymptotics for the Haezendonck-Goovaerts risk measure as $q \uparrow 1$. As a subsidiary, we also consider the case with an exponentially distributed risk variable and a general Young function, and we obtain an analytical expression for the Haezendonck-Goovaerts risk measure.

Keywords: Asymptotics; Haezendonck-Goovaerts risk measure; Max-domain of attraction; Regular/rapid variation; Young function

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1 Introduction

Throughout the paper, let X be a real-valued random variable, representing a risk variable in loss-profit style, with a distribution function $F = 1 - \overline{F}$ on $\mathbb{R} = (-\infty, \infty)$. Let $\varphi(\cdot)$ be a non-negative and convex function on $[0, \infty)$ with $\varphi(0) = 0$, $\varphi(1) = 1$ and $\varphi(\infty) = \infty$. This function is called a normalized Young function, which, due to the convexity of $\varphi(\cdot)$, is continuous and strictly increasing. Recall that the Orlicz space associated with the Young function $\varphi(\cdot)$ is defined as

$$L^{\varphi} = \{X : \mathrm{E}\left[\varphi(cX)\right] < \infty \text{ for some } c > 0\}$$

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and the Orlicz heart as

$$L_0^\varphi = \{X : \mathrm{E}[\varphi(cX)] < \infty \text{ for all } c > 0\}.$$

It is easy to see that L^φ and L_0^φ coincide with each other if $\limsup_{x\uparrow\infty} \varphi(2x)/\varphi(x) < \infty$; see also page 77 of Rao and Ren (1991).

For a Young function $\varphi(\cdot)$ and a risk variable $X \in L_0^\varphi$, let $H_q[X, x]$ be the unique solution h to the equation

$$\mathrm{E}\left[\varphi\left(\frac{(X-x)_+}{h}\right)\right] = 1 - q, \qquad q \in (0,1), \tag{1.1}$$

if $\overline{F}(x) > 0$ and let $H_q[X, x] = 0$ otherwise, where and throughout the paper, denote by $Y_+ = Y1_{(Y\geq 0)} = Y \vee 0$ the positive part of a random variable Y and by 1_A the indicator of an event A. The existence and uniqueness of the solution h to equation (1.1) for the case $\overline{F}(x) > 0$ can be seen from the fact that the left-hand side of (1.1) is a continuous and strictly decreasing function of $h > 0$ with limits $+\infty$ as $h \downarrow 0$ and 0 as $h \uparrow \infty$. For $q \in (0, 1)$, the Haezendonck-Goovaerts risk measure for X is defined as

$$H_q[X] = \inf_{x\in\mathbb{R}} \left(x + H_q[X, x]\right). \tag{1.2}$$

This risk measure was first introduced by Haezendonck and Goovaerts (1982) and was named as the Haezendonck risk measure by Goovaerts et al. (2004). Based on a recent conversation with Bellini and Rosazza Gianin during the 15th International Congress on Insurance: Mathematics and Economics in Trieste, we think that it is more proper to call it the Haezendonck-Goovaerts risk measure in order to acknowledge the contribution of both authors in their seminal paper Haezendonck and Goovaerts (1982). This risk measure has recently been studied by Bellini and Rosazza Gianin (2008a, 2008b, 2011), Nam et al. (2011), Krätschmer and Zähle (2011), Goovaerts et al. (2011) and Ahn and Shyamalkumar (2011). We have followed the style of Bellini and Rosazza Gianin (2008a, 2008b) to define this risk measure. As pointed out by Bellini and Rosazza Gianin (2008a, 2011), the Haezendonck-Goovaerts risk measure $H_q[X]$ is a law invariant and coherent risk measure. We remark that, due to its very definition, an analytic expression for $H_q[X]$ is not possible in general.

The simplest, yet interesting, special case is when $\varphi(t) = t$ for $t \geq 0$. In this case,

$$H_q[X] = \inf_{x\in\mathbb{R}} \left(x + \frac{\mathrm{E}\left[(X-x)_+\right]}{1-q}\right) = F^{\leftarrow}(q) + \frac{\mathrm{E}[(X - F^{\leftarrow}(q))_+]}{1-q}, \tag{1.3}$$

where, and throughout the paper, $F^{\leftarrow}(q) = \inf\{x \in \mathbb{R} : F(x) \geq q\}$ denotes the inverse function of F, also called the quantile of F or the Value at Risk of X. Thus, the Haezendonck-Goovaerts risk measure is reduced to $\mathrm{CTE}_q[X]$, the well-known Conditional Tail Expectation of X.

The parameter q in the definition of the Haezendonck-Goovaerts risk measure vaguely represents a confidence level. This is demonstrated by the special case above corresponding to $\varphi(t) = t$ for $t \geq 0$. In the post financial crisis era, risk managers become more and more concerned with the tail area of risks due to the excessive prudence of nowadays regulatory framework.

The majority of this paper focuses on the asymptotic behavior of $H_q[X]$ as $q \uparrow 1$. For a risk variable X with a distribution function F on $\mathbb{R}$, denote by $\hat{x} = F^{\leftarrow}(1) \leq \infty$ its upper endpoint and by $\hat{p} = \Pr(X = \hat{x})$ the probability assigned to the upper endpoint. Note that $\hat{p} = 0$ holds automatically if $\hat{x} = \infty$ or if F is continuous at $\hat{x}$. Theorem 3.1 of Goovaerts et al. (2004) shows that $F^{\leftarrow}(q) \leq H_q[X] \leq \hat{x}$. Thus, if $0 < \hat{p} \leq 1$, then $H_q[X] = \hat{x}$ for all $1 - \hat{p} < q < 1$, while if $\hat{p} = 0$, then

$$\lim_{q \uparrow 1} H_q[X] = \hat{x}.$$

We only need to consider the latter case with $\hat{p} = 0$. When $\hat{x} = \infty$ we shall establish exact asymptotics for $H_q[X]$ diverging to ∞ as $q \uparrow 1$, while when $\hat{x} < \infty$ we shall establish exact asymptotics for $\hat{x} - H_q[X]$ decaying to 0 as $q \uparrow 1$. Hence, the notion of extreme value theory becomes relevant.

We shall assume that the risk variable X follows a distribution function F from the max-domain of attraction of an extreme value distribution function. Due to the complexity of the problem, we shall only consider a power Young function, $\varphi(t) = t^k$ for $k \geq 1$. We prove the following:

- If F is from the Fréchet max-domain of attraction, then $H_q[X] \sim c_1 F^{\leftarrow}(q)$;
- If F is from the Gumbel max-domain of attraction, then $H_q[X] \sim F^{\leftarrow}(1-c_2 q)$ provided $\hat{x} = \infty$ or $\hat{x} - H_q[X] \sim (\hat{x} - F^{\leftarrow}(1 - c_2 q))$ provided $\hat{x} < \infty$;
- If F is from the Weibull max-domain of attraction, then $\hat{x} - H_q[X] \sim c_3 (\hat{x} - F^{\leftarrow}(q))$.

In these assertions, the notation $\sim$ means that the quotient of both sides tends to 1 as $q \uparrow 1$ and the coefficients c_1, c_2 and c_3 are explicitly given.

As a subsidiary, we also consider the case with an exponentially distributed risk variable X and a general Young function $\varphi(\cdot)$. For this case, we obtain an analytical expression for the Haezendonck-Goovaerts risk measure $H_q[X]$.

However, at this stage we cannot extend the study to the case of both a generally distributed risk variable X and a general Young function $\varphi(\cdot)$.

The rest of this paper consists of six sections. In Section 2 we establish a general result for the Haezendonck-Goovaerts risk measure with a power Young function. This result forms the theoretical basis of our asymptotic analysis. Then after preparing some preliminaries on max-domains of attraction and regular/rapid variation in Section 3, we derive exact asymptotic formulas for the Haezendonck-Goovaerts risk measure with a power Young function for the Fréchet, Gumbel and Weibull cases in Sections 4, 5 and 6, respectively. For each case, numerical studies are also carried out to examine the accuracies of the asymptotic formulas. In Section 7 we consider the case of an exponentially distributed risk variable and a general Young function.

2 General discussions with a power Young function

Let X be a risk variable distributed by F with an upper endpoint $\hat{x} \leq \infty$. Assume a power Young function, $\varphi(t) = t^k$ for some $k \geq 1$. For this case, the Orlicz space and Orlicz heart coincide with each other, both equal to $\left\{X : \mathrm{E}[X_+^k] < \infty\right\}$.

As we mentioned before, if $k = 1$, then the Haezendonck-Goovaerts risk measure for X is equal to $\mathrm{CTE}_q[X]$ given by (1.3). Thus, we only consider $k > 1$ in the following theorem in which we analyze the Haezendonck-Goovaerts risk measure for a general risk variable X:

Theorem 2.1 *Let the Young function be $\varphi(t) = t^k$ for some $k > 1$ and let X be a risk variable with $\mathrm{E}[X_+^k] < \infty$, $\hat{x} \leq \infty$ and $\Pr(X = \hat{x}) = 0$. Then the Haezendonck-Goovaerts risk measure for X is equal to*

$$H_q[X] = x + \left(\frac{\mathrm{E}\left[(X-x)_+^k\right]}{1-q} \right)^{1/k}, \qquad q \in (0,1), \tag{2.1}$$

where $x = x(q) \in (-\infty, \hat{x})$ is the unique solution to the equation

$$\frac{\left(\mathrm{E}\left[(X-x)_+^{k-1}\right]\right)^k}{\left(\mathrm{E}\left[(X-x)_+^k\right]\right)^{k-1}} = 1 - q. \tag{2.2}$$

The proof of Theorem 2.1 relies on the following elementary result:

Lemma 2.1 *Consider the function $g(x) = \mathrm{E}\left[(X-x)_+^k\right]$ for $x \in \mathbb{R}$, where $k \geq 1$ is a constant and X is a random variable with $\mathrm{E}[X_+^k] < \infty$.*

(a) If $k > 1$, then $g(x)$ is continuously differentiable with

$$\frac{\mathrm{d}}{\mathrm{d}x} g(x) = -k\mathrm{E}\left[(X-x)_+^{k-1}\right], \qquad x \in \mathbb{R}; \tag{2.3}$$

(b) If $k = 1$, then $\frac{\mathrm{d}}{\mathrm{d}x} g_+(x) = -\overline{F}(x)$ and $\frac{\mathrm{d}}{\mathrm{d}x} g_-(x) = -\overline{F}(x-0)$ for each $x \in \mathbb{R}$.

