SOCIETY OF ACTUARIES

Article from:

ARCH 2013.1 Proceedings

August 1- 4, 2012

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First- and Second-order Asymptotics for the Tail Distortion Risk Measure of Extreme Risks

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June 21, 2012

Abstract

The tail distortion risk measure at level p was first introduced in Zhu and Li (2012), where the parameter $p \in (0, 1)$ indicates the confidence level. They established first-order asymptotics for this risk measure, as $p \uparrow 1$, for the Fréchet case. In this paper, we extend their work by establishing both first-order and second-order asymptotics for the Fréchet, Weibull and Gumbel cases. Numerical studies are also carried out to examine the accuracy of both asymptotics.

Keywords: Asymptotics; Extended regular variation; Max-domain of attraction; Regular variation; Second-order condition; Tail distortion risk measure

MSC: Primary 62P05; Secondary 60G70, 62E20

1 Introduction

Let X be a real-valued random variable, with distribution function $F = 1 - \overline{F}$ on $\mathbb{R} = (-\infty, \infty)$. Let $g(\cdot) : [0, 1] \to [0, 1]$ be a distortion function, namely, a nondecreasing function with $g(0) = 0$ and $g(1) = 1$. The distortion risk measure of X associated with $g(\cdot)$ is defined as

$$D_g[X] = \int_{-\infty}^{0} \left[g\left(\overline{F}(x)\right) - 1\right] \mathrm{d}x + \int_{0}^{\infty} g(\overline{F}(x))\mathrm{d}x.$$

This risk measure can be viewed as a distorted expectation of X in the sense of Choqet integral; see Denneberg (1994). The distortion risk measure was first introduced by Wang

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(1996) and has been applied to deciding insurance premiums, capital requirements and capital allocations. For more details, the reader is referred to Wang (2002), Valdez and Chernih (2003) and Tsanakas (2004), among others. Important properties of the distortion risk measure, such as coherence and second-order stochastic dominance, have been well studied; see, for example, Hardy and Wirch (2003) and Dhaene et al. (2006).

The distortion risk measure takes into account both loss and profit sides. Zhu and Li (2012) proposed the tail distortion risk measure, which focuses on the loss side only. With a distortion function $g(\cdot)$, introduce $g_p(\cdot)$ with $0 < p < 1$ as

$$g_p(u) = \begin{cases} g\left(\frac{u}{1-p}\right), & 0 \le u < 1-p, \\ 1, & 1-p \le u \le 1, \end{cases}$$

which again is a distortion function. Then the tail distortion risk measure at level p of a risk variable X is defined as

$$T_p[X] = \int_{-\infty}^{0} \left[g_p\left(\overline{F}(x)\right) - 1\right] \mathrm{d}x + \int_0^{\infty} g_p\left(\overline{F}(x)\right) \mathrm{d}x.$$

Note that this definition differs from the one given by Zhu and Li (2012), but they are identical when the risk variable X is continuous.

Clearly, if $g(\cdot)$ is concave, then $g_p(\cdot)$ is concave as well, which leads to the coherence of the tail distortion risk measure. For the special case with $g(x) = x$ and a continuous risk variable X, for $p > F(0)$ the tail distortion risk measure becomes the well-known the expected shortfall, $T_p[X] = \mathrm{E}\left[X | X > F^{\leftarrow}(p)\right]$, where, and throughout the paper,

$$F^{\leftarrow}(p) = \inf\{x : F(x) \ge p\}$$

is the value at risk of X or the quantile of F with the usual convention $\inf \varnothing = \infty$.

Define the function $U(\cdot)$ as the quantile function of $1/\overline{F}$, namely,

$$U(t) = \left(\frac{1}{\overline{F}}\right)^{\leftarrow}(t) = F^{\leftarrow}\left(1 - \frac{1}{t}\right).$$

Then we can rewrite $T_p[X]$ as

$$T_p[X] = \int_0^1 F^{\leftarrow}(1-q)\mathrm{d}g_p(q) = \int_0^1 U\left(\frac{1}{q(1-p)}\right) \mathrm{d}g(q). \tag{1.1}$$

Relation (1.1) will be the starting point of our derivation of asymptotics for $T_p[X]$.

In the definition of tail distortion risk measure, the parameter p clearly represents the confidence level. Nowadays, people are keen to measuring the tail area of a risk. The tail area of a risk corresponds to a large loss, whose occurrence is often companied by disastrous consequences. Therefore, in this paper we will study the asymptotic behavior of the tail distortion risk measure as $p \uparrow 1$. From this point of view, the use of extreme value theory becomes appropriate.

For a risk variable X with distribution function F, we denote by $\hat{x}$ its upper endpoint $F^{\leftarrow}(1)$. From (1.1), we see that $T_p[X] \uparrow \hat{x}$ if and only if $p \uparrow 1$. We will assume that $\Pr(X = \hat{x}) = 0$ because otherwise $T_p[X] = \hat{x}$ when p is close to 1. We will consider risk variables with distributions from the max-domain of attraction of an extreme value distribution. For the Fréchet and Gumbel cases with $\hat{x} = \infty$, we derive first-order asymptotics for $T_p[X]$ diverging to ∞ as $p \uparrow 1$. For the Weibull and Gumbel cases with $\hat{x} < \infty$, we derive first-order asymptotics for $\hat{x} - T_p[X]$ converging to 0 as $p \uparrow 1$. Furthermore, we derive second-order asymptotics for the three cases. In this paper, we follow the methodology of Mao and Hu (2012), who studied the second-order properties of Haezendonck–Goovaerts risk measure.

The rest of this paper consists of four sections. In section 2, we introduce the extreme value theory. In section 3, we derive first-order asymptotics for $T_p[X]$ for the Fréchet, Weibull and Gumbel cases separately. In section 4, we derive second-order asymptotics for $T_p[X]$ for all three cases. In section 5, we numerically examine the accuracy of first-order and second-order asymptotics.

2 Extreme Value Theory

A distribution function F is said to belong to the max-domain of attraction of a distribution function G, denoted by $F \in \mathrm{MDA}(G)$, if for a sequence of independent and identically distributed random variables, $\{X_n, n = 1, 2, \ldots\}$, with common distribution function F, the normalized block maximum $M_n = \max_{1 \le i \le n} X_i$, has a distribution weakly converging to G. The classical Fisher-Tippett theorem (see Fisher and Tippett (1928)) states that G has to be one of the Fréchet, Weibull and Gumbel distributions whose standard forms are given by, respectively,

$\Phi_\alpha(x) = \exp\{-x^{-\alpha}\}$ for $x > 0$,

$\Psi_\alpha(x) = \exp\{-|x|^\alpha\}$ for $x \le 0$, and

$\Lambda(x) = \exp\{-\mathrm{e}^{-x}\}$ for $x \in \mathbb{R}$.

