RECORD OF SOCIETY OF ACTUARIES
1984 VOL. 10 NO. 4B

PRACTICAL APPLICATIONS OF STATISTICS AND OPERATIONS RESEARCH FOR ACTUARIES

Moderator: DAVID M. HOLLAND. Panelists: ROBERT P. CLANCY, JAMES C. HICKMAN, EDWARD L. ROBBINS. Recorder: JAMES L. SWEENEY

(1) - A nontechnical introduction to Applied Statistics and Operations Research
(2) - Practical applications and examples for financial reporting and asset management
(3) - Overview of the potential use in actuarial applications of these management tools

This session is sponsored by the Education and Examination Committee of the Society.

MR. DAVID M. HOLLAND:

There was an old fellow named Harry,
Who, of statistics, was wary.
He felt that statistics
were used by sadistics,
and not the real actuary.

This is a discussion of existentialism. Namely, does the existence of Applied Statistics and Operations Research (OR) on the Society syllabus precede any practical essence or is there a practical essence which precedes their inclusion on the syllabus? Stated more simply, there is a common complaint that statistics and OR are tested on the early syllabus but they have very little to do with real actuarial practice.

The purpose of today's session is to point out how useful these topics are to the practicing actuary. The person on the front line who has to answer some difficult and important questions for management needs to be aware of the advantages of these tools. Our profession has evolved beyond the question of "have the reserves been correctly calculated" to "is the reserve correct". There is a big difference between a calculation procedure and the correct level of reserves. Classical actuarial models offer demonstrations over long periods of time but an actuary may live or die based on projections of short range economic financial events. Actuaries construct mortality tables and set assumptions, but it is equally important for the actuary to be able to monitor and manage these assumptions on a timely basis. For instance, there is a lot of interest in smoker/nonsmoker mortality and it has had an influence on pricing. We need to be able to statistically monitor the mix of business that is being developed to determine the validity of our original assumptions.

As background, you may be interested in the principles which guided the Task Force on Operations Research and Applied Statistics, chaired by Mr. Jim Tilley, which is the basis for the current material on the syllabus. This report stated:

"The mathematical syllabus for the associateship examinations should be viewed as an introduction to selected topics in a number of disciplines with emphasis on fundamental principles and concepts. The educational goal should be to present a wide range of topics in mathematics which have, or potentially have, useful applications to practical actuarial problems, or which help the actuary to communicate more effectively with those in allied professions. . . We wish to stress that our work was guided by the principle that topics included on the syllabus should have demonstrable applicability to actuarial problems. Moreover, our recommendations concerning the extent to which any topic is developed in the course of reading, recognize our belief that the Society's educational role relating to the development of mathematical skills is to train generalists and not specialists."

The point is that a practicing actuary may face a number of problems. The syllabus, particularly as it relates to Applied Statistics and Operations Research, does not necessarily make you an expert in these areas but it gives you an awareness of the tools that are available. So if you encounter a problem which has to do with maximization or projections, etc., you will know that an area of expertise exists and there is already a well defined body of methodology. Then you can pursue it appropriately for your project.

Another factor that came after Mr. Tilley's Task Force in developing the syllabus, was something called the "Strategic Premise for Actuarial Education". This was set forth to the Board of Governors from the Education & Examination Committee. Mr. Michael Cowell was then the Chairman. This Strategic Premise is both a goal and plan for actuarial education. I would like to share part of this with you to give you a flavor of some of the thinking which went into the development of the mathematical concepts on the syllabus.

"In a broad sense, the mathematical elements of actuarial education in the associateship syllabus prepare the actuary to 'measure' the impact of contingent events on financial arrangements while the practice oriented subjects in the fellowship syllabus prepare the actuary to 'manage' that impact and those arrangements, and to 'communicate' their predicted outcome in a dynamic environment. To that end, a strategic premise for actuarial education and the ongoing development of the evolution of the syllabus could be summed up as:

(1) To provide the actuary with an understanding of fundamental mathematical concepts and how they are applied, with recognition of the dynamic nature of these fundamental concepts and that they must remain consistent with developments in mathematical knowledge.
(2) To provide the actuary with an accurate picture of the sociodemographical, political and legal, and economic environments within which financial arrangements operate, along with an understanding of the changing nature and potential future directions of these environments.
(3) To expose a broad range of techniques that the actuary can recognize and identify as to their application and as to their inherent limitations, with appropriate new techniques introduced into this range as they are developed.
(4) To expose a broad range of relevant actuarial practice, including current and potential applications of mathematical concepts

and techniques to the various and specialized areas of actuarial practice.