Proof. Our derivation for $\frac{\mathrm{d}}{\mathrm{d}x}g_+(x)$ below is good for both $k>1$ and $k=1$. Observe that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}g_+(x) &= \lim_{\Delta x\downarrow 0}\frac{\mathrm{E}\left[(X-(x+\Delta x))_+^k\right]-\mathrm{E}\left[(X-x)_+^k\right]}{\Delta x}\\
&= \lim_{\Delta x\downarrow 0}\mathrm{E}\left[\frac{-(X-x)^k}{\Delta x}1_{(x<X\le x+\Delta x)}+\frac{(X-(x+\Delta x))^k-(X-x)^k}{\Delta x}1_{(X>x+\Delta x)}\right]\\
&= \lim_{\Delta x\downarrow 0}\mathrm{E}\left[I_1(\Delta x)+I_2(\Delta x)\right]. \qquad (2.4)
\end{aligned}$$

Clearly,

$$|I_1(\Delta x)|\le(\Delta x)^{k-1}1_{(x<X\le x+\Delta x)}.$$

Moreover, there is some ξ between $X-(x+\Delta x)$ and $X-x$ such that

$$I_2(\Delta x)=-k\xi^{k-1}1_{(X>x+\Delta x)}.$$

These estimates for $I_1(\Delta x)$ and $I_2(\Delta x)$ enable us to apply the dominated convergence theorem to interchange the order of the limit and expectation in (2.4). Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}g_+(x)=\mathrm{E}\left[\lim_{\Delta x\downarrow 0}I_1(\Delta x)+\lim_{\Delta x\downarrow 0}I_2(\Delta x)\right]=-k\mathrm{E}\left[(X-x)_+^{k-1}\right].$$

When $k=1$, this gives $\frac{\mathrm{d}}{\mathrm{d}x}g_+(x)=-\overline{F}(x)$.

For $\frac{\mathrm{d}}{\mathrm{d}x}g_-(x)$, we need to distinguish the cases $k>1$ and $k=1$. For $k>1$, going along the same lines as above we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}g_-(x)=-k\mathrm{E}\left[(X-x)_+^{k-1}\right].$$

For $k=1$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}g_-(x) &= \lim_{\Delta x\uparrow 0}\frac{\mathrm{E}\left[(X-(x+\Delta x))_+\right]-\mathrm{E}\left[(X-x)_+\right]}{\Delta x}\\
&= \lim_{\Delta x\uparrow 0}\mathrm{E}\left[\frac{X-(x+\Delta x)}{\Delta x}1_{(x+\Delta x<X<x)}-1_{(X\ge x)}\right].
\end{aligned}$$

Clearly, the first term after the expectation, denoted by $I_3(\Delta x)$, satisfies

$$|I_3(\Delta x)|\le 1_{(x+\Delta x<X<x)}.$$

Hence, by the dominated convergence theorem,

$$\frac{\mathrm{d}}{\mathrm{d}x}g_-(x)=\mathrm{E}\left[\lim_{\Delta x\uparrow 0}I_3(\Delta x)-1_{(X\ge x)}\right]=-\overline{F}(x-0).$$

Finally, for $k>1$, the expression for $\frac{\mathrm{d}}{\mathrm{d}x}g(x)$ given by (2.3) is obviously continuous in $x\in\mathbb{R}$. This ends the proof of Lemma 2.1. ■

Proof of Theorem 2.1. For $\varphi(t) = t^k$, it follows straightforwardly from (1.1) that

$$H_q[X, x] = \left(\frac{\mathrm{E}\left[(X - x)_+^k\right]}{1 - q} \right)^{1/k}.$$

By virtue of Minkowski's inequality, one can easily verify that $H_q[X, \cdot]$ is convex over $\mathbb{R}$ and is strictly convex over $(-\infty, \hat{x})$; see also Proposition 3 of Bellini and Rosazza Gianin (2011). Write

$$g^*(x) = x + \left(\frac{\mathrm{E}\left[(X - x)_+^k\right]}{1 - q} \right)^{1/k}, \qquad x \in \mathbb{R},$$

so that $H_q[X] = \inf_{x \in \mathbb{R}} g^*(x)$. The function $g^*(\cdot)$ inherits the convexity of $H_q[X, \cdot]$ and $g^*(x)$ diverges to $+\infty$ as $x \to \pm\infty$. Hence, its overall infimum is attainable. To obtain this infimum, we naturally consider the equation $\frac{\mathrm{d}}{\mathrm{d}x} g^*(x) = 0$. By Lemma 2.1(a),

$$\frac{\mathrm{d}}{\mathrm{d}x} g^*(x) = 1 - \frac{\left(\mathrm{E}\left[(X - x)_+^k\right]\right)^{(1/k)-1} \mathrm{E}\left[(X - x)_+^{k-1}\right]}{(1 - q)^{1/k}}. \tag{2.5}$$

Thus, the equation $\frac{\mathrm{d}}{\mathrm{d}x} g^*(x) = 0$ is equivalent to (2.2).

For every $q \in (0, 1)$, the existence of a solution $x \in (-\infty, \hat{x})$ to (2.2) can be verified as follows. The left-hand side of (2.2) is a continuous function of $x \in \mathbb{R}$. As $x \downarrow -\infty$, applying the dominated convergence theorem to both the numerator and denominator yields that

$$\frac{\left(\mathrm{E}\left[(X - x)_+^{k-1}\right]\right)^k}{\left(\mathrm{E}\left[(X - x)_+^k\right]\right)^{k-1}} = \frac{\left(\mathrm{E}\left[\frac{(X-x)_+^{k-1}}{(-x)^{k-1}}\right]\right)^k}{\left(\mathrm{E}\left[\frac{(X-x)_+^k}{(-x)^k}\right]\right)^{k-1}} \to 1;$$

while as $x \uparrow \hat{x}$, applying Hölder's inequality to the numerator yields that

$$\begin{aligned}
\frac{\left(\mathrm{E}\left[(X - x)_+^{k-1}\right]\right)^k}{\left(\mathrm{E}\left[(X - x)_+^k\right]\right)^{k-1}} &= \frac{\left(\mathrm{E}\left[(X - x)_+^{k-1} 1_{(X > x)}\right]\right)^k}{\left(\mathrm{E}\left[(X - x)_+^k\right]\right)^{k-1}} \\
&\leq \frac{\left(\mathrm{E}\left[(X - x)_+^k\right]\right)^{k-1} \mathrm{E}\left[1_{(X > x)}\right]}{\left(\mathrm{E}\left[(X - x)_+^k\right]\right)^{k-1}} \\
&= \overline{F}(x) \to 0.
\end{aligned} \tag{2.6}$$

Moreover, the uniqueness of the solution to equation (2.2), or, equivalently, to the equation $\frac{\mathrm{d}}{\mathrm{d}x} g^*(x) = 0$, is ensured by the strict convexity of the function $g^*(\cdot)$ on $(-\infty, \hat{x})$. This ends the proof of Theorem 2.1. ∎

When proceeding our asymptotic analysis we shall need the following lemma too:

Lemma 2.2 *Consider equation (2.2) in which $k \geq 1$ is a constant, X is a random variable with $\mathrm{E}[X_+^k] < \infty$, $\hat{x} \leq \infty$ and $\Pr(X = \hat{x}) = 0$, and $q \in (0, 1)$, $x \in (-\infty, \hat{x})$ are two deterministic variables. Then $q \uparrow 1$ if and only if $x \uparrow \hat{x}$.*

Proof. For $k = 1$, equation (2.2) is simplified to $\overline{F}(x) = 1 - q$. Thus, the equivalence of $q \uparrow 1$ and $x \uparrow \hat{x}$ is obvious.

Now consider $k > 1$ only. The derivation in (2.6) shows that the left-hand side of (2.2) is bounded by $\overline{F}(x)$. Thus, $1 - q \leq \overline{F}(x)$, from which we easily infer that $x \uparrow \hat{x}$ implies $q \uparrow 1$. Conversely, the proof of Theorem 2.1 shows that the function $g^*(\cdot)$ is strictly convex over $(-\infty, \hat{x})$. Thus, $\frac{\mathrm{d}}{\mathrm{d}x} g^*(x)$ given by (2.5) is strictly increasing in $x \in (-\infty, \hat{x})$, or, equivalently, the left-hand side of (2.2) is strictly decreasing in $x \in (-\infty, \hat{x})$. Thus, $q \uparrow 1$ must lead to $x \uparrow \hat{x}$. ■

3 Max-domains of attraction and regular variation

In this section we highlight some basic concepts in extreme value theory. Monographs on extreme value theory in the context of insurance and finance are Resnick (1987, 2007), Embrechts et al. (1997), McNeil et al. (2005) and Malevergne and Sornette (2006), among others. In this paper we follow the main methodology of Hashorva et al. (2010) and Asimit et al. (2011), who studied some insurance problems using extreme value theory.

A distribution function F on $\mathbb{R}$ is said to belong to the max-domain of attraction of an extreme value distribution function G, written as $F \in \mathrm{MDA}(G)$, if

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| F^n (c_n x + d_n) - G(x) \right| = 0$$

holds for some norming constants $c_n > 0$ and $d_n \in \mathbb{R}$, $n \in \mathbb{N} = \{0, 1, \ldots\}$. By the classical Fisher-Tippett theorem (see Fisher and Tippett (1928) and Gnedenko (1943)), only three choices for G are possible, namely the Fréchet, Gumbel and Weibull distributions, which we denote by Φ_γ, Λ and Ψ_γ, respectively, with $\gamma > 0$ indexing members of the Fréchet and Weibull max-domains of attraction.

The Fréchet distribution function is given by $\Phi_\gamma(x) = \exp\{-x^{-\gamma}\}$ for $x > 0$. A distribution function F belongs to $\mathrm{MDA}(\Phi_\gamma)$ if and only if its upper endpoint $\hat{x}$ is infinite and the relation

$$\lim_{x\uparrow\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\gamma}, \qquad y > 0, \tag{3.1}$$

holds; see Theorem 3.3.7 of Embrechts et al. (1997).

