SOCIETY OF ACTUARIES

Article from:

ARCH 2014.1 Proceedings

July 31-August 3, 2013

Option Pricing Without Tears:
Valuing Equity-Linked Death Benefits

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Let T_x denote the time-until-death random variable for a life aged x.
Let S(t) be the time-t price of a stock or mutual fund.
Consider death benefits that depend on the value of $S(T_x)$, i.e., consider $b(S(T_x))$ for some function b(.).
Examples:

$$b(s) = \text{Max}(s, K)$$

$$b(s) = (s - K)_+$$

Problem: Evaluate

$$E[e^{-\delta T_x} b(S(T_x))]$$

where the expectation is taken with respect to an *appropriate* probability distribution and
δ is a continuously compounded interest rate.

$$E[e^{-\delta T_x} b(S(T_x))]$$

$$= E[E[e^{-\delta T_x} b(S(T_x)) \mid T_x]]$$

$$= \int_0^\infty E[e^{-\delta t} b(S(t)) \mid T_x = t] \, f_{T_x}(t) \, dt$$

$$= \int_0^\infty E[e^{-\delta t} b(S(t))] \, f_{T_x}(t) \, dt$$

if T_x is independent of $\{S(t)\}$.

So we want to calculate

$$\int_0^\infty E[e^{-\delta t} b(S(t))] f_{T_x}(t)\,dt.$$

If

$$f_{T_x}(t) = \sum_j c_j\, f_{\tau_j}(t)\;,$$

then

$$\int_0^\infty E[e^{-\delta t} b(S(t))] f_{T_x}(t)\,dt$$

$$= \sum_j c_j \int_0^\infty E[e^{-\delta t} b(S(t))] f_{\tau_j}(t)\,dt$$

$$= \sum_j c_j\, E[e^{-\delta \tau_j} b(S(\tau_j))].$$

The time-until-death density function can be *approximated* by linear combinations of *exponential* density functions

$$f_{T_x}(t) \approx \sum_j c_j \times f_{\tau_j}(t) = \sum_j c_j \times \lambda_j e^{-\lambda_j t}.$$

Thus, our valuation problem becomes finding

$$E[e^{-\delta\tau} b(S(\tau))],$$

where τ is an *exponential* random variable independent of $\{S(t)\}$. It turns out to be an elementary calculus exercise for *geometric Brownian motion* $\{S(t)\}$.

Let $S(t) = S(0)e^{\mu t + \sigma Z(t)}$, $t \geq 0$,

where $\{Z(t)\}$ is a standard Brownian motion.

Let τ be an independent exponential random variable with mean $1/\lambda$. Then,

$$E[e^{-\delta\tau}\, b(S(\tau), \text{Max}\{S(t);\, 0 \leq t \leq \tau\})]$$

$$= \frac{2\lambda}{\sigma^2}\int_0^{\infty}\left[\int_{-\infty}^{m} b(S(0)e^{x}, S(0)e^{m})e^{-\alpha x}dx\right]e^{-(\beta-\alpha)m}dm$$

where $\alpha < 0$ and $\beta > 0$ are the solutions of

$$\tfrac{1}{2}\sigma^2 x^2 + \mu x - (\lambda + \delta) = 0.$$

$$E[e^{-\delta\tau}\, b(S(\tau), \mathrm{Max}\{S(t); 0 \le t \le \tau\})]$$

$$= \frac{2\lambda}{\sigma^2} \int_0^{\infty} \left[\int_{-\infty}^{m} b(S(0)e^{x}, S(0)e^{m})\, e^{-\alpha x} dx \right] e^{-(\beta-\alpha)m} dm.$$

Examples:

$b(s, u) = (s - K)_+$	call option
$b(s, u) = (K - s)_+$	put option
$b(s, u) = u$	high water mark payoff

Barrier Options

Assume $S(0) < B$, a barrier.

$b(s, u) = I(u < B) \times \pi(s)$

Up-and-out option

$b(s, u) = I(u \geq B) \times \pi(s)$

Up-and-in option

Useful for incorporating lapses or surrenders.

