Computing Tight Bounds for Insurance Payments with Nonlinear Risk

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Semidefinite Programming

\[(SDP) \quad \inf \quad C \bullet X \]
\[\text{s.t.} \quad A_i \bullet X \leq b_i \quad \forall i = 1, \ldots, n \]
\[X \succeq 0 \]

where \(A \bullet B := tr(A^TB)\)

- whole matrix \(X\) is a variable
- \(X \succeq 0\) means \(X\) is a semidefinite matrix (all eigenvalues of \(X\) are nonnegative)
- applications in engineering and finance
- any problem arriving at this form can be solved efficiently!
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**Motivation**

\[ \gamma \leq \mathbb{E}[\psi(x)] \leq \gamma \]

- when distribution is not known
- difficult to estimate the distribution, e.g. extreme events
- only some realizations of \( x \) exist \( \rightarrow \) moments can be estimated
- efficiently find the numerical bounds?

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- analytical form: $\psi(x)$ is (piecewise) linear:
- numerical ways with semidefinite programming (SDP):
- $\psi(x)$ is nonlinear:
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An example on mortgage payment

Recall

\[ P = A \left( \frac{1}{1 + r} + \cdots + \frac{1}{(1 + r)^t} \right) = A \frac{(1 + r)^t - 1}{r(1 + r)^t} \]

\[ f_{P,t}(r) \coloneqq A = \frac{Pr(1 + r)^t}{(1 + r)^t - 1} \]

- How worst can \( \mathbb{E}(f_{P,t}(r)) \) be? \( \rightarrow \) \( \sup \mathbb{E}[f_{P,t}(r)] \)?
- bound for stop-loss insurance? \( \rightarrow \) \( \sup \mathbb{E}[(f_{P,t}(r) - h)_+] \)
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Experiential Scenario

- loan $1000, 20 periodic payments in return
- floating rate (assume latest 2.5%, so $f_{1000,20}(0.025) = 51.32$.)
- 12-month Hong Kong Dollar Interest Rate (take 5 years, 10 years and 20 years samples)

<table>
<thead>
<tr>
<th>period</th>
<th>$\mu$</th>
<th>$\sigma$</th>
<th>$\sup \mathbb{E}[f_{1000,20}(r)]$</th>
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<tbody>
<tr>
<td>5-year</td>
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- consider a threshold $h$ in terms of quantifying $\sigma$ above $\mu$

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- interest rate (in a broad sense)
  - mortgage payments
    - $x$ is mortgage rate
  - annuity life insurance
    - $x$ is discounted rate
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- ... may be more!
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Q&A
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