It is usually convenient to characterize the three max-domains of attraction in terms of regular variation; see, for example, Embrechts et al. (1997). A positive measurable function $f(\cdot)$ is said to be regularly varying at ∞, with a regularity index $\alpha \in \mathbb{R}$, denoted by $f(\cdot) \in \mathcal{R}_\alpha$, if

$$\lim_{t\to\infty} \frac{f(tx)}{f(x)} = x^\alpha, \qquad x > 0. \tag{2.1}$$

For the extreme case of (2.1) with $\alpha = \pm\infty$, the function $f(\cdot)$ is said to be rapidly varying at ∞, denoted by $f(\cdot) \in \mathcal{R}_{\pm\infty}$.

Using the concept of extended regular variation leads to a unified description for the three max-domains of attraction. By definition, a positive measurable function $f(\cdot)$ is said to be extended regularly varying with index $\gamma \in \mathbb{R}$, denoted by $f(\cdot) \in \mathrm{ERV}_\gamma$, if there exists an auxiliary function $a(\cdot) > 0$ such that, for all $x > 0$,

$$\lim_{t\to\infty} \frac{f(tx) - f(t)}{a(t)} = \frac{x^\gamma - 1}{\gamma}, \tag{2.2}$$

where the right-hand side is interpreted as $\log x$ when $\gamma = 0$. We will use this usual convention and throughout the paper. The auxiliary function $a(\cdot)$ appearing in (2.2) is often chosen to be

$$a_0(t) = \begin{cases} \gamma f(t), & \gamma > 0, \\ -\gamma(f(\infty) - f(t)), & \gamma < 0, \\ f(t) - t^{-1}\int_0^t f(s)\mathrm{d}s, & \gamma = 0. \end{cases} \tag{2.3}$$

Note that for $\gamma = 0$, if $f(\infty) = \infty$ then $a_0(t) = o(f(t))$ as $t \to \infty$ while if $f(\infty) < \infty$ then $a_0(t) = o(f(\infty) - f(t))$ as $t \to \infty$. Theorem 1.1.6 of de Haan and Ferreira (2006) essentially proves that $F \in \mathrm{MDA}(G_\gamma)$ if and only if $U \in \mathrm{ERV}_\gamma$, where

$$G_\gamma = \begin{cases} \Phi_{1/\gamma}, & \gamma > 0, \\ \Psi_{-1/\gamma}, & \gamma < 0, \\ \Lambda, & \gamma = 0, \end{cases}$$

and U is the quantile function of $1/\overline{F}$ as defined above.

Very often we need not only to derive first-order asymptotics such as (2.1) and (2.2) but also to know their convergence speed. For the latter purpose, we introduce the concepts of second-order regular variation and second-order extended regular variation.

By definition, a positive measurable function $f(\cdot)$ is said to be second-order regularly varying with first-order index $\gamma \in \mathbb{R}$ and second-order index $\rho \le 0$, denoted by $f(\cdot) \in 2\mathrm{RV}_{\gamma,\rho}$, if there exists an auxiliary function $A(\cdot)$, which does not change sign eventually and converges to 0, such that, for all $x > 0$,

$$\lim_{t\to\infty} \frac{\frac{f(tx)}{f(t)} - x^\gamma}{A(t)} = x^\gamma \frac{x^\rho - 1}{\rho}.$$

More generally, a positive measurable function $f(\cdot)$ is said to be second-order extended regularly varying with first-order index $\gamma \in \mathbb{R}$ and second-order index $\rho \le 0$, denoted by $f(\cdot) \in 2\mathrm{ERV}_{\gamma,\rho}$, if there exist a first-order auxiliary function $a(\cdot)$ and a second-order auxiliary function $A(\cdot)$, where $A(\cdot)$ does not change sign eventually and converges to 0, such that, for all $x > 0$,

$$\lim_{t\to\infty} \frac{\frac{f(tx)-f(t)}{a(t)} - \frac{x^\gamma - 1}{\gamma}}{A(t)} = \frac{1}{\rho}\left(\frac{x^{\gamma+\rho} - 1}{\gamma+\rho} - \frac{x^\gamma - 1}{\gamma}\right), \qquad x > 0. \tag{2.4}$$

Some immediate explanations of (2.4) follow. The right-hand side of (2.4), written as $H_{\gamma,\rho}(x)$, is equal to

$$H_{\gamma,\rho}(x) = \begin{cases} \frac{1}{\rho}\left(\frac{x^{\gamma+\rho}-1}{\gamma+\rho} - \frac{x^\gamma-1}{\gamma}\right), & \rho \neq 0, \\ \frac{1}{\gamma}\left(x^\gamma \log x - \frac{x^\gamma-1}{\gamma}\right), & \rho = 0, \gamma \neq 0, \\ \frac{1}{2}(\log x)^2, & \gamma = \rho = 0. \end{cases}$$

In (2.4), $a(\cdot)$ is itself second-order regularly varying, $a(\cdot) \in 2\mathrm{RV}_{\gamma,\rho}$ with auxiliary function $A(\cdot) \in \mathcal{R}_\gamma$. The two auxiliary functions $a(\cdot)$ and $A(\cdot)$ are unique up to asymptotic equivalence. Moreover, according to Corollary 2.3.5 of de Haan and Ferreira (2006), if $a(\cdot)$ and $A(\cdot)$ are chosen appropriately, then $H_{\gamma,\rho}(x)$ is reduced to $\Psi_{\gamma,\rho}(x)$ as

$$\Psi_{\gamma,\rho}(x) = \begin{cases} \frac{x^{\gamma+\rho}-1}{\gamma+\rho}, & \rho < 0, \\ \frac{1}{\gamma} x^\gamma \log x, & \gamma \neq \rho = 0, \\ \frac{1}{2} (\log x)^2, & \gamma = \rho = 0. \end{cases} \tag{2.5}$$

3 First-order Asymptotics

Firstly, we derive first-order asymptotics for the tail distortion risk measure of a general risk variable X.