Assume $S(t) = S(0)e^{X(t)}$, $t \geq 0$, where

$$X(t) = \mu t + \sigma Z(t) + \sum_{j=1}^{N_{\nu}(t)} J_j - \sum_{k=1}^{N_{\omega}(t)} K_k$$

$$f_J(x) = \sum_{i=1}^{m} A_i v_i e^{-v_i x}, \qquad x > 0$$

$$f_K(x) = \sum_{i=1}^{n} B_i w_i e^{-w_i x}, \quad x > 0$$

$$\sum_{i=1}^{m} A_i = 1, \qquad \sum_{i=1}^{n} B_i = 1$$

$$S(t) = S(0)e^{X(t)}, \quad t \geq 0$$

Running maximum $M(t) := \text{Max}\{X(u); 0 \leq u \leq t\}$

Running minimum $m(t) := \text{Min}\{X(u); 0 \leq u \leq t\}$

Because $\{X(u)\}$ is a Levy process,

(i) $M(\tau)$ and $[X(\tau) - M(\tau)]$ are independent random variables,

(ii) $[X(\tau) - M(\tau)]$ has the same distribution as $m(\tau)$.

(i) is hard to prove; (ii) is easy.

In fact, (ii) is true for each fixed t.

$$
\begin{aligned}
X(t) - M(t) &= X(t) - \mathrm{Max}\{X(s);\ 0 \le s \le t\} \\
&= X(t) + \mathrm{Min}\{-X(s);\ 0 \le s \le t\} \\
&= \mathrm{Min}\{X(t) - X(s);\ 0 \le s \le t\} \\
&= \mathrm{Min}\{X(t - s);\ 0 \le s \le t\} \text{ in distribution} \\
&= \mathrm{Min}\{X(s);\ 0 \le s \le t\} \\
&= m(t)
\end{aligned}
$$

Running maximum $M(t) := \mathrm{Max}\{X(u); 0 \le u \le t\}$

Running minimum $m(t) := \mathrm{Min}\{X(u); 0 \le u \le t\}$

(i) $M(\tau)$ and $[X(\tau) - M(\tau)]$ are independent r.v.'s.

(ii) $[X(\tau) - M(\tau)]$ and $m(\tau)$ have the same distribution.

Then,

$$
\begin{aligned}
E[e^{zX(\tau)}] &= E[e^{z[X(\tau)-M(\tau)+M(\tau)]}] \\
&= E[e^{z[X(\tau)-M(\tau)]}] \times E[e^{zM(\tau)}] \\
&= E[e^{zm(\tau)}] \times E[e^{zM(\tau)}],
\end{aligned}
$$

which is a version of *Wiener-Hopf factorization*.

Assume

$$X(t) = \mu t + \sigma Z(t) + \sum_{j=1}^{N_\nu(t)} J_j - \sum_{k=1}^{N_\omega(t)} K_k$$

where

$$f_J(x) = \sum_{i=1}^{m} A_i v_i e^{-v_i x}, \qquad x > 0$$

$$f_K(x) = \sum_{i=1}^{n} B_i w_i e^{-w_i x}, \quad x > 0$$

Then, $E[e^{zX(t)}] = e^{t\Psi(z)}$ for each $t \geq 0$, with

$$\Psi(z) = \mu z + ½\sigma^2 z^2 + \nu \sum_{i=1}^{m} A_i \frac{z}{v_i - z} - \omega \sum_{i=1}^{n} B_i \frac{z}{w_i + z}$$

$$\Psi(z) = \mu z + ½\sigma^2 z^2 + \nu \sum_{i=1}^{m} A_i \frac{z}{v_i - z} - \omega \sum_{i=1}^{n} B_i \frac{z}{w_i + z}$$

can be extended by *analytic continuation*.

The moment-generating function of $X(\tau)$ is

$$E[e^{zX(\tau)}] = E[E[e^{zX(\tau)} | \tau]]$$

$$= E[e^{\Psi(z)\tau}]$$

$$= \frac{\lambda}{\lambda - \Psi(z)}.$$

The zeros of the RHS are the poles of $\Psi(z)$.

The poles of the RHS are the zeros of $\lambda - \Psi(z)$.