The standard Gumbel distribution function is given by $\Lambda(x) = \exp\{-\mathrm{e}^{-x}\}$ for $x \in \mathbb{R}$. A distribution function F with an upper endpoint $\hat{x} \leq \infty$ belongs to $\mathrm{MDA}(\Lambda)$ if and only if the relation

$$\lim_{x\uparrow\hat{x}} \frac{\overline{F}(x + ya(x))}{\overline{F}(x)} = \mathrm{e}^{-y}, \qquad y \in \mathbb{R}, \tag{3.2}$$

holds for some positive auxiliary function $a(\cdot)$ on $(-\infty, \hat{x})$. Recall that $a(\cdot)$ is unique up to asymptotic equivalence and a commonly-used choice for $a(x)$ is the mean excess function,

$a(x) = \mathrm{E}[X - x | X > x]$ for $x < \hat{x}$. Recall also that

$$\begin{cases} a(x) = o(x), & \text{if } \hat{x} = \infty, \\ a(x) = o(\hat{x} - x), & \text{if } \hat{x} < \infty. \end{cases} \tag{3.3}$$

See Resnick (1987) and Embrechts et al. (1997) for more details. The following representation theorem is useful; see Balkema and de Haan (1972), Proposition 1.4 of Resnick (1987) or relation (3.35) of Embrechts et al. (1997). For $F \in \mathrm{MDA}(\Lambda)$ with $\hat{x} \le \infty$, there is some $x_0 < \hat{x}$ such that

$$\overline{F}(x) = b(x) \exp\left\{ - \int_{x_0}^{x} \frac{1}{a(y)} \mathrm{d}y \right\}, \qquad x_0 < x < \hat{x}, \tag{3.4}$$

where $a(\cdot)$ is an auxiliary function, chosen to be positive and absolutely continuous with $\lim_{x \uparrow \hat{x}} \frac{\mathrm{d}}{\mathrm{d}x} a(x) = 0$, and $b(\cdot)$ is a positive measurable function with $\lim_{x \uparrow \hat{x}} b(x) = b > 0$.

The Weibull distribution function is given by $\Psi_\gamma(x) = \exp\{-|x|^\gamma\}$ for $x \le 0$. A distribution function F belongs to $\mathrm{MDA}(\Psi_\gamma)$ if and only if its upper endpoint $\hat{x}$ is finite and

$$\lim_{x \downarrow 0} \frac{\overline{F}(\hat{x} - xy)}{\overline{F}(\hat{x} - x)} = y^\gamma, \qquad y > 0; \tag{3.5}$$

see Theorem 3.3.12 of Embrechts et al. (1997).

The following elementary result might be known somewhere but we cannot suitably address a reference:

Lemma 3.1 *Let F on $\mathbb{R}$ belong to the max-domain of attraction of a non-degenerate distribution function. Then*

(a) $\overline{F}(x - 0) \sim \overline{F}(x)$ as $x \uparrow \hat{x}$;

(b) $\overline{F}(F^{\leftarrow}(q)) \sim 1 - q$ as $q \uparrow 1$.

Proof. (a) The result for the Gumbel case is known in Corollary 1.6 of Resnick (1987). The result for the Fréchet and Weibull cases easily follows from the equivalent conditions (3.1) and (3.5), respectively.

(b) This follows from the two-sided inequality $\overline{F}(F^{\leftarrow}(q)) \le 1 - q \le \overline{F}(F^{\leftarrow}(q) - 0)$ and the result in (a). ■

It is often convenient and useful to restate the equivalent conditions for the max-domains of attraction in terms of regular or rapid variation. A positive measurable function $r(\cdot)$ is said to be regularly varying at $x_0 = \pm 0$ or $\pm\infty$ with a regularity index $\alpha \in (-\infty, \infty)$, denoted by $r(\cdot) \in \mathcal{R}_\alpha(x_0)$, if

$$\lim_{x \to x_0} \frac{r(xy)}{r(x)} = y^\alpha, \qquad y > 0.$$

The class $\mathcal{R}_0(x_0)$ consists of functions slowly varying at x_0. Moreover, a positive measurable function $r(\cdot)$ is said to be rapidly varying at $x_0 = \pm 0$ or $\pm\infty$, denoted by $r(\cdot) \in \mathcal{R}_\infty(x_0)$ or $1/r(\cdot) \in \mathcal{R}_{-\infty}(x_0)$, if

$$\lim_{x\to x_0} \frac{r(xy)}{r(x)} = \begin{cases} \infty, & \text{for } y > 1, \\ 0, & \text{for } 0 < y < 1. \end{cases}$$

Relation (3.1) shows that $F \in \mathrm{MDA}(\Phi_\gamma)$ if and only if $\overline{F}(\cdot) \in \mathcal{R}_{-\gamma}(+\infty)$, while relation (3.5) shows that $F \in \mathrm{MDA}(\Psi_\gamma)$ if and only if $\overline{F}(\hat{x} - \cdot) \in \mathcal{R}_\gamma(+0)$. Furthermore, for $F \in \mathrm{MDA}(\Lambda)$, it easily follows from relations (3.2) and (3.3) that $\overline{F}(\cdot) \in \mathcal{R}_{-\infty}(+\infty)$ provided $\hat{x} = \infty$ or $\overline{F}(\hat{x} - \cdot) \in \mathcal{R}_\infty(+0)$ provided $\hat{x} < \infty$.

Recall Potter's bounds as summarized by Theorem 1.5.6 of Bingham et al. (1987):

Lemma 3.2 *Let $r(\cdot)$ be a function regularly varying at $x_0 = +0$ or $+\infty$ with index $\alpha \in (-\infty, \infty)$. It holds for arbitrary $0 < \varepsilon < 1$ and all x, y sufficiently close to x_0 that*

$$(1-\varepsilon)\left(\left(\frac{y}{x}\right)^{\alpha+\varepsilon} \wedge \left(\frac{y}{x}\right)^{\alpha-\varepsilon}\right) \le \frac{r(y)}{r(x)} \le (1+\varepsilon)\left(\left(\frac{y}{x}\right)^{\alpha+\varepsilon} \vee \left(\frac{y}{x}\right)^{\alpha-\varepsilon}\right).$$

The following lemma is motivated by Proposition 1.1 of Davis and Resnick (1988):

Lemma 3.3 *Let $F \in \mathrm{MDA}(\Lambda)$ with the representation given by (3.4). Then, for arbitrary $0 < \varepsilon < 1$, there is some $x_0 < \hat{x}$ such that, for all $x_0 < x < \hat{x}$ and all $y \ge 0$,*

$$\frac{\overline{F}(x + ya(x))}{\overline{F}(x)} \le (1+\varepsilon)(1+\varepsilon y)^{-1/\varepsilon}.$$

Proof. Since $\lim_{x\uparrow\hat{x}} \frac{\mathrm{d}}{\mathrm{d}x} a(x) = 0$, there is some $x_0 < \hat{x}$ such that the inequality

$$a(x + za(x)) - a(x) \le \varepsilon z a(x)$$

holds for all $x_0 < x < \hat{x}$ and all $z \ge 0$. It follows that

$$\frac{a(x)}{a(x+za(x))} \ge \frac{1}{1+\varepsilon z}.$$

Hence, for all $x_0 < x < \hat{x}$ and all $y \ge 0$,

$$\begin{aligned} \frac{\overline{F}(x + ya(x))}{\overline{F}(x)} &= \frac{b(x+ya(x))}{b(x)} \exp\left\{-\int_x^{x+ya(x)} \frac{1}{a(z)} \mathrm{d}z\right\} \\ &= \frac{b(x+ya(x))}{b(x)} \exp\left\{-\int_0^y \frac{a(x)}{a(x+za(x))} \mathrm{d}z\right\} \\ &\le (1+\varepsilon) \exp\left\{-\int_0^y \frac{1}{1+\varepsilon z} \mathrm{d}z\right\} \\ &= (1+\varepsilon)(1+\varepsilon y)^{-1/\varepsilon}. \end{aligned}$$

This proves Lemma 3.3. ■

4 The Fréchet case with a power Young function

Our asymptotic analysis in the next three sections is based on Theorem 2.1. In this section we consider the Fréchet case. As usual, denote by $\mathrm{B}(\cdot,\cdot)$ the beta function, namely,

$$\mathrm{B}(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x, \qquad a,b>0.$$

Theorem 4.1 *Let $\varphi(t)=t^k$ for some $k\geq 1$ and let $F\in \mathrm{MDA}(\Phi_\gamma)$ for some $\gamma>k$. Then, as $q\uparrow 1$,*

$$H_q[X]\sim \frac{\gamma\left(\gamma-k\right)^{(k/\gamma)-1}}{k^{(k-1)/\gamma}}\left(\mathrm{B}\left(\gamma-k,k\right)\right)^{1/\gamma}F^{\leftarrow}(q). \tag{4.1}$$

We first prepare an elementary result:

Lemma 4.1 *If $F\in \mathrm{MDA}(\Phi_\gamma)$ for some $\gamma>0$, then it holds for all $0<k<\gamma$ that*

$$\lim_{x\uparrow\infty}\frac{\mathrm{E}\left[(X-x)_+^k\right]}{x^k\overline{F}(x)}=k\mathrm{B}(\gamma-k,k). \tag{4.2}$$

Proof. Since $\overline{F}(\cdot)\in\mathcal{R}_{-\gamma}(+\infty)$, by Lemma 3.2, for arbitrary $0<\varepsilon<\gamma-k$, there is some $x_0>0$ such that, for all $x>x_0$ and all $y\geq 0$,

$$(1-\varepsilon)\left(\frac{x+y}{x}\right)^{-\gamma-\varepsilon}\leq\frac{\overline{F}(x+y)}{\overline{F}(x)}\leq(1+\varepsilon)\left(\frac{x+y}{x}\right)^{-\gamma+\varepsilon}.$$

By the second inequality above, it holds for all $x>x_0$ that

$$\begin{aligned}\frac{\mathrm{E}\left[(X-x)_+^k\right]}{\overline{F}(x)} &= \int_0^\infty\frac{\overline{F}(x+y)}{\overline{F}(x)}\mathrm{d}y^k\\ &\leq(1+\varepsilon)\int_0^\infty\left(\frac{x+y}{x}\right)^{-\gamma+\varepsilon}\mathrm{d}y^k\\ &=(1+\varepsilon)x^k\int_0^\infty(z+1)^{-\gamma+\varepsilon}\mathrm{d}z^k.\end{aligned}$$

By the arbitrariness of ε, it follows that

$$\limsup_{x\uparrow\infty}\frac{\mathrm{E}\left[(X-x)_+^k\right]}{x^k\overline{F}(x)}\leq\int_0^\infty(z+1)^{-\gamma}\mathrm{d}z^k.$$

In the same way we can establish the corresponding inequality for the lower limit. In addition, using the change of variables $u=(z+1)^{-1}$ we have

$$\int_0^\infty(z+1)^{-\gamma}\mathrm{d}z^k=k\int_0^1 u^{\gamma-k-1}\left(1-u\right)^{k-1}\mathrm{d}u=k\mathrm{B}(\gamma-k,k).$$

Thus, relation (4.2) holds. ■

Proof of Theorem 4.1. Recall Lemma 2.2, which shows that $q \uparrow 1$ if and only if $x \uparrow \hat{x}$. In the proof below we shall tacitly use this equivalence.