Theorem 3.1 *Let X be a random variable with $\Pr(X = \hat{x}) = 0$. Assume that $U \in \mathrm{ERV}_\gamma$ with $\gamma \in \mathbb{R}$ and the first-order auxiliary function $a_0(\cdot)$. In case $\gamma \geq 0$, assume that $\int_1^\infty g\left(x^{-1/(\gamma+\delta)}\right) \mathrm{d}x < \infty$ for some $\delta > 0$. Then when $\hat{x} = \infty$*

$$T_p[X] \sim F^{\leftarrow}(p) + a_0\left(\frac{1}{1-p}\right) \int_0^1 \frac{q^{-\gamma}-1}{\gamma} \mathrm{d}g(q); \tag{3.1}$$

when $\hat{x} < \infty$

$$\hat{x} - T_p[X] \sim \hat{x} - F^{\leftarrow}(p) - a_0\left(\frac{1}{1-p}\right) \int_0^1 \frac{q^{-\gamma}-1}{\gamma} \mathrm{d}g(q). \tag{3.2}$$

Proof. By Theorem B.2.18 in de Haan and Ferreira (2006), there exists some $0 < p_0 = p_0(\delta) < 1$ such that for $p_0 \leq p < 1$ and $0 < q < 1$,

$$\left| \frac{U\left(\frac{1}{q(1-p)}\right) - U\left(\frac{1}{1-p}\right)}{a_0\left(\frac{1}{1-p}\right)} - \frac{q^{-\gamma}-1}{\gamma} \right| \leq q^{-\gamma-\delta}.$$

Therefore, if the inequalities

$$\int_0^1 q^{-\gamma-\delta} \mathrm{d}g(q) < \infty \qquad \text{for some } \delta > 0, \tag{3.3}$$

and

$$\int_0^1 \frac{q^{-\gamma}-1}{\gamma} \mathrm{d}g(q) < \infty \tag{3.4}$$

hold, then applying the dominated convergence theorem we obtain the following: for $\hat{x} = \infty$,

$$\lim_{p \uparrow 1} \int_0^1 \frac{U\left(\frac{1}{q(1-p)}\right) - U\left(\frac{1}{1-p}\right)}{a_0\left(\frac{1}{1-p}\right)} \mathrm{d}g(q) = \int_0^1 \frac{q^{-\gamma}-1}{\gamma} \mathrm{d}g(q),$$

which gives the desired result (3.1); similarly, for $\hat{x} < \infty$,

$$\lim_{p\uparrow 1}\int_0^1 \frac{\hat{x} - U\left(\frac{1}{q(1-p)}\right) - \left(\hat{x} - U\left(\frac{1}{1-p}\right)\right)}{a_0\left(\frac{1}{1-p}\right)}\mathrm{d}g(q) = -\int_0^1 \frac{q^{-\gamma}-1}{\gamma}\mathrm{d}g(q),$$

which gives the desired result (3.2).

It remains to verify inequalities (3.3) and (3.4). We consider the three cases, $\gamma < 0$, $\gamma = 0$ and $\gamma > 0$, respectively. For $\gamma < 0$, inequality (3.3) with $\delta = -\gamma/2$ and inequality (3.4) hold obviously. For $\gamma = 0$, inequality (3.3) is verified by using the condition $\int_1^\infty g(x^{-1/\delta})\mathrm{d}x < \infty$ and integration by parts, while inequality (3.4) can be verified as

$$\int_0^1 \log q^{-1}\mathrm{d}g(q) \le \int_0^1 q^{-\delta}\mathrm{d}g(q) < \infty,$$

where the last step is due to (3.3). For $\gamma > 0$, inequality (3.3) is verified the same as before, while inequality (3.4) is verified as

$$\int_0^1 \frac{q^{-\gamma}-1}{\gamma}\mathrm{d}g(q) = \frac{1}{\gamma}\int_1^\infty g(x^{-1/\gamma})\mathrm{d}x \le \frac{1}{\gamma}\int_1^\infty g(x^{-1/(\gamma+\delta)})\mathrm{d}x < \infty.$$

This ends the proof. ■

Plugging the expression for $a_0(\cdot)$ given in (2.3) into Theorem 3.1, we immediately obtain first-order asymptotics for the three max-domains of attraction.

Corollary 3.1 *Under the conditions of Theorem 3.1, we have the following:*

(a) the Fréchet case: if $\gamma > 0$ *then* $T_p[X] \sim F^{\leftarrow}(p)\left(1 + \int_1^\infty g\left(x^{-1/\gamma}\right)\mathrm{d}x\right)$*;*

(b) the Weibull case: if $\gamma < 0$ *then* $\hat{x} - T_p[X] \sim (\hat{x} - F^{\leftarrow}(p))\left(1 - \int_0^1 g\left(x^{-1/\gamma}\right)\mathrm{d}x\right)$*;*

(c) the Gumbel case: if $\gamma = 0$ *and* $\hat{x} = \infty$ *then* $T_p[X] \sim F^{\leftarrow}(p)$*, while if* $\gamma = 0$ *and* $\hat{x} < \infty$ *then* $\hat{x} - T_p[X] \sim \hat{x} - F^{\leftarrow}(p)$*.*

The result for the Fréchet case coincides with the one earlier obtained by Zhu and Li (2012).

4 Second-order Asymptotics

To derive second-order asymptotics, similar as in the proof of Theorem 3.1, we need a uniform inequality for the second-order extended regularly varying function, which is a restatement of Theorem B.3.10 of de Haan and Ferreira (2006).

Lemma 4.1 *Suppose $f(\cdot) \in 2\mathrm{ERV}_{\gamma,\rho}$ with $\gamma \in \mathbb{R}$, $\rho \le 0$, first-order auxiliary function $a(\cdot)$ and second-order auxiliary function $A(\cdot)$. Then there exist functions $a_0(\cdot)$ and $A_0(\cdot)$ such that for every $\varepsilon, \delta > 0$, some $t_0 = t_0(\varepsilon, \delta)$ and all $t, tx \ge t_0$,*

$$\left| \frac{\frac{f(tx)-f(t)}{a_0(t)} - \frac{x^\gamma - 1}{\gamma}}{A_0(t)} - \Psi_{\gamma,\rho}(x) \right| \le \varepsilon \max\left\{x^{\gamma+\rho+\delta}, x^{\gamma+\rho-\delta}\right\},$$

where $\Psi_{\gamma,\rho}(\cdot)$ is defined in (2.5).