(5) To develop the actuary's sense of inquisitiveness so as to encourage exploration into areas where traditional methods and practices do not appear to work effectively."

Basically, these are the goals that direct us in the development of the syllabus. We think that we have created tools which will be very useful to the actuary but it is as though we have invented the car but told no one where they are trying to go. We hope that this session will give people a perspective of how these topics can be practical and useful in their day to day responsibilities as actuaries.

Our panelists today are Jim Hickman, who is Professor of Statistics of the University of Wisconsin, Ed Robbins who is Manager for Peat, Marwick, Mitchell & Company in Chicago and Bob Clancy who is Associate Actuary in the Treasury Department at John Hancock Mutual Life Insurance Company. Jim will go first and will discuss actuarial applications of Applied Statistics in general. Ed will discuss a specific statistical example with emphasis on the design of samples for auditing and actuarial modeling. Then Bob will discuss some very important actuarial applications of Operations Research.

MR. JAMES C. HICKMAN: Dave has set such a very high standard by starting out with a limerick. I sat there and thought "what on earth can I do?" The only limerick I could think of was that old one that goes,

"There was an odd fellow from Trinity
who solved the square root of infinity,
counting up the digits
gave him such fidgets
that he chucked math and took up divinity".

I hope that what we do today will not cause you to chuck Applied Statistics and Operations Research, although perhaps divinity could use your talents.

Applied Statistics has had an increasing role in our educational efforts. In the last few years there have been several movements to bring statistics and stochastic models into actuarial education. Part 2 has definitely moved, over the last decade, to a greater emphasis of ideas useful in data analysis, and not just those cute things that solve combinatorial problems. Applied Statistics actually entered Part 3 as a separate topic in 1983. A more statistical approach was introduced in life contingencies in 1984. And in Part 5, statistical models have become important. (They have always been there.) But in 1983, came survival models and in 1984 new notes on graduation were written for Part 5, all of which were rooted in statistical ideas.

I would like to talk about the reasons for these changes. One of the reasons is intellectual. This has been a century in which stochastic models have entered first physics, then agriculture, economics and many other fields. Today the physical, biological, social and management sciences all make extensive use of probabilistic rather than mechanistic or deterministic models. To keep in the intellectual mainstream, it was necessary that actuaries bring more of these models into our syllabus. Another reason is educational coherence. The idea is that the basic concepts that you learn in Part 2 are now repeated in Part 3 - Applied Statistics, Part 4 - Life Contingencies, and Part 5 - Risk Theory and the Construction of Tables. This leads to coherence and reinforcement. There is another good reason for introducing Applied Statistics on the syllabus from the viewpoint of educational policy. We compete in the world of ideas. We like to view ourselves as being preeminent in the area of creating models for insurance systems, but there are a lot of other people who also believe themselves to be competent in that area. In order to maintain that preeminence, we must be able to talk their language and to beat them on their own grounds.

What are we talking about? The topics actually on Part 3 under the topic of Applied Statistics are really three in number. These three are: Analysis of Variance, Regression, and Auto-Regressive Moving Average Time Series Models. These are the three topics that form the bulk of the Applied Statistics section on Part 3 and amazingly they are closely related. As a matter of fact, they may all be summarized in simple expression.

A. General Model

Response = Fit + Error

$$Y_T = F(X_{1,T}, \ldots X_{P,T}) + E_T$$

Where $X_{i,T}$ $i = 1,2,\ldots P$ are

explanatory variables at time T.

Example: Loss Ratio for Quarter T = Function of earlier loss ratios and general economic variables + Error

The error term should appear to be "white noise", containing no information about the process generating the response.

All of the topics fit into this model. There is a response variable that we would like to understand and we find a function that may be a function of a lot of explanatory variables. In the equation, the "Y" is the response variable observed at time "T" which is a function of a lot of "X"'s or explanatory variables plus an error term, an "E" term, at the end. For example, that response might be a loss ratio for quarter "T" and those "X"'s might be a bunch of economic variables in quarter "T", or they might be values of a loss ratio of earlier quarters plus, of course, that error term.

That error term, after we have completed our modeling, should contain no information. It should appear as "white noise", which, if you like to play stereos, is that sound which contains no music or message. Our chore in applied statistics is to squeeze out all information except white noise.

Now, let's take a look at that general model and those three specific examples. First let us look at the analysis of variance. Now, we were admonished to suppress equations and keep to words, so that is what I am going to try to do.

B. Analysis of Variance

In the analysis of variance, that fit (remember response is equal to fit plus error) is of the form of some kind of overall constant plus the effect of a factor or factors which may exist at several levels.