We distinguish the cases $k = 1$ and $k > 1$. For $k = 1$, applying Lemmas 4.1 and 3.1(b) to relation (1.3) we have

$$H_q[X] \sim F^{\leftarrow}(q) + \frac{F^{\leftarrow}(q)}{\gamma - 1} \frac{\overline{F}\left(F^{\leftarrow}(q)\right)}{1 - q} \sim \frac{\gamma}{\gamma - 1} F^{\leftarrow}(q).$$

This proves relation (4.1) for $k = 1$.

Now turn to the case $k > 1$. Starting from Theorem 2.1 we need to approximate the optimal value of x that solves equation (2.2). By Lemma 4.1,

$$\begin{aligned} 1 - q &= \frac{\left(\mathrm{E}\left[(X - x)_+^{k-1}\right]\right)^k}{\left(\mathrm{E}\left[(X - x)_+^{k}\right]\right)^{k-1}} \\ &\sim \frac{\left((k-1)\mathrm{B}(\gamma - k + 1, k - 1)\right)^k}{\left(k\mathrm{B}(\gamma - k, k)\right)^{k-1}} \overline{F}(x) \\ &= \frac{(\gamma - k)^k \mathrm{B}(\gamma - k, k)}{k^{k-1}} \overline{F}(x), \end{aligned}$$

or, equivalently,

$$\overline{F}(x) \sim \frac{k^{k-1}}{(\gamma - k)^k \mathrm{B}(\gamma - k, k)} (1 - q). \tag{4.3}$$

By Proposition 0.8(V) of Resnick (1987), it is easy to verify that $\overline{F}(\cdot) \in \mathcal{R}_{-\gamma}(+\infty)$ if and only if $F^{\leftarrow}(1 - \cdot) \in \mathcal{R}_{-1/\gamma}(+0)$. It follows from (4.3) that

$$x = F^{\leftarrow}\left(1 - \frac{(1 + o(1))k^{k-1}}{(\gamma - k)^k \mathrm{B}(\gamma - k, k)}(1 - q)\right) \sim \left(\frac{(\gamma - k)^k \mathrm{B}(\gamma - k, k)}{k^{k-1}}\right)^{1/\gamma} F^{\leftarrow}(q). \tag{4.4}$$

Now, substituting (4.2), (4.3) and (4.4) into (2.1) yields that

$$\begin{aligned} H_q[X] &= x + \left(\frac{\mathrm{E}\left[(X - x)_+^k\right]}{1 - q}\right)^{1/k} \\ &\sim x + \left(\frac{k\mathrm{B}(\gamma - k, k)}{1 - q} \cdot x^k \overline{F}(x)\right)^{1/k} \\ &\sim x + \left(\frac{k\mathrm{B}(\gamma - k, k)}{1 - q} \cdot \frac{k^{k-1}}{(\gamma - k)^k \mathrm{B}(\gamma - k, k)} (1 - q)\right)^{1/k} x \\ &= \frac{\gamma}{\gamma - k} x \\ &\sim \frac{\gamma\left(\gamma - k\right)^{(k/\gamma) - 1}}{k^{(k-1)/\gamma}} \left(\mathrm{B}\left(\gamma - k, k\right)\right)^{1/\gamma} F^{\leftarrow}(q). \end{aligned}$$

This proves relation (4.1) for $k > 1$. ∎

Let us use R to numerically examine the accuracy of the asymptotic relation (4.1). Assume that F is a Pareto distribution given by

$$F(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha}, \qquad x, \alpha, \theta > 0.$$

Thus, $F \in \mathrm{MDA}(\Phi_\gamma)$ with $\gamma = \alpha$. We use `uniroot` to find the root x to (2.2) and then compute (2.1) to get the exact value of the Haezendonck-Goovaerts risk measure $H_q[X]$. Moreover, we compute the asymptotic formula given by (4.1).

In both graphs below, we compare the asymptotic estimate to the exact value on the left and show their ratio on the right. For Graph 4.1, we set $k = 1.1$ and 1.2, $\alpha = 1.6$, and $\theta = 1$. Apparently, the ratio converges to 1 as $q \uparrow 1$. We also find that the accuracy improves as k decreases.

Graph 4.1 is inserted here.

Similarly, for Graph 4.2, we set $k = 1.1$, $\alpha = 1.5$ and 1.6, and $\theta = 1$. We find that the ratio converges to 1 as $q \uparrow 1$ and that the accuracy improves gradually as α decreases.

Graph 4.2 is inserted here.

5 The Gumbel case with a power Young function

Now we consider the Gumbel case. Note that, for a random variable X distributed by $F \in \mathrm{MDA}(\Lambda)$, the requirement $\mathrm{E}\left[X_+^k\right] < \infty$ for all $k \geq 1$ holds automatically since either $\overline{F} \in \mathcal{R}_{-\infty}(+\infty)$ or $\hat{x} < \infty$. As usual, denote by $\Gamma(\cdot)$ the gamma function, namely,

$$\Gamma(a) = \int_0^\infty x^{a-1} \mathrm{e}^{-x} \mathrm{d}x, \qquad a > 0.$$

Theorem 5.1 *Let $\varphi(t) = t^k$ for some $k \geq 1$ and let $F \in \mathrm{MDA}(\Lambda)$ with an upper endpoint $0 < \hat{x} \leq \infty$. Then, as $q \uparrow 1$,*

(i) when $\hat{x} = \infty$ we have

$$H_q[X] \sim F^{\leftarrow}\left(1 - \frac{k^k}{\Gamma(k+1)}(1-q)\right); \tag{5.1}$$

(ii) when $\hat{x} < \infty$ we have

$$\hat{x} - H_q[X] \sim \hat{x} - F^{\leftarrow}\left(1 - \frac{k^k}{\Gamma(k+1)}(1-q)\right). \tag{5.2}$$

To prove Theorem 5.1, we first prepare an elementary result:

Lemma 5.1 *If $F \in \mathrm{MDA}(\Lambda)$ with an upper endpoint $0 < \hat{x} \leq \infty$, then it holds for all $k > 0$ that*

$$\lim_{x \uparrow \hat{x}} \frac{\mathrm{E}\left[(X-x)_+^k\right]}{a^k(x)\overline{F}(x)} = \Gamma(k+1), \tag{5.3}$$

where $a(\cdot)$ is the auxiliary function appearing in the representation given by (3.4).

Proof. When $\hat{x} = \infty$, we have

$$\begin{aligned}\mathrm{E}\left[(X-x)_+^k\right] &= \int_0^\infty \overline{F}(x+y)\mathrm{d}y^k \\ &= a^k(x)\overline{F}(x)\int_0^\infty \frac{\overline{F}(x+za(x))}{\overline{F}(x)}\mathrm{d}z^k \\ &\sim a^k(x)\overline{F}(x)\int_0^\infty \mathrm{e}^{-z}\mathrm{d}z^k,\end{aligned}$$

where in the last step we applied the dominated convergence theorem justified by Lemma 3.3. Similarly, when $\hat{x} < \infty$, recalling (3.3) we have

$$\begin{aligned}\mathrm{E}\left[(X-x)_+^k\right] &= \int_0^{\hat{x}-x} \overline{F}(x+y)\mathrm{d}y^k \\ &= a^k(x)\overline{F}(x)\int_0^{(\hat{x}-x)/a(x)} \frac{\overline{F}(x+za(x))}{\overline{F}(x)}\mathrm{d}z^k \\ &\sim a^k(x)\overline{F}(x)\int_0^\infty \mathrm{e}^{-z}\mathrm{d}z^k.\end{aligned}$$

Thus, for both cases, relation (5.3) holds. ■

Proof of Theorem 5.1. In the proof below, we shall tacitly use the equivalence of $q \uparrow 1$ and $x \uparrow \hat{x}$, as shown by Lemma 2.2. The auxiliary function $a(\cdot)$ appearing below corresponds to the one in the representation given by (3.4).

(i) We distinguish the cases $k = 1$ and $k > 1$. For $k = 1$, applying Lemma 5.1 to relation (1.3) and then applying Lemma 3.1 and relation (3.3), we have

$$H_q[X] \sim F^{\leftarrow}(q) + a(F^{\leftarrow}(q))\frac{\overline{F}(F^{\leftarrow}(q))}{1-q} \sim F^{\leftarrow}(q).$$

This proves relation (5.1) for $k = 1$.

For $k > 1$, applying Lemma 5.1 to relation (2.2) leads to

$$1 - q = \frac{\left(\mathrm{E}\left[(X-x)_+^{k-1}\right]\right)^k}{\left(\mathrm{E}\left[(X-x)_+^k\right]\right)^{k-1}} \sim \frac{\left(\Gamma(k)a^{k-1}(x)\overline{F}(x)\right)^k}{\left(\Gamma(k+1)a^k(x)\overline{F}(x)\right)^{k-1}} = \frac{\Gamma(k+1)}{k^k}\overline{F}(x). \tag{5.4}$$

Then, substituting (5.3) and (5.4) to (2.1) yields that

$$H_q[X] - x = \left(\frac{\mathrm{E}\left[(X-x)_+^k\right]}{1-q}\right)^{1/k} \sim ka(x), \tag{5.5}$$

which implies that $H_q[X] \sim x$. By Proposition 0.8(V) of Resnick (1987), $\overline{F}(\cdot) \in \mathcal{R}_{-\infty}(+\infty)$ implies $F^{\leftarrow}(1-\cdot) \in \mathcal{R}_0(+0)$. By this and (5.4) we have

$$x = F^{\leftarrow}\left(1-(1+o(1))\frac{k^k}{\Gamma(k+1)}(1-q)\right) \sim F^{\leftarrow}\left(1-\frac{k^k}{\Gamma(k+1)}(1-q)\right). \tag{5.6}$$

This leads to relation (5.1) for $k > 1$.

(ii) As before, we still distinguish the cases $k = 1$ and $k > 1$. For $k = 1$, applying Lemma 5.1 to relation (1.3) and then applying Lemma 3.1 and relation (3.3), we have

$$\begin{aligned}\hat{x} - H_q[X] &= (\hat{x} - F^{\leftarrow}(q)) - \frac{\mathrm{E}[(X - F^{\leftarrow}(q))_+]}{1-q} \\ &\sim (\hat{x} - F^{\leftarrow}(q)) - a(F^{\leftarrow}(q))\frac{\overline{F}(F^{\leftarrow}(q))}{1-q} \\ &\sim \hat{x} - F^{\leftarrow}(q).\end{aligned}$$

This proves relation (5.2) for $k = 1$.