The next lemma establishes the relation between 2RV and 2ERV:

Lemma 4.2 *Let X be a random variable and let $\rho \le 0$.*

(a) When $\gamma > 0$, if $U \in \mathrm{ERV}_\gamma$ with auxiliary function $a_0(\cdot)$ and $U \in 2\mathrm{RV}_{\gamma,\rho}$ with auxiliary function $A(\cdot)$, then $U \in 2\mathrm{ERV}_{\gamma,\rho}$;

(b) When $\gamma < 0$, if $U \in \mathrm{ERV}_\gamma$ with auxiliary function $a_0(\cdot)$ and $\hat{x} - U \in 2\mathrm{RV}_{\gamma,\rho}$ with auxiliary function $A(\cdot)$, then $\hat{x} - U \in 2\mathrm{ERV}_{\gamma,\rho}$.

Proof. (a) In this case, $a_0(t) = \gamma U(t)$. We have

$$\begin{aligned} \frac{\frac{U(tx)-U(t)}{a_0(t)} - \frac{x^\gamma - 1}{\gamma}}{A(t)} &= \frac{U(t)\frac{\frac{U(tx)}{U(t)} - 1}{\gamma U(t)} - \frac{x^\gamma - 1}{\gamma}}{A(t)} \\ &= \frac{\frac{U(tx)}{U(t)} - x^\gamma}{\gamma A(t)} \\ &\to \frac{x^\gamma}{\gamma}\frac{x^\rho - 1}{\rho}, \qquad \text{as } t \to \infty. \end{aligned}$$

Thus, $U \in 2\mathrm{ERV}_{\gamma,\rho}$ by definition.

(b) In this case, $a_0(t) = -\gamma(\hat{x} - U(t))$. Similarly as above, we can prove that $\hat{x} - U \in 2\mathrm{ERV}_{\gamma,\rho}$ with the same limit function. ■

We conclude that, under the conditions of Lemma 4.2 and for $\gamma \neq 0$, the limit function $H_{\gamma,\rho}(x)$ in (2.4) is $H_{\gamma,\rho}(x) = \frac{x^\gamma}{\gamma}\frac{x^\rho - 1}{\rho}$. Now we derive second-order asymptotics for the tail distortion risk measure of a general variable X.

Theorem 4.1 *Let X be a random variable with $\Pr(X = \hat{x}) = 0$. Assume that $U \in 2\mathrm{ERV}_{\gamma,\rho}$ with $\gamma \neq 0$, $\rho \le 0$, the first-order auxiliary function $a_0(\cdot)$ and the second-order auxiliary function $A(\cdot)$. In case $\gamma > 0$, assume that $\int_1^\infty g\left(x^{-1/(\gamma+\delta)}\right)\mathrm{d}x < \infty$ for some $\delta > 0$. Then when $\hat{x} = \infty$,*

$$\begin{aligned} T_p[X] = {} & F^{\leftarrow}(p) + a_0\left(\frac{1}{1-p}\right)\int_0^1 \frac{q^{-\gamma} - 1}{\gamma}\mathrm{d}g(q) \\ & + a_0\left(\frac{1}{1-p}\right) A\left(\frac{1}{1-p}\right)\left(\int_0^1 H_{\gamma,\rho}\left(q^{-1}\right)\mathrm{d}g(q) + o(1)\right); \qquad (4.1) \end{aligned}$$

when $\hat{x} < \infty$,

$$\hat{x} - T_p[X] = \hat{x} - F^{\leftarrow}(p) - a_0\left(\frac{1}{1-p}\right)\int_0^1 \frac{q^{-\gamma}-1}{\gamma}\mathrm{d}g(q)$$
$$-a_0\left(\frac{1}{1-p}\right)A\left(\frac{1}{1-p}\right)\left(\int_0^1 H_{\gamma,\rho}\left(q^{-1}\right)\mathrm{d}g(q) + o(1)\right). \qquad (4.2)$$

Proof. Note that $H_{\gamma,\rho}(x) = \frac{x^\gamma}{\gamma}\frac{x^\rho - 1}{\rho} = c_2^{-1}\left(\Psi_{\gamma,\rho}(x) - c_1\frac{x^\gamma-1}{\gamma}\right)$, where $c_1 = \gamma/(\gamma+\rho)$ and $c_2 = \gamma\rho/(\gamma+\rho)$. Thus

$$\left|\frac{\frac{U\left(\frac{1}{q(1-p)}\right)-U\left(\frac{1}{1-p}\right)}{a_0\left(\frac{1}{1-p}\right)} - \frac{q^{-\gamma}-1}{\gamma}}{A\left(\frac{1}{1-p}\right)} - H_{\gamma,\rho}(q^{-1})\right|$$
$$= \left|\frac{\frac{U\left(\frac{1}{q(1-p)}\right)-U\left(\frac{1}{1-p}\right)}{a_0\left(\frac{1}{1-p}\right)} - \frac{q^{-\gamma}-1}{\gamma}}{A_0\left(\frac{1}{1-p}\right)} \cdot \frac{A_0\left(\frac{1}{1-p}\right)}{A\left(\frac{1}{1-p}\right)} - H_{\gamma,\rho}(q^{-1})\right|$$
$$\leq \left|\frac{A_0\left(\frac{1}{1-p}\right)}{A\left(\frac{1}{1-p}\right)}\left(\frac{\frac{U\left(\frac{1}{q(1-p)}\right)-U\left(\frac{1}{1-p}\right)}{a_0\left(\frac{1}{1-p}\right)} - \frac{q^{-\gamma}-1}{\gamma}}{A_0\left(\frac{1}{1-p}\right)} - \Psi_{\gamma,\rho}\left(q^{-1}\right)\right)\right|$$
$$+\left|\left(\frac{A_0\left(\frac{1}{1-p}\right)}{A\left(\frac{1}{1-p}\right)} - \frac{1}{c_2}\right)\Psi_{\gamma,\rho}\left(q^{-1}\right) + \frac{c_1}{c_2}\cdot\frac{q^{-\gamma}-1}{\gamma}\right|$$
$$\leq (1+\varepsilon)\, q^{-(\gamma+\rho+\delta)} + Cq^{-(\gamma+\delta)}.$$