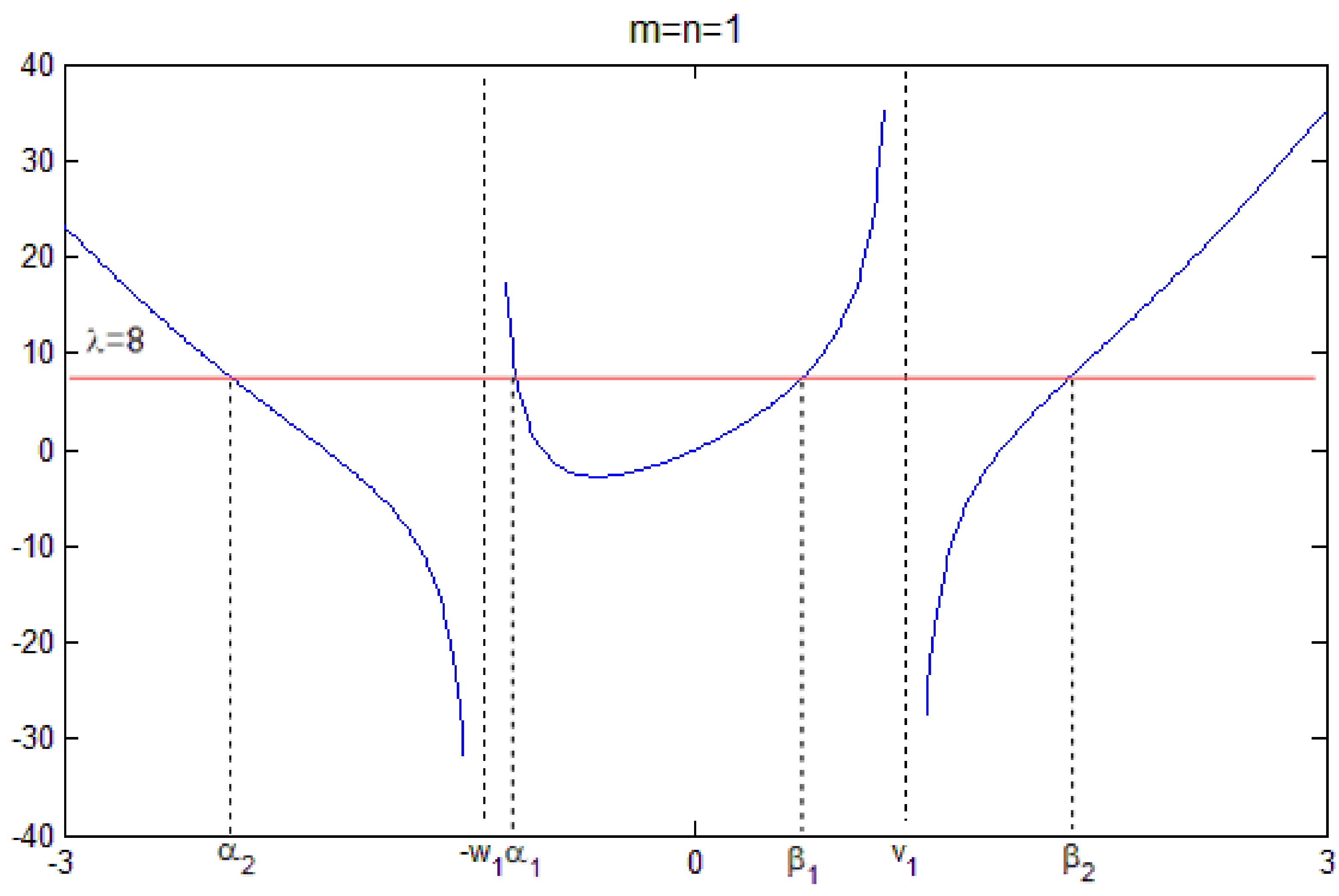
Label the parameters (the poles of $\Psi(z)$) such that