Next consider $k > 1$. For this case the relations in (5.4) still hold. Then, substituting (5.3) and (5.4) to (2.1) yields that

$$(\hat{x} - H_q[X]) - (\hat{x} - x) = -\left(\frac{\mathrm{E}\left[(X-x)_+^k\right]}{1-q}\right)^{1/k} \sim -ka(x), \tag{5.7}$$

which implies that $\hat{x} - H_q[X] \sim \hat{x} - x$. By Proposition 0.8(V) of Resnick (1987), $\overline{F}(\hat{x} - \cdot) \in \mathcal{R}_{\infty}(+0)$ implies $\hat{x} - F^{\leftarrow}(1-\cdot) \in \mathcal{R}_0(+0)$. By this and (5.4) we have

$$\hat{x} - x = \hat{x} - F^{\leftarrow}\left(1-(1+o(1))\frac{k^k}{\Gamma(k+1)}(1-q)\right) \sim \hat{x} - F^{\leftarrow}\left(1-\frac{k^k}{\Gamma(k+1)}(1-q)\right). \tag{5.8}$$

This proves relation (5.2) for $k > 1$. ∎

In the proof above, the asymptotics for x given by relation (5.6) can actually be changed to $x \sim F^{\leftarrow}(1-c(1-q))$ for any $c > 0$ because $F^{\leftarrow}(1-\cdot) \in \mathcal{R}_0(+0)$. Similarly, the asymptotics for $\hat{x} - x$ given by relation (5.8) can be changed to $\hat{x} - x \sim \hat{x} - F^{\leftarrow}(1-c(1-q))$ for any $c > 0$. However, the most rational choice for c in both places should be $c = k^k/\Gamma(k+1)$.

We would like to point out that relations (5.5) and (5.7) give second-order asymptotics for $H_q[X]$. They become more powerful than (5.1) and (5.2) provided that it is possible to get the exact value of x solving (2.2) or a good approximation for x.

Recall that a distribution function F belongs to the class $\mathcal{L}(\lambda)$ for some $\lambda > 0$ if $\hat{x} = \infty$ and

$$\lim_{x\uparrow\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = \mathrm{e}^{-\lambda y}, \qquad y \in \mathbb{R}. \tag{5.9}$$

The class $\mathcal{L}(\lambda)$ contains many well-known light-tailed distributions such as the exponential, gamma and inverse Gaussian distributions. Relation (5.9) directly shows that $F \in \mathrm{MDA}(\Lambda)$ with $a(\cdot) \equiv 1/\lambda$. Thus, by (5.5) we arrive at the following:

Corollary 5.1 *Let $\varphi(t) = t^k$ for some $k > 1$ and let $F \in \mathcal{L}(\lambda)$ for some $\lambda > 0$. Then*

$$\lim_{q\uparrow 1} (H_q[X] - x) = \frac{k}{\lambda},$$

where x is determined by (2.2) and satisfies $x \sim F^{\leftarrow}\left(1 - (1-q)\frac{k^k}{\Gamma(k+1)}\right)$.

Finally, we use R to numerically examine the accuracy of the asymptotics given by (5.5). Assume that F is a lognormal distribution given by

$$F(x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right), \qquad x > 0, -\infty < \mu < \infty, \sigma > 0,$$

where $\Phi(\cdot)$ denotes the standard normal distribution function. Note that

$$a(x) = \frac{\overline{\Phi}(\sigma^{-1}(\ln x - \mu))\sigma x}{\phi(\sigma^{-1}(\ln x - \mu))},$$

where ϕ is the standard normal density function; see, e.g. page 150 of Embrechts et al. (1997).

For Graph 5.1, we set $k = 1.5$ and 2, $\mu = 2$, and $\sigma = 0.5$. We compare the asymptotic estimate $x + ka(x)$ given by (5.5) to the exact value of $H_q[X]$ on the left and show their ratio on the right. Apparently, the ratio converges to 1 as $q \uparrow 1$. We find that the accuracy is particularly good when σ is smaller than 1.

Graph 5.1 is inserted here.

6 The Weibull case with a power Young function

In the last section of asymptotic analysis we consider the Weibull case. For this case the requirement $\mathrm{E}\left[X_+^k\right] < \infty$ for all $k \geq 1$ holds automatically since $\hat{x} < \infty$.

Theorem 6.1 *Let $\varphi(t) = t^k$ for some $k \geq 1$ and let $F \in \mathrm{MDA}(\Psi_\gamma)$ with $\gamma > 0$ and $0 < \hat{x} < \infty$. Then, as $q \uparrow 1$,*

$$\hat{x} - H_q[X] \sim \frac{\gamma}{\gamma + k}\left(\frac{k^{k-1}}{\mathrm{B}(\gamma+1, k)(\gamma+k)^k}\right)^{1/\gamma}(\hat{x} - F^{\leftarrow}(q)). \tag{6.1}$$

We first prepare an elementary result:

Lemma 6.1 *If $F \in \mathrm{MDA}(\Psi_\gamma)$ with $\gamma > 0$ and $0 < \hat{x} < \infty$, then it holds for all $k > 0$ that*

$$\lim_{x\uparrow\hat{x}} \frac{\mathrm{E}\left[(X-x)_+^k\right]}{(\hat{x}-x)^k\,\overline{F}(x)} = k\mathrm{B}(\gamma+1, k). \tag{6.2}$$

Proof. Similarly as in the proof of Lemma 5.1, for $x < \hat{x}$,

$$\mathrm{E}\left[(X-x)_+^k\right] = \int_0^{\hat{x}-x} \overline{F}(x+y)\mathrm{d}y^k = \overline{F}(x)\int_0^{\hat{x}-x} \frac{\overline{F}\left(\hat{x}-(\hat{x}-x-y)\right)}{\overline{F}\left(\hat{x}-(\hat{x}-x)\right)}\mathrm{d}y^k. \tag{6.3}$$

Since $\overline{F}\left(\hat{x}-\cdot\right) \in \mathcal{R}_\gamma(+0)$, by Lemma 3.2, for arbitrary $0<\varepsilon<1$, there is some $x_0 < \hat{x}$ such that, for all $x_0 < x < \hat{x}$ and all $0 < y < \hat{x}-x$,

$$(1-\varepsilon)\left(\frac{\hat{x}-x-y}{\hat{x}-x}\right)^{\gamma+\varepsilon} \le \frac{\overline{F}\left(\hat{x}-(\hat{x}-x-y)\right)}{\overline{F}\left(\hat{x}-(\hat{x}-x)\right)} \le (1+\varepsilon)\left(\frac{\hat{x}-x-y}{\hat{x}-x}\right)^{\gamma-\varepsilon}.$$

Applying the second inequality above to (6.3), it holds for all $x_0 < x < \hat{x}$ that

$$\begin{aligned}\mathrm{E}\left[(X-x)_+^k\right] &\le (1+\varepsilon)\overline{F}(x)\int_0^{\hat{x}-x}\left(\frac{\hat{x}-x-y}{\hat{x}-x}\right)^{\gamma-\varepsilon}\mathrm{d}y^k\\ &= (1+\varepsilon)\overline{F}(x)\left(\hat{x}-x\right)^k\int_0^1 (1-z)^{\gamma-\varepsilon}\,\mathrm{d}z^k.\end{aligned}$$

By the arbitrariness of ε, it follows that

$$\limsup_{x\uparrow\hat{x}} \frac{\mathrm{E}\left[(X-x)_+^k\right]}{\left(\hat{x}-x\right)^k\overline{F}(x)} \le \int_0^1 (1-z)^\gamma\,\mathrm{d}z^k.$$

A corresponding lower bound can be obtained similarly. Thus, relation (6.2) holds. ■

Proof of Theorem 6.1. In the proof below, we shall tacitly use the equivalence of $q \uparrow 1$ and $x \uparrow \hat{x}$, as shown by Lemma 2.2.

We distinguish the cases $k=1$ and $k>1$. For $k=1$, starting from relation (1.3) and then applying Lemmas 6.1 and 3.1, we have

$$\begin{aligned}\hat{x} - H_q[X] &= (\hat{x}-F^{\leftarrow}(q)) - \frac{\mathrm{E}[(X-F^{\leftarrow}(q))_+]}{1-q}\\ &\sim (\hat{x}-F^{\leftarrow}(q)) - \frac{1}{\gamma+1}\left(\hat{x}-F^{\leftarrow}(q)\right)\frac{\overline{F}(F^{\leftarrow}(q))}{1-q}\\ &\sim \frac{\gamma}{\gamma+1}(\hat{x}-F^{\leftarrow}(q)).\end{aligned}$$

This proves relation (6.1) for $k=1$.

Now consider $k>1$. Applying Lemma 6.1 to relation (2.2), we have

$$\begin{aligned}1-q &= \frac{\left(\mathrm{E}\left[(X-x)_+^{k-1}\right]\right)^k}{\left(\mathrm{E}\left[(X-x)_+^k\right]\right)^{k-1}}\\ &\sim \frac{\left((k-1)\mathrm{B}\left(\gamma+1,k-1\right)\left(\hat{x}-x\right)^{k-1}\overline{F}(x)\right)^k}{\left(k\mathrm{B}\left(\gamma+1,k\right)\left(\hat{x}-x\right)^k\overline{F}(x)\right)^{k-1}}\\ &= \mathrm{B}\left(\gamma+1,k\right)\frac{\left(\gamma+k\right)^k}{k^{k-1}}\overline{F}(x).\end{aligned} \tag{6.4}$$

Substituting (6.2) and (6.4) into (2.1) yields that

$$H_q[X] - x = \left(\frac{\mathrm{E}\left[(X-x)_+^k \right]}{1-q} \right)^{1/k} \sim \frac{k}{\gamma + k} \left(\hat{x} - x \right). \tag{6.5}$$

By Proposition 0.8(V) of Resnick (1987), $\overline{F}(\hat{x} - \cdot) \in \mathcal{R}_\gamma(+0)$ implies $\hat{x} - F^{\leftarrow}(1 - \cdot) \in \mathcal{R}_{1/\gamma}(+0)$. Thus, after rewriting (6.4) as

$$\overline{F}\left(\hat{x} - (\hat{x} - x) \right) \sim \frac{(1-q)\, k^{k-1}}{\mathrm{B}(\gamma+1, k)\, (\gamma+k)^k},$$

we see that

$$\begin{aligned} \hat{x} - x &= \hat{x} - F^{\leftarrow}\left(1 - \frac{(1+o(1))\, k^{k-1}}{\mathrm{B}(\gamma+1, k)\, (\gamma+k)^k} (1-q) \right) \\ &\sim \left(\frac{k^{k-1}}{\mathrm{B}(\gamma+1, k)\, (\gamma+k)^k} \right)^{1/\gamma} \left(\hat{x} - F^{\leftarrow}(q) \right). \end{aligned} \tag{6.6}$$

Finally, by (6.5) and (6.6),

$$\begin{aligned} \hat{x} - H_q[X] &= (\hat{x} - x) - (H_q[X] - x) \\ &\sim \frac{\gamma}{\gamma+k} (\hat{x} - x) \\ &\sim \frac{\gamma}{\gamma+k} \left(\frac{k^{k-1}}{\mathrm{B}(\gamma+1, k)\, (\gamma+k)^k} \right)^{1/\gamma} \left(\hat{x} - F^{\leftarrow}(q) \right). \end{aligned}$$

This proves relation (6.1) for $k > 1$. ■

Similarly as before, we use R to numerically examine the accuracy of relation (6.1). Assume that F is a beta distribution with probability density function given by

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{\mathrm{B}(a,b)}, \qquad 0 < x < 1, a, b > 0.$$

Thus, $F \in \mathrm{MDA}(\Psi_\gamma)$ with $\gamma = b$. We compare the asymptotic estimate for $\hat{x} - H_q[X]$ given by relation (6.1) to its exact value on the left and show their ratio on the right.