In the last step, we used Lemma 4.1 and the following facts: for every $\gamma \neq 0$ and $\rho \leq 0$, there exists some positive constant C such that, for all $0 < q < 1$,

$$\frac{q^{-(\gamma+\rho)}-1}{\gamma+\rho} \leq Cq^{-(\gamma+\rho+\delta)}, \qquad \frac{q^{-\gamma}-1}{\gamma} \leq Cq^{-(\gamma+\delta)}, \qquad \log q^{-1} < q^{-\delta},$$

and $A_0(t) \sim A(t)$. Since one easily checks that $\int_0^1 q^{-\gamma-\delta}\mathrm{d}g(q) < \infty$ hold for some $\delta > 0$, applying the dominated convergence theorem we obtain the following: for $\hat{x} = \infty$,

$$\lim_{p\uparrow 1}\int_0^1 \frac{\frac{U\left(\frac{1}{q(1-p)}\right)-U\left(\frac{1}{1-p}\right)}{a_0\left(\frac{1}{1-p}\right)} - \frac{q^{-\gamma}-1}{\gamma}}{A\left(\frac{1}{1-p}\right)}\mathrm{d}g(q) = \int_0^1 H_{\gamma,\rho}(q^{-1})\mathrm{d}g(q),$$

which gives (4.1); similarly, for $\hat{x} < \infty$,

$$\lim_{p\uparrow 1}\int_0^1 \frac{\frac{\hat{x}-U\left(\frac{1}{q(1-p)}\right)-\left(\hat{x}-U\left(\frac{1}{1-p}\right)\right)}{a_0\left(\frac{1}{1-p}\right)} + \frac{q^{-\gamma}-1}{\gamma}}{A\left(\frac{1}{1-p}\right)}\mathrm{d}g(q) = -\int_0^1 H_{\gamma,\rho}(q^{-1})\mathrm{d}g(q),$$

which gives (4.2). This ends the proof. ■

Next we develop second-order asymptotics for the three max-domains of attraction.

Corollary 4.1 *Let X be a random variable with $\Pr(X = \hat{x}) = 0$. In case $\gamma \geq 0$, assume that $\int_1^\infty g\left(x^{-1/(\gamma+\delta)}\right) \mathrm{d}x < \infty$ for some $\delta > 0$.*

(a) The Fréchet case: If for some $\gamma > 0$ and $\rho \leq 0$, $U \in \mathrm{ERV}_\gamma$ with auxiliary function $a_0(\cdot)$ and $U \in 2\mathrm{RV}_{\gamma,\rho}$ with auxiliary function $A(\cdot)$, then we have

$$T_p[X] = F^{\leftarrow}(p)\left(1 + \int_1^\infty g\left(x^{-1/\gamma}\right) \mathrm{d}x\right) + F^{\leftarrow}(p) A\left(\frac{1}{1-p}\right)\left(I_{\gamma,\rho} + o(1)\right),$$

where

$$I_{\gamma,\rho} = \begin{cases} -\rho^{-1}\left(\int_0^1 g\left(x^{-1/(\gamma+\rho)}\right) \mathrm{d}x + \int_1^\infty g\left(x^{-1/\gamma}\right) \mathrm{d}x\right), & \gamma \leq |\rho|, \\ \rho^{-1}\int_1^\infty \left(g\left(x^{-1/(\gamma+\rho)}\right) - g\left(x^{-1/\gamma}\right)\right) \mathrm{d}x, & \gamma > |\rho|. \end{cases}$$

(b) The Weibull case: If for some $\gamma < 0$ and $\rho \leq 0$, $U \in \mathrm{ERV}_\gamma$ with auxiliary function $a_0(\cdot)$ and $\hat{x} - U \in 2\mathrm{RV}_{\gamma,\rho}$ with auxiliary function $A(\cdot)$, then we have

$$\begin{aligned}\hat{x} - T_p[X] &= \left(\hat{x} - F^{\leftarrow}(p)\right)\left(1 - \int_0^1 g\left(x^{-1/\gamma}\right) \mathrm{d}x\right) \\ &\quad + \left(\hat{x} - F^{\leftarrow}(p)\right) A\left(\frac{1}{1-p}\right)\left(\rho^{-1}\int_0^1 \left(g\left(x^{-1/\gamma}\right) - g\left(x^{-1/(\gamma+\rho)}\right)\right) \mathrm{d}x + o(1)\right).\end{aligned}$$

(c) The Gumbel case: Assume $\gamma = 0$, $\rho \leq 0$ and $U \in \mathrm{ERV}_0$ with auxiliary function $a_0(\cdot)$. Define $I_{0,\rho}$ by

$$I_{0,\rho} = \begin{cases} -\rho^{-1}\int_0^1 g\left(x^{-1/\rho}\right) \mathrm{d}x, & \rho < 0, \\ \frac{1}{2}\int_0^1 g\left(e^{-\sqrt{x}}\right) \mathrm{d}x, & \rho = 0. \end{cases}$$

(c1) When $\hat{x} = \infty$ further assume $U \in 2\mathrm{ERV}_{0,\rho}$ with auxiliary functions $a_0(\cdot)$ and $A_0(\cdot)$. Then we have

$$T_p[X] = F^{\leftarrow}(p) + a_0\left(\frac{1}{1-p}\right)\int_0^\infty g\left(e^{-x}\right) \mathrm{d}x + a_0\left(\frac{1}{1-p}\right) A_0\left(\frac{1}{1-p}\right)\left(I_{0,\rho} + o(1)\right).$$

(c2) When $\hat{x} < \infty$, further assume $\hat{x} - U \in 2\mathrm{ERV}_{0,\rho}$ with auxiliary functions $a_0(\cdot)$ and $A_0(\cdot)$. Then we have

$$\begin{aligned}\hat{x} - T_p[X] &= \hat{x} - F^{\leftarrow}(p) - a_0\left(\frac{1}{1-p}\right)\int_0^\infty g\left(e^{-x}\right) \mathrm{d}x \\ &\quad - a_0\left(\frac{1}{1-p}\right) A_0\left(\frac{1}{1-p}\right)\left(I_{0,\rho} + o(1)\right).\end{aligned}$$

Proof. (a) By Lemma 4.2 and Theorem 4.1, we obtain

$$T_p[X] = F^{\leftarrow}(p)\left(1+\int_0^1 q^{-\gamma}\mathrm{d}g(q)\right) + F^{\leftarrow}(p)A\left(\frac{1}{1-p}\right)\left(\int_0^1 q^{-\gamma}\frac{q^{-\rho}-1}{\rho}\mathrm{d}g(q)+o(1)\right).$$