Fit = Overall mean + Effect of a
factor(s) which
exists at several levels

Example:

$$Y_{ijk} = M + L_i + B_j + E_{ijk}$$

Claims incurred by insured auto K in a fixed period	= Overall mean	+ Deviations for territories	+ Deviations for driver classes	+ Error

Chang, L., and Fairley, W. (1979), "Pricing Automobile Insurance Under a Multivariate Classification," Journal of Risk and Insurance, 46, 75-93.

As an example, suppose that the response function that you are interested in was claims incurred by insured auto "K". K is a label that we will put on a particular auto and let's suppose that the factors for determining the response function may be territorial division or driving record divisions. The analysis of variance is a device, as in this example, for analyzing the response variable that is made up of a constant and other terms that can bump around at several different levels.

C. Regression

The second of those models is regression which is probably a non-informative name. It goes back to Francis Galton, one of those remarkable Victorians who seemed to be able to dabble successfully in everything. The difference between regression and analysis of variance is really very small. The main point is that those explanatory variables no longer have to be confined to just a few levels. Those explanatory variables can take on a continuum of values. An example is loss ratios for autos in various groups which might be a constant, plus terms that have to do with horsepower of the automobile, the size of the hometown and the driving record. And, as a matter of fact, there is a paper by Hilary Seal in the eighteenth Transactions of the International Congress of Actuaries addressing this topic.

Fit = Function of several variables where
the value of the variables need
not be confined to a few levels

Example:

Loss Ratio of Auto Insurance for Groups	= Constant	+ Terms for horsepower, mileage, size of home city, driving record, etc.	+ Error

Seal, H. L. (1964), "The Use of Multiple Regression in Risk Classification Based on Proportionate Losses," 18th TICA, 2, 659-669.

or transformed:

$$Y_i = \log\left(\frac{P_i}{1 - P_i}\right) = B_o + \sum_{j=1} B_j X_j + e_i$$

Loss ratios do not fit well into the regression model but that is alright since you can transform them. Transformation is one of the ways that you can expand the range of applicability of these models. In his paper, Mr. Seal actually takes a look at loss ratios which have been transformed by logistic transformation. Logistical transformation is used to fit loss ratios into a regression model. Why go to the extra effort? Well, we know much about the regression model. There are many canned programs and techniques by which we can study these models. So, rather than punt, we will transform our variables so that we can use them. In this example, our transformed function consists of a constant term and those "X"'s, or explanatory variables, which might be terms for horsepower, mileage, size of hometown, etc., which will contribute explanatory information. We will model until what is left is white noise.

D. Time Series

The third type of model studied on Part 3 is the time series model. In time series models the function, or the fit, is a function of earlier values of the same variable rather than other explanatory variables. For example, if you are interested in hog prices, you try to predict tomorrow's hog prices based on today's and yesterday's and a few days' before that, so that the basis of the function is earlier values of the same response variable plus perhaps earlier realizations of error terms. As an insurance example, average paid claims in a quarter might be a function of average paid claims in earlier quarters plus an error term.

Fit = Function of earlier values of the response variable and earlier random errors

Example:

Average paid claim in quarter T = Function of average paid claim in earlier quarters + Error

Cummins, J. D. and Powell, A. (1980) "The Performance of Alternative Models for Forecasting Automobile Insurance Paid Claim Costs," ASTIN Bulletin, 11, 91-106.

If you are interested in forecasting average paid claims, you will want to read a paper by Dave Cummins and Alyn Powell. In that paper they examine several time series models in which the fit is the time series type. They also examine some regression models where the response variable is the average paid claim cost, but the explanatory variables, instead of being earlier values of the same variable, are various economic variables like the CPI component for auto repair costs. It is an interesting paper and I hope it gives you some idea of the applicability of these time series models.

Time series models are the third of this general class of models that we are studying now in Part 3.

Now, there is a certain commonality in the approach to all of these models. Here are some examples.

Moving Average - First Difference

$$Y_t - Y_{t-1} = \theta e_{t-1} + e_t$$

Here we look at the first difference of a response variable, Y, at time "t" and time "t - 1" which may be a function of an earlier error term and a parameter, θ, which is a constant. This is called moving average because we are averaging, in a certain way, previous error terms. That model happens to be of some note in some business areas. It is the basic model for what is called exponential smoothing. It is only one of a broad class of time series models that are now studied in Part 3.

Auto-Regressive 2

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$$

This model, where today's response is a function of yesterday's and the day before plus a random error term, is called an auto-regressive 2 model. This is called auto-regressive because it looks like a regression equation except the "Y"'s are earlier values of the same variable, automatically regressed onto itself. These are simply examples of the classes of models, all of which fit into that same general model that we set out in the beginning.