For Graph 6.1, we set $k = 3$ and 6, $a = 2$, and $b = 6$. Apparently, the ratio converges to 1 as $q \uparrow 1$. We also find that the varying value of k does not affect much the convergence rate.

Graph 6.1 is inserted here.

For Graph 6.2, we set $k = 3$, $a = 2$ and $b = 6$ or 10. The same as before, the ratio converges to 1 as $q \uparrow 1$. We also find that the accuracy slightly improves as b increases.

Graph 6.2 is inserted here.

7 The exponential case with a general Young function

In this section we consider the case in which the risk variable X is exponentially distributed and the Young function is general. We seek an analytical expression for the Haezendonck-Goovaerts risk measure.

Theorem 7.1 *Let $\varphi(\cdot)$ be a general Young function such that $\int_0^\infty \mathrm{e}^{-st}\mathrm{d}\varphi(t) < \infty$ for all $s > 0$ and let X follow an exponential distribution function with rate $\lambda > 0$, namely, $\overline{F}(x) = \mathrm{e}^{-\lambda x}$ for $x \geq 0$. Then*

$$H_q[X] = \frac{1}{\lambda} \ln \frac{\int_0^\infty \mathrm{e}^{-\lambda ht}\mathrm{d}\varphi(t)}{1-q} + h, \qquad q \in (0,1), \tag{7.1}$$

where $h \in (0,\infty)$ is the unique solution to the equation

$$\int_0^\infty \mathrm{e}^{-\lambda ht}\mathrm{d}\varphi(t) = \int_0^\infty t\mathrm{e}^{-\lambda ht}\mathrm{d}\varphi(t). \tag{7.2}$$

The condition on the Young function $\varphi(\cdot)$ ensures that the exponential risk variable X belongs to the Orlicz heart of $\varphi(\cdot)$. It is interesting to notice that the solution h to equation (7.2) is invariant with both q and x.

In order to prove Theorem 7.1, we first prepare an elementary result below:

Lemma 7.1 *Let $\varphi(\cdot)$ be a general Young function such that $\int_0^\infty \mathrm{e}^{-st}\mathrm{d}\varphi(t) < \infty$ for all $s > 0$. Then*

$$\lim_{s\downarrow 0} \frac{\int_0^\infty t\mathrm{e}^{-st}\mathrm{d}\varphi(t)}{\int_0^\infty \mathrm{e}^{-st}\mathrm{d}\varphi(t)} = \infty \qquad \text{and} \qquad \lim_{s\uparrow \infty} \frac{\int_0^\infty t\mathrm{e}^{-st}\mathrm{d}\varphi(t)}{\int_0^\infty \mathrm{e}^{-st}\mathrm{d}\varphi(t)} = 0. \tag{7.3}$$

Proof. With arbitrary $M > 0$ we derive

$$\begin{aligned}
\frac{\int_0^\infty t\mathrm{e}^{-st}\mathrm{d}\varphi(t)}{\int_0^\infty \mathrm{e}^{-st}\mathrm{d}\varphi(t)} &\geq \frac{\int_M^\infty t\mathrm{e}^{-st}\mathrm{d}\varphi(t)}{\int_0^M \mathrm{e}^{-st}\mathrm{d}\varphi(t) + \int_M^\infty \mathrm{e}^{-st}\mathrm{d}\varphi(t)} \\
&\geq \frac{M\int_M^\infty \mathrm{e}^{-st}\mathrm{d}\varphi(t)}{\varphi(M) + \int_M^\infty \mathrm{e}^{-st}\mathrm{d}\varphi(t)} \\
&= \frac{M}{\frac{\varphi(M)}{\int_M^\infty \mathrm{e}^{-st}\mathrm{d}\varphi(t)} + 1} \to M, \qquad \text{as } s \downarrow 0,
\end{aligned}$$

where the last step is due to the monotone convergence theorem and $\varphi(\infty) = \infty$. Hence, the first relation in (7.3) holds by the arbitrariness of M.

Notice that, with arbitrary $0 < \varepsilon < 1$,

$$\frac{\int_\varepsilon^\infty t\mathrm{e}^{-st}\mathrm{d}\varphi(t)}{\int_0^\varepsilon t\mathrm{e}^{-st}\mathrm{d}\varphi(t)} \leq \frac{\int_\varepsilon^\infty t\mathrm{e}^{-st}\mathrm{d}\varphi(t)}{\int_{\varepsilon/2}^\varepsilon t\mathrm{e}^{-st}\mathrm{d}\varphi(t)} \leq \frac{\int_\varepsilon^\infty t\mathrm{e}^{-st}\mathrm{d}\varphi(t)}{\frac{\varepsilon}{2}\mathrm{e}^{-s\varepsilon}\left(\varphi(\varepsilon) - \varphi\left(\frac{\varepsilon}{2}\right)\right)} \to 0, \qquad \text{as } s \uparrow \infty.$$

Similarly,

$$\lim_{s\uparrow\infty} \frac{\int_\varepsilon^\infty \mathrm{e}^{-st}\mathrm{d}\varphi(t)}{\int_0^\varepsilon \mathrm{e}^{-st}\mathrm{d}\varphi(t)} = 0.$$

It follows that

$$\limsup_{s\uparrow\infty} \frac{\int_0^\infty t\mathrm{e}^{-st}\mathrm{d}\varphi(t)}{\int_0^\infty \mathrm{e}^{-st}\mathrm{d}\varphi(t)} = \limsup_{s\uparrow\infty} \frac{\int_0^\varepsilon t\mathrm{e}^{-st}\mathrm{d}\varphi(t)}{\int_0^\varepsilon \mathrm{e}^{-st}\mathrm{d}\varphi(t)} \le \varepsilon.$$

Hence, the second relation in (7.3) holds by the arbitrariness of ε. ■

Proof of Theorem 7.1. First of all, let us modify the definition of the Haezendonck-Goovaerts risk measure in the following way. We think of x in (1.1) as $x = x_q[X,h]$, a function of h and q, and introduce $g^*(h) = x + h$. Then we can rewrite (1.2) as

$$H_q[X] = \inf_{0<h<\infty} g^*(h). \tag{7.4}$$

Our idea is to find the optimal value of h at which the infimum in (7.4) is attained. Relation (1.1) is rewritten as $\int_0^\infty \mathrm{e}^{-\lambda ht}\mathrm{d}\varphi(t) = (1-q)\,\mathrm{e}^{\lambda x}$, which implies that

$$x = \frac{1}{\lambda}\ln\frac{\int_0^\infty \mathrm{e}^{-\lambda ht}\mathrm{d}\varphi(t)}{1-q}. \tag{7.5}$$

Under the condition on $\varphi(\cdot)$, it is easy to see that $\int_0^\infty \mathrm{e}^{-\lambda ht}\mathrm{d}\varphi(t)$ is infinitely differentiable with respect to $h \in (0,\infty)$. By (7.5),

$$\frac{\mathrm{d}x}{\mathrm{d}h} = -\frac{\int_0^\infty t\mathrm{e}^{-\lambda ht}\mathrm{d}\varphi(t)}{\int_0^\infty \mathrm{e}^{-\lambda ht}\mathrm{d}\varphi(t)}.$$

Furthermore, by Hölder's inequality,

$$\frac{\mathrm{d}^2x}{\mathrm{d}h^2} = \lambda\frac{\int_0^\infty t^2\mathrm{e}^{-\lambda ht}\mathrm{d}\varphi(t)\int_0^\infty \mathrm{e}^{-\lambda ht}\mathrm{d}\varphi(t) - \left(\int_0^\infty t\mathrm{e}^{-\lambda ht}\mathrm{d}\varphi(t)\right)^2}{\left(\int_0^\infty \mathrm{e}^{-\lambda ht}\mathrm{d}\varphi(t)\right)^2} > 0,$$

where the strict inequality is due to the non-degeneracy of the Young function $\varphi(\cdot)$. This means that the function $x = x_q[X,h]$ and, hence, the function $g^*(h) = x + h$ are strictly convex over $h \in (0,\infty)$. Thus, the infimum in (7.4) is attained at h such that

$$\frac{\mathrm{d}}{\mathrm{d}h}g^*(h) = -\frac{\int_0^\infty t\mathrm{e}^{-\lambda ht}\mathrm{d}\varphi(t)}{\int_0^\infty \mathrm{e}^{-\lambda ht}\mathrm{d}\varphi(t)} + 1 = 0 \tag{7.6}$$

provided that this last equation admits a solution. Equations (7.6) and (7.2) are equivalent. The existence of a solution $h \in (0,\infty)$ to equation (7.6) is justified by Lemma 7.1, while the uniqueness of the solution h to equation (7.6) results from the strict convexity of the function $g^*(\cdot)$ over $(0,\infty)$. Finally, substituting this optimal value h and the expression for x given by (7.1) into (7.4), we obtain the desired expression for $H_q[X]$ in (7.1). ■

By Theorem 7.1, the calculation of $H_q[X]$ is reduced to solving equation (7.2). For a general Young function $\varphi(\cdot)$, it is not possible to solve this equation analytically. We consider the following special cases:

(i) Let $\varphi(t) = t^k$ for some $k \geq 1$. Then equation (7.2) has a unique positive solution $h = k/\lambda$. Thus,

$$H_q[X] = \frac{1}{\lambda} \ln \frac{\Gamma(k+1)}{(1-q)k^k} + \frac{k}{\lambda},$$

which is consistent with Corollary 5.1.