Write $I_{\gamma,\rho} = \int_0^1 q^{-\gamma}\frac{q^{-\rho}-1}{\rho}\mathrm{d}g(q)$. If $\gamma \le |\rho|$, we continue to derive

$$I_{\gamma,\rho} = \rho^{-1}\int_0^1 \left(q^{-\gamma-\rho}-q^{-\gamma}\right)\mathrm{d}g(q) = -\rho^{-1}\left(\int_0^1 g\left(x^{-1/(\gamma+\rho)}\right)\mathrm{d}x + \int_1^{\infty} g\left(x^{-1/\gamma}\right)\mathrm{d}x\right).$$

Notice that if $\gamma = -\rho$ then $\int_0^1 g\left(x^{-1/(\gamma+\rho)}\right)\mathrm{d}x = 0$, where we used the dominated convergence theorem. If $\gamma > |\rho| > 0$ then

$$\int_0^1 q^{-\gamma}\frac{q^{-\rho}-1}{\rho}\mathrm{d}g(q) = \rho^{-1}\int_1^{\infty}\left(g\left(x^{-1/(\gamma+\rho)}\right)-g\left(x^{-1/\gamma}\right)\right)\mathrm{d}x.$$

When $\rho = 0$, by the dominated convergence theorem we obtain

$$\begin{aligned}\int_0^1 q^{-\gamma}\log q^{-1}\mathrm{d}g(q) &= \lim_{\rho\to 0}\int_0^1 q^{-\gamma}\frac{q^{-\rho}-1}{\rho}\mathrm{d}g(q)\\ &= \lim_{\rho\to 0}\rho^{-1}\int_1^{\infty}\left(g\left(x^{-1/(\gamma+\rho)}\right)-g\left(x^{-1/\gamma}\right)\right)\mathrm{d}x.\end{aligned}$$

Thus for $\gamma > |\rho| \ge 0$, we have $I_{\gamma,\rho} = \rho^{-1}\int_1^{\infty}\left(g\left(x^{-1/(\gamma+\rho)}\right)-g\left(x^{-1/\gamma}\right)\right)\mathrm{d}x$.

(b) By Lemma 4.2 and Theorem 4.1, we obtain

$$\begin{aligned}\hat{x} - T_p[X] &= \left(\hat{x}-F^{\leftarrow}(p)\right)\left(1-\int_0^1 g\left(x^{-1/\gamma}\right)\mathrm{d}x\right)\\ &\quad + \left(\hat{x}-F^{\leftarrow}(p)\right)A\left(\frac{1}{1-p}\right)\left(\int_0^1 q^{-\gamma}\frac{q^{-\rho}-1}{\rho}\mathrm{d}g(q)+o(1)\right).\end{aligned}$$

Similar as in (a), it holds for all $\rho \le 0$ that

$$\int_0^1 q^{-\gamma}\frac{q^{-\rho}-1}{\rho}\mathrm{d}g(q) = \rho^{-1}\int_0^1\left(g\left(x^{-1/\gamma}\right)-g\left(x^{-1/(\gamma+\rho)}\right)\right)\mathrm{d}x.$$

Then we obtain the desired result.

(c) Notice that $U \in 2\mathrm{ERV}_{0,\rho}$ with auxiliary functions $a_0(\cdot)$ and $A_0(\cdot)$ such that

$$\lim_{t\to\infty}\frac{\frac{U(tx)-U(t)}{a_0(t)}-\log x}{A_0(t)} = \begin{cases}\frac{x^{\rho}-1}{\rho}, & \rho < 0,\\ \frac{1}{2}\left(\log x\right)^2, & \rho = 0.\end{cases}$$

Then the asymptotics for the Gumbel case can be derived similarly by using Lemma 4.1. ■

5 Numerical Examples

In this section, we use `R` to numerically examine the accuracy of first-order and second-order asymptotics derived in section 3 and 4.

Example 5.1 (The Fréchet case) Assume that F is a Pareto distribution given by

$$F(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha}, \qquad x, \alpha, \theta > 0.$$

Thus $F \in \mathrm{MDA}(\Phi_\alpha)$ and $U(t) = \theta(t^\gamma - 1)$. Easily one can check $U \in \mathrm{ERV}_\gamma$ with $\gamma = 1/\alpha$ and $U \in 2\mathrm{RV}_{\gamma,-\gamma}$ with the second-order auxiliary function $A(t) = \gamma t^{-\gamma}$. We choose the distortion function $g(x) = \sqrt{x}$ and set $\alpha = 2.1$ and $\theta = 1$. In Graph 5.1, we compare the exact value, first-order and second-order asymptotic values for $T_p[X]$ on the left and show the ratios of exact value to both asymptotic values on the right. We find that both ratios converge to 1 as $p \uparrow 1$ and second-order asymptotics is more accurate than first-order asymptotics.

$\boxed{\text{Graph 5.1 about here.}}$

Example 5.2 (The Weibull case) Assume that F is a beta distribution with probability density function given by

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{\mathrm{B}(a,b)}, \qquad 0 < x < 1, a, b > 0.$$

Thus, $F \in \mathrm{MDA}(\Psi_b)$. One can check $U \in \mathrm{ERV}_\gamma$ with $\gamma = -1/b$ and $1 - U \in 2\mathrm{RV}_{\gamma,\gamma}$ with the second-order auxiliary function

$$A(t) = -\frac{a-1}{b(b+1)}\left(\frac{t}{b\mathrm{B}(a,b)}\right)^{-\frac{1}{b}};$$

see, for example, Mao and Hu (2012). We choose the distortion function $g(x) = \sqrt{x}$ and set $a = 2$ and $b = 6$. Similarly as in Graph 5.1, we compare the exact value for $1 - T_p[x]$ and its asymptotic values of beta distribution in Graph 5.2. Again we can see that both ratios converge to 1 as $p \uparrow 1$ and second-order asymptotics is more accurate than first-order asymptotics.