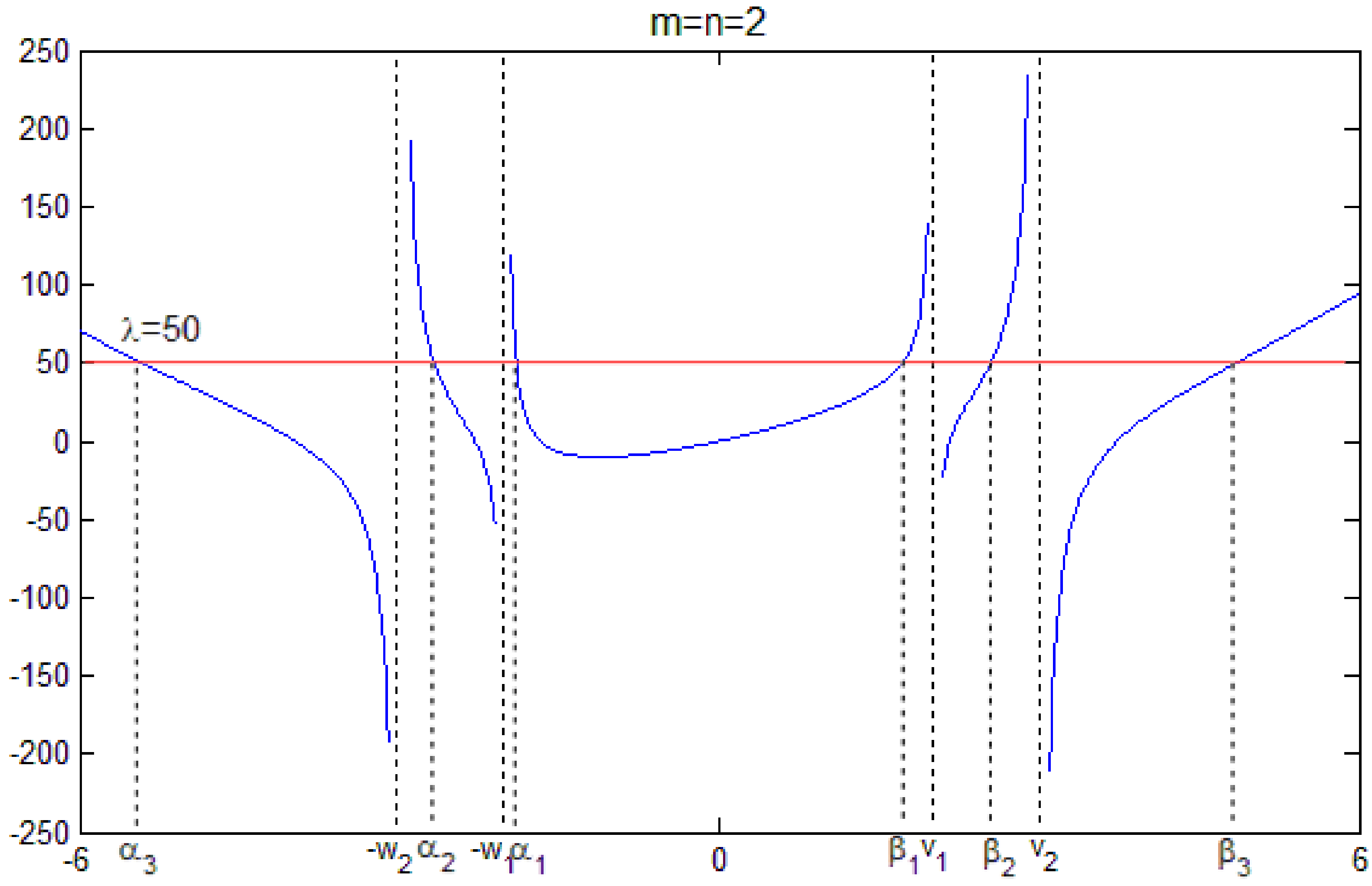
$$v_1 < v_2 < \ldots < v_m$$

$$w_1 < w_2 < \ldots < w_n$$

If the weights A's and B's are positive, then

$$-\infty < \alpha_{n+1} < -w_n < \ldots < -w_1 < \alpha_1 < 0 < \beta_1 < v_1 < \ldots < v_m < \beta_{m+1} < \infty$$


m=n=1
40
30
20
10
0
-10
-20
-30
-40
λ=8
-3
α2
-w1
α1
0
β1
v1
β2
3



m=n=2
250
200
150
100
50
0
-50
-100
-150
-200
-250
λ=50
-6
α3
-w2
α2
-w1
α1
0
β1
v1
β2
v2
β3
6

Label the parameters (the poles of $\Psi(z)$) such that

$$v_1 < v_2 < \ldots < v_m$$

$$w_1 < w_2 < \ldots < w_n$$

If the weights A's and B's are positive, then

$$-\infty < \alpha_{n+1} < -w_n < \ldots < -w_1 < \alpha_1 < 0 < \beta_1 < v_1 < \ldots < v_m < \beta_{m+1} < \infty$$

Wiener-Hopf: $E[e^{zX(\tau)}] = E[e^{zm(\tau)}] \times E[e^{zM(\tau)}]$.