(ii) Let $\varphi(t) = \sum_{k=1}^{n} a_k t^k$ for some $n \in \mathbb{N}$ and some real-valued coefficients $a_1, \ldots, a_n$ fulfilling a certain condition such that $\varphi(\cdot)$ is a normalized Young function. Then equation (7.2) becomes

$$a_1(\lambda h)^n + \sum_{k=1}^{n-1} ((k+1)a_{k+1} - ka_k)\,\Gamma(k+1)(\lambda h)^{n-k} - na_n\Gamma(n+1) = 0. \qquad (7.7)$$

We can always numerically solve this polynomial equation (7.7) in R. For example, let $\varphi(t) = (2t^5 + 3t^4 - 2t^3 + 3t^2 + t)/7$, $q = 0.95$ and $\lambda = 1$. Equation (7.7) becomes

$$h^5 + 5h^4 - 24h^3 + 108h^2 - 48h - 1200 = 0.$$

We use `polyroot` to find its unique positive solution $h = 3.349538$. Plugging it in (7.1) gives $H_q[X] = 4.4558760$.

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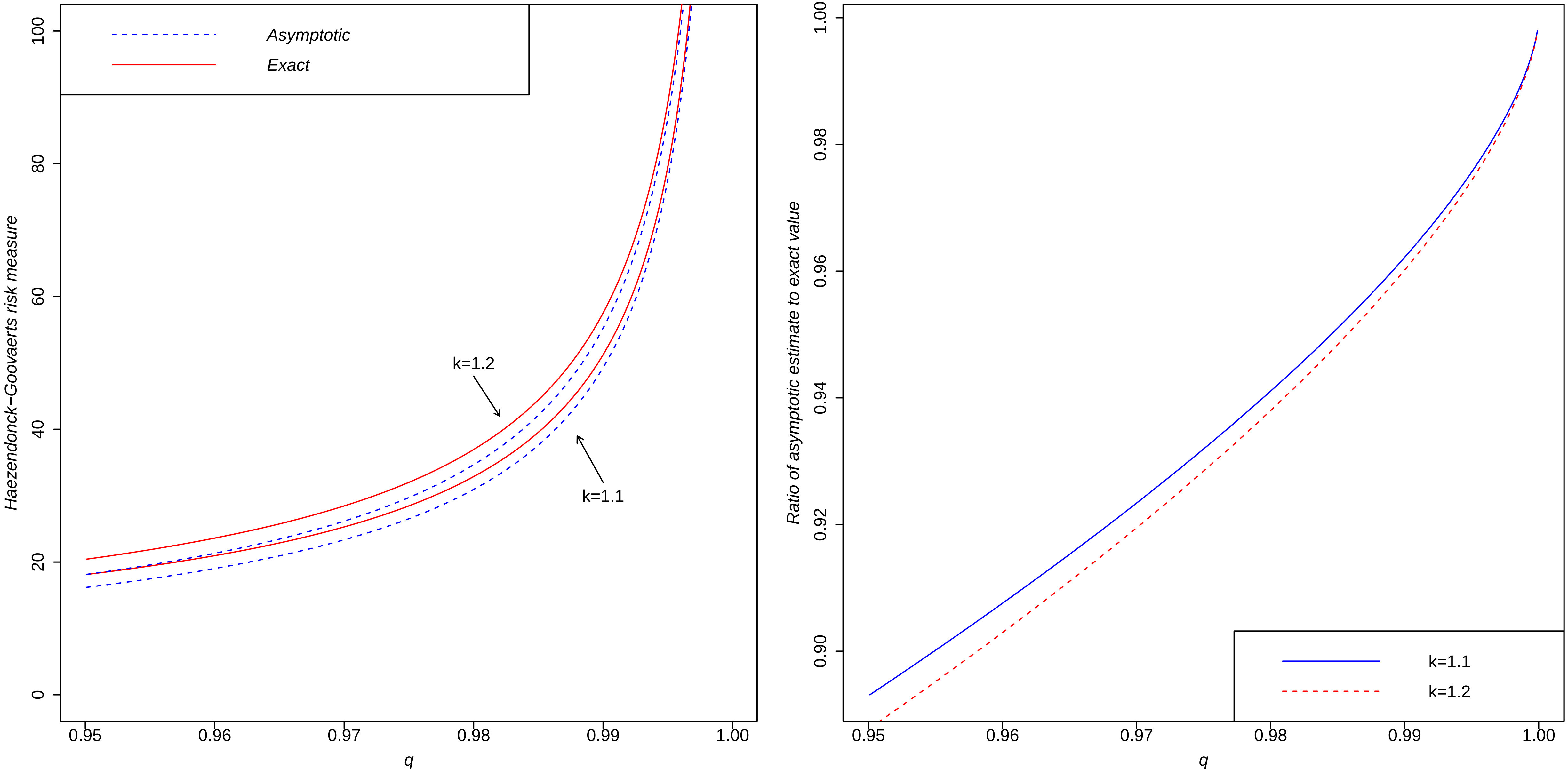
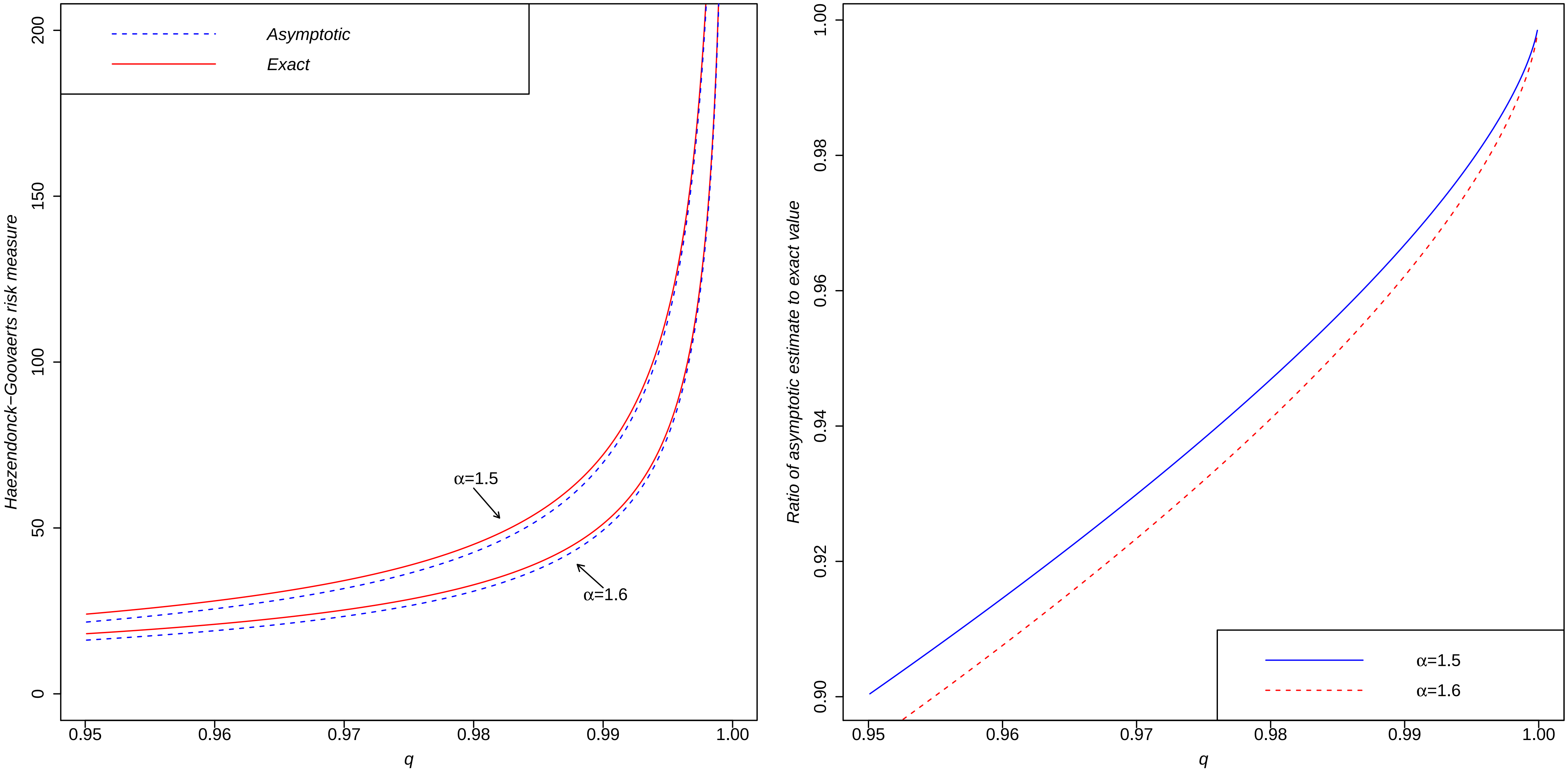
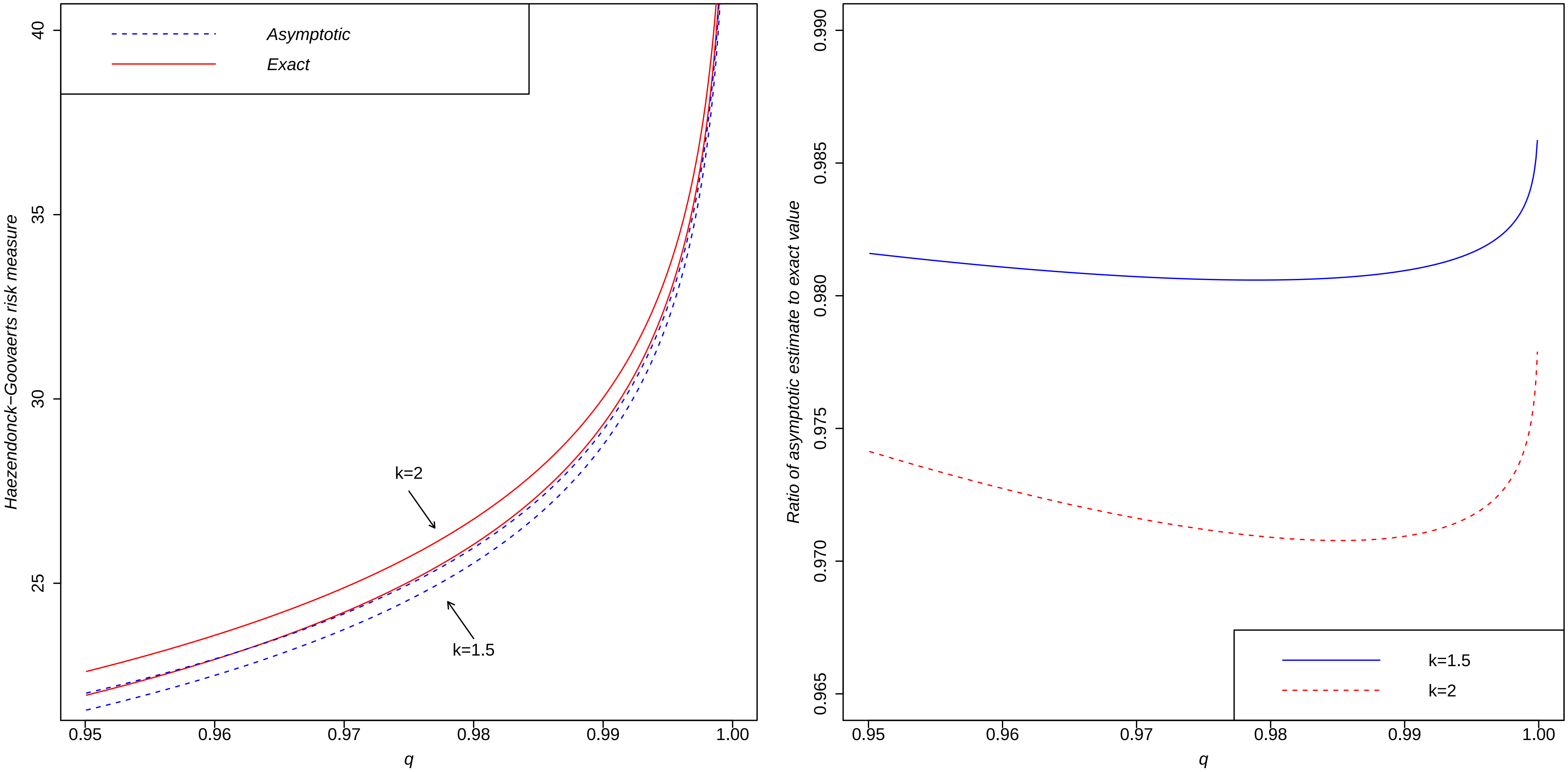