$\boxed{\text{Graph 5.2 about here.}}$

Example 5.3 (The Gumbel case) Assume that F is a Weibull distribution given by

$$F(x) = 1 - e^{-\left(\frac{x}{b}\right)^a}, \qquad x > 0, 0 < a < 1, b > 0.$$

Thus, $F \in \mathrm{MDA}(\Lambda)$ and $U(t) = b\,(\log t)^{1/a}$. If we set $a = 1/2$ and $b = 1$, then $U \in 2\mathrm{ERV}_{0,0}$ with first-order auxiliary function $a(t) = 2\log t$ and second-order auxiliary function $A(t) =$

$1/\log t$. We choose the distortion function $g(x) = x^2$. Similarly as in Graph 5.1, we compare the exact value and asymptotic values for $T_p[x]$ of Weibull distribution in Graph 5.3. Again we find that both ratios converge to 1 as $p \uparrow 1$ and second-order asymptotics is more accurate than first-order asymptotics.

Graph 5.3 about here.

Acknowledgments. The author would like to thank Professor Qihe Tang for his useful comments on this paper. The project was supported by a Centers of Actuarial Excellence (CAE) Research Grant from the Society of Actuaries.

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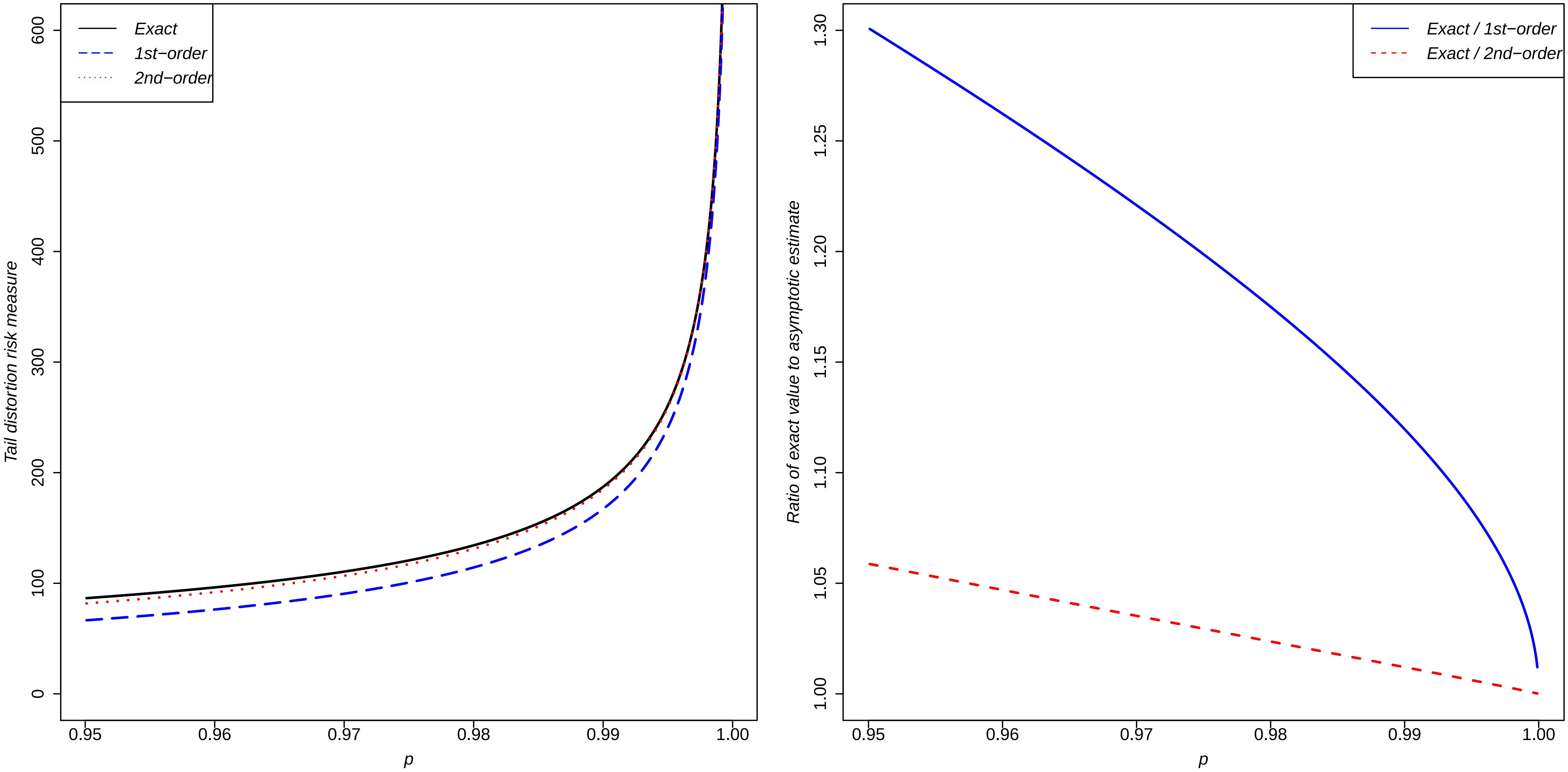
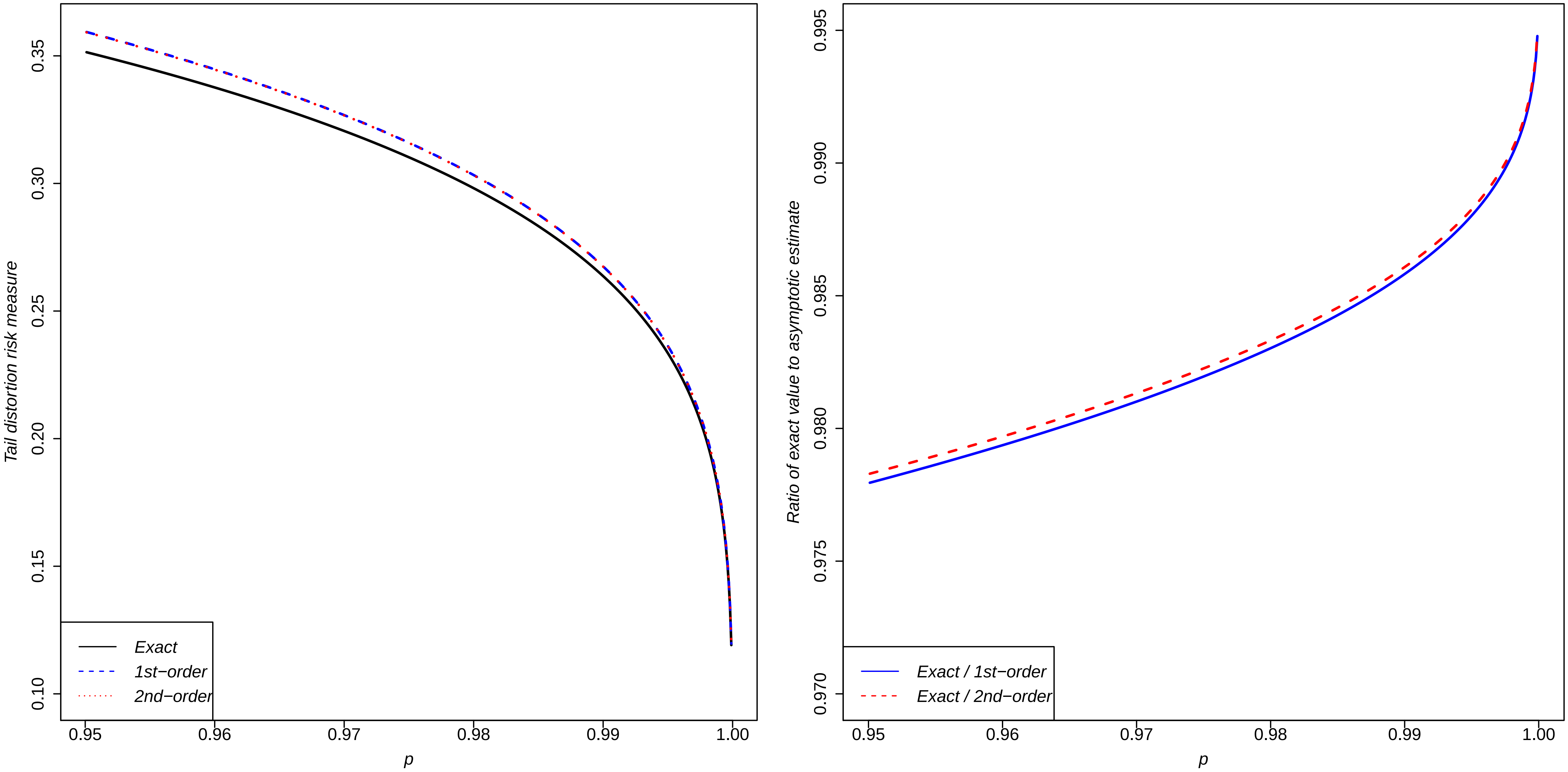
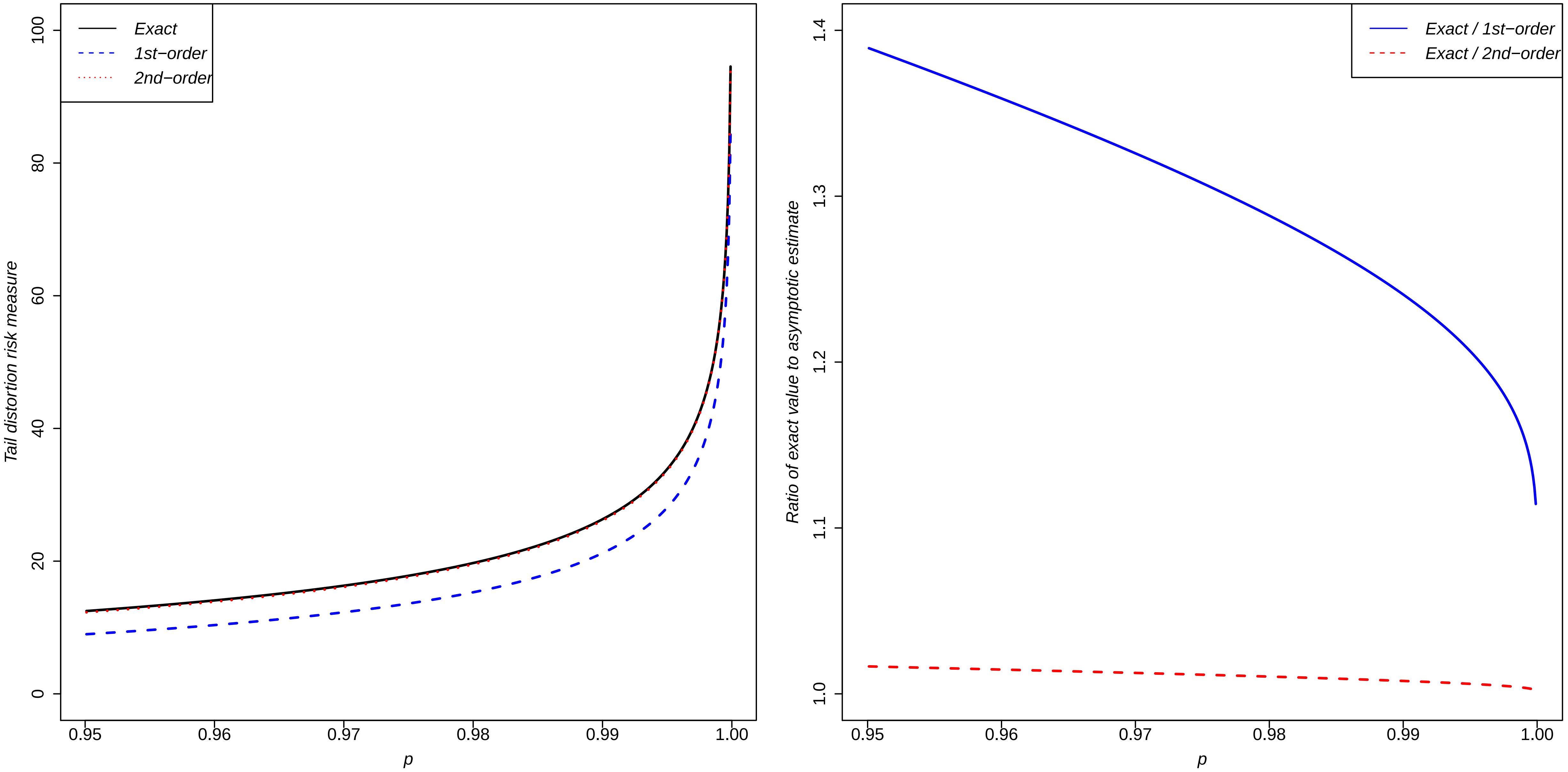


Graph 5.1 Frechet case: Pareto distribution



Graph 5.2 Weibull case: Beta distribution



Graph 5.3 Gumbel case: Weibull distribution