For $z > 0$, $0 < E[e^{zm(\tau)}] \leq 1$. No positive zeros or poles.

For $z < 0$, $0 < E[e^{zM(\tau)}] \leq 1$. No negative zeros or poles.

$$-\infty < \alpha_{n+1} < -w_n < \ldots < -w_1 < \alpha_1 < 0 < \beta_1 < v_1 < \ldots < v_m < \beta_{m+1} < \infty$$

$$E[e^{zm(\tau)}] \propto \left(\prod_{j=1}^{n} (z + w_j) \right) \left(\prod_{j=1}^{n+1} \frac{1}{z - \alpha_j} \right)$$

$$E[e^{zM(\tau)}] \propto \left(\prod_{j=1}^{m} (z - v_j) \right) \left(\prod_{j=1}^{m+1} \frac{1}{z - \beta_j} \right)$$

$$E[e^{zm(\tau)}] = \left(\prod_{j=1}^{n} \frac{z + w_j}{w_j} \right) \left(\prod_{j=1}^{n+1} \frac{-\alpha_j}{z - \alpha_j} \right)$$

$$E[e^{zM(\tau)}] = \left(\prod_{j=1}^{m} \frac{v_j - z}{v_j} \right) \left(\prod_{j=1}^{m+1} \frac{\beta_j}{\beta_j - z} \right)$$

$$E[e^{zM(\tau)}]$$

$$= \left(\prod_{j=1}^{m} \frac{v_j - z}{v_j} \right) \left(\prod_{j=1}^{m+1} \frac{\beta_j}{\beta_j - z} \right)$$

$$= \sum_{k=1}^{m+1} \left(\prod_{j=1}^{m} \frac{v_j - \beta_k}{v_j} \right) \left(\prod_{j=1, j\neq k}^{m+1} \frac{\beta_j}{\beta_j - \beta_k} \right) \frac{\beta_k}{\beta_k - z}.$$

Thus, $f_{M(\tau)}(x) = \sum_{k=1}^{m+1} b_k e^{-\beta_k x}, \qquad x > 0,$

where $b_k = \left(\prod_{j=1}^{m} \frac{v_j - \beta_k}{v_j} \right) \left(\prod_{j=1, j\neq k}^{m+1} \frac{\beta_j}{\beta_j - \beta_k} \right) \beta_k$

$$E[e^{zm(\tau)}]$$

$$= \left(\prod_{j=1}^{n} \frac{z + w_j}{w_j} \right) \left(\prod_{j=1}^{n+1} \frac{-\alpha_j}{z - \alpha_j} \right)$$

$$= \sum_{k=1}^{n+1} \left(\prod_{j=1}^{n} \frac{\alpha_k + w_j}{w_j} \right) \left(\prod_{j=1, j \neq k}^{n+1} \frac{-\alpha_j}{\alpha_k - \alpha_j} \right) \frac{-\alpha_k}{z - \alpha_k}$$

Thus, $f_{m(\tau)}(x) = \sum_{k=1}^{n+1} a_k e^{-\alpha_k x}, \qquad x < 0,$

where $a_k = \left(\prod_{j=1}^{n} \frac{\alpha_k + w_j}{w_j} \right) \left(\prod_{j=1, j \neq k}^{n+1} \frac{-\alpha_j}{\alpha_k - \alpha_j} \right) (-\alpha_k)$

For $y \geq \max(x, 0)$,

$$
\begin{aligned}
& f_{X(\tau),\, M(\tau)}(x, y) \\
&= f_{M(\tau),\, X(\tau)-M(\tau)}(y, x-y) \times 1 \\
&= f_{M(\tau)}(y)\, f_{X(\tau)-M(\tau)}(x-y) \\
&= f_{M(\tau)}(y)\, f_{m(\tau)}(x-y) \\
&= \left(\sum_{k=1}^{m+1} b_k e^{-\beta_k y} \right) \left(\sum_{j=1}^{n+1} a_j e^{-\alpha_j (x-y)} \right) \\
&= \sum_{k=1}^{m+1} \sum_{j=1}^{n+1} a_j b_k \times e^{-\alpha_j x} e^{-(\beta_k - \alpha_j) y}
\end{aligned}
$$

Further Work

1. Use SOA CAE research grant to hire graduate students to estimate the parameters and to program the formulas.

2. Binomial tree version.

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