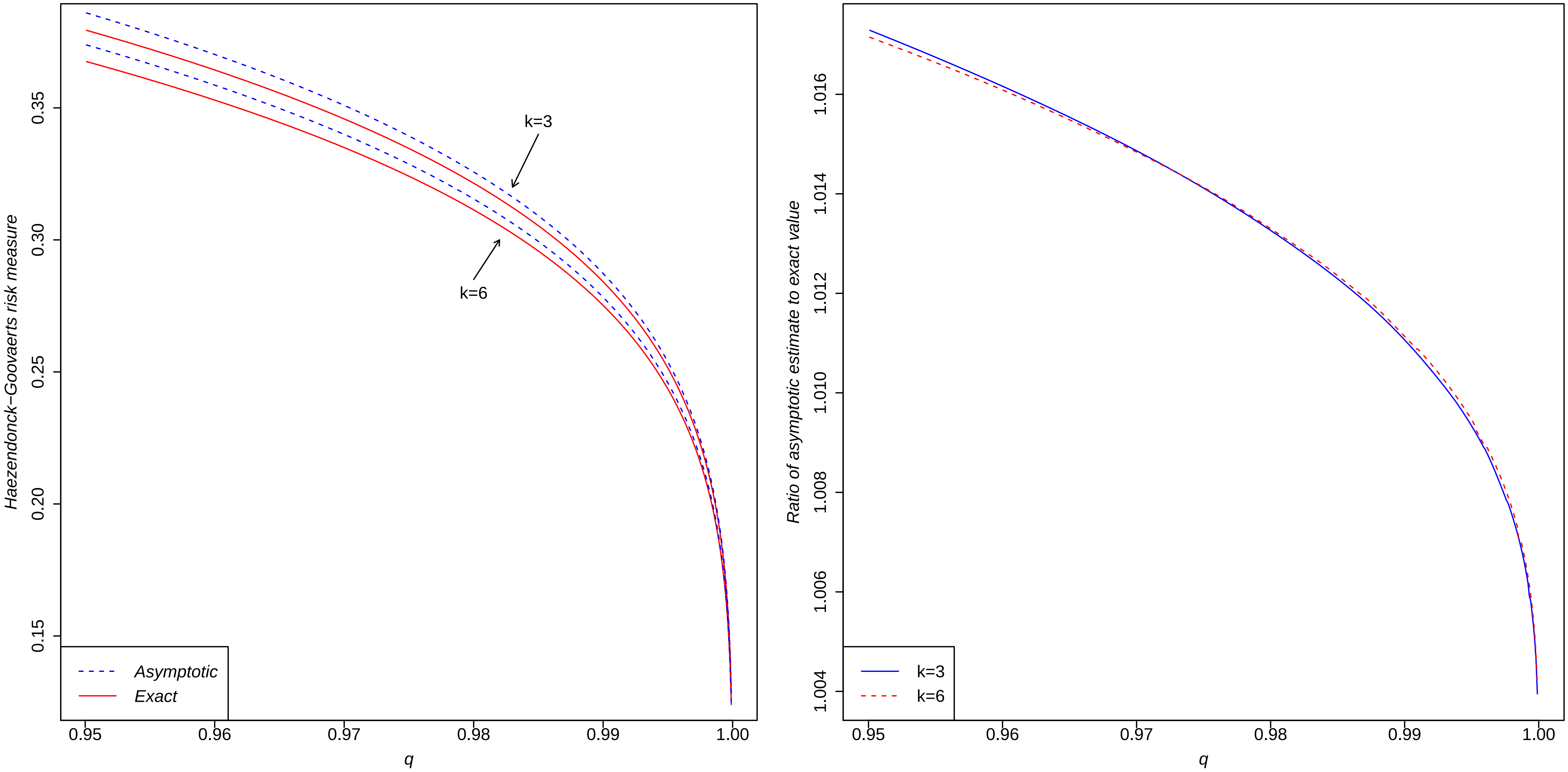
Graph 4.1 Young function with k=1.1 and 1.2, and Pareto distribution with α=1.6 and θ=1



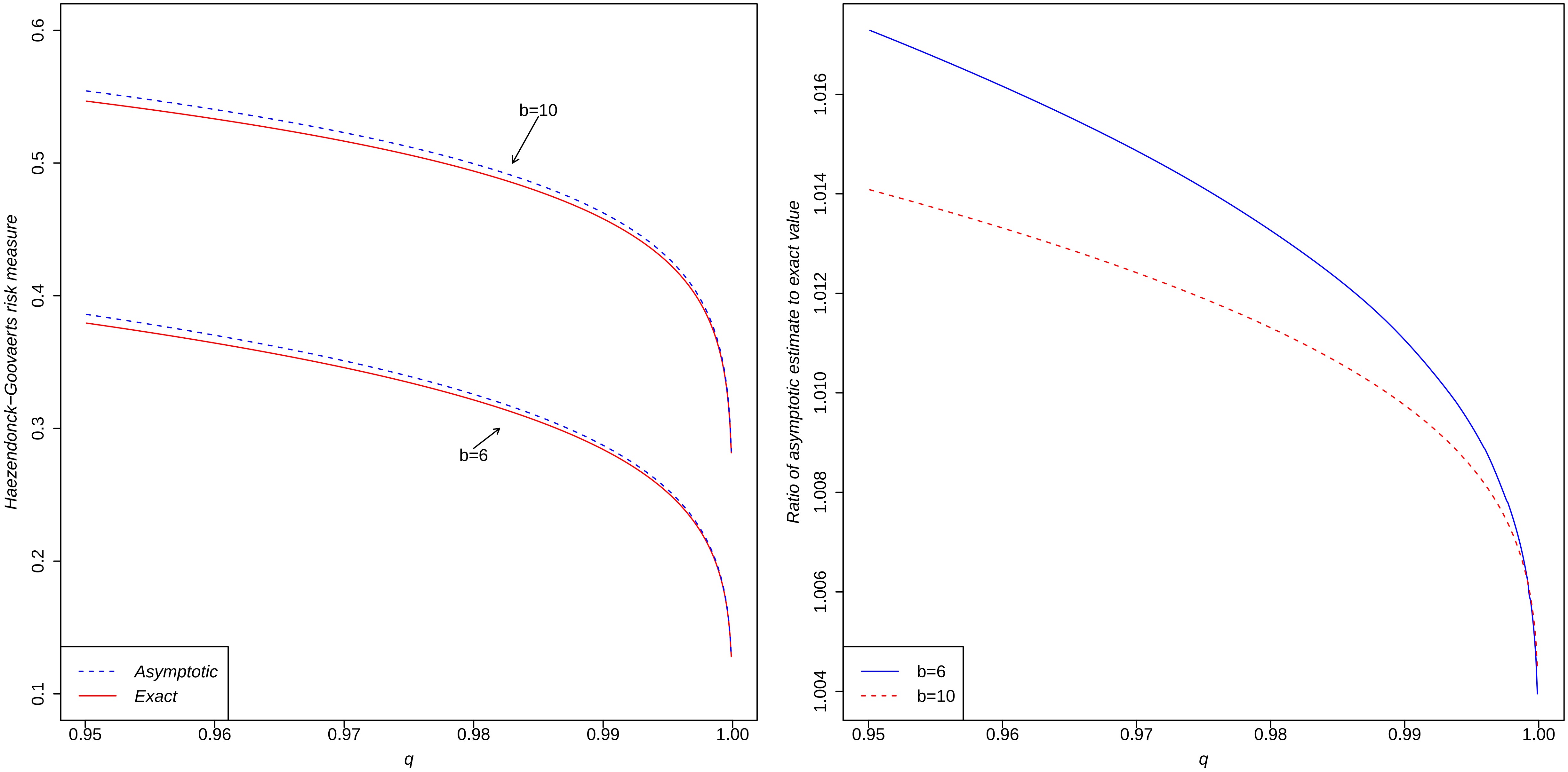
Graph 4.2 Young function with k=1.1, and Pareto distribution with α=1.5 and 1.6 and θ=1



Graph 5.1 Young function with k=1.5 and 2, and lognormal distribution with μ=2 and σ=0.5



Graph 6.1 Young function with k=3 and 6, and beta distribution with a=2 and b=6



Graph 6.2 Young function with k=3, and beta distribution with a=2 and b=6 and 10