Now, probably more important than the models themselves is the general philosophy of applied statistics. That philosophy starts with a goal. The goal is to learn about a real world process for the purpose of prediction or control. We try to have a systematic and coherent method for attempting to learn about that process. Now, often the difference between prediction and control is very great; you and I as private citizens or as businessmen cannot control that greater world. If we were on the Federal Reserve Board or the Council of Economic Advisors, we might use different models to try to control the economy than the models we could use if we were interested in simply short-term prediction to protect ourselves.

The idea in this coherent process is first of all to tentatively identify a model. As you might guess, the tricks to doing that are partially graphic. You will want to plot your data in many different ways to try to get an idea as to what model might work. There are also analytic tools, some of which you already know: computing coefficents of correlations, computing coefficients of auto-correlation, and correlation of successive or lagged items of the same series. There are other tricks, both graphic and analytic, to help you tentatively identify a model.

The next thing to do is to estimate that model. That is the black box of this process. That is what the computer does, and does well. Part of the growth in Applied Statistics in recent years occurred because we can now estimate very complicated models thanks to the computer and also to what students are learning in Numerical Analysis.

The third step is by far the most important. Criticize the model by looking usually at the response minus the fit,

$$Y_T - F(X_{1,T},\ldots X_{P,T})$$

This difference is called the residual and the residual should look like "white noise". It should contain no information.

Once again you have both graphical (just looking at them to see if you can see patterns) and analytic techniques to see if you have squeezed out of the data all of the useful information. If the answer is "no", go back and use this new information to identify a new model and work back through the cycle. After you have squeezed the useful information out, you are ready to go ahead with forecasting what the process may generate for the future.

Now, let's turn to a few examples of this process that we hope will give some ideas of its use in insurance. We can explore man-made worlds. Man-made worlds are those that we create in our own companies in accounting and valuation. These examples might include the use of multiple regression to estimate projected benefits of a pension scheme where the records are scanty. I was once involved in a problem where the basic employee records existed only on tax forms stored away in an attic. In order to get some feeling about the liabilities of this pension system, we felt that a regression formula could estimate projected benefits. We could also use multiple regression to establish expense standards where the response variable might be total expenses in year "t" and the "X"'s are various activity levels in that year. There is an example of that in a paper by Bob Miller and Bill Fortney that use multiple regression in expense analysis. (Miller, R. B. and Fortney, W. G. (1984) "Industry Wide Expense Standards using Random Coefficient Regression," Insurance: Mathematics and Economics 3, 19-34.)

We could also explore the economy, not just the man-made part that you and I create, but in general what is going on with the outer world. As an example, we could use time series analysis to forecast average health claims per insured. Don Jones has an interesting discussion of using time series analysis for these one, two and three step month ahead forecasts using fit functions that depend upon earlier values of the same variable. (Jones,

D. A. (1973) "Discussion of Time Series Analysis and Forecasting," TSA XXV Part 2, D210-D223.) We could use this same technique to estimate sales. Frank Reynolds, almost eleven years ago, had a discussion which shows the use of time series models in forecasting life insurance sales, but those models are a little bit more complicated than the ones I have shown you. They have a seasonal component. (Reynolds, F. G. (1973) "Discussion of Time Series Analysis and Forecasting," TSA XXV Part 1, 303-321.) I used to work for a Midwestern company and our sales did bunch up in the fall because that is when the crops were harvested. Such facts can be built into the time series models very easily.

We can explore society and nature itself in the insurance world. For example, there is a paper by Peter Ellis that uses time series models to estimate monthly auto deaths in the United States. The purpose of Mr. Ellis' paper was to measure the effectiveness or possible noneffectiveness of the 55 m.p.h. speed limit. He constructed a time series model of monthly data and found out, as you probably already would have guessed, that the 55 m.p.h. speed limit was a roaring success in reducing auto deaths. (Ellis, P. M. (1977) "Motor Vehicle Mortality Reductions since the Energy Crisis," Journal of Risk and Insurance, 44.) Other examples are the use of regression analysis in the analysis of causes of mortality or transition probabilities. For example, the transition rate from one state to another from having or not having cancer may be influenced by a lot of explanatory variables. Regression may be the way that we can learn more about what influences the response rate. You will see another example of that in the case of radiation in a book published by the National Academy of Sciences, where the explanatory variables are the dose of radiation, the age of the person at the time of treatment and the sex of the person.

We have tried to build a case based on competition, educational coherence and staying in the intellectual mainstream, as to why actuaries need to understand up-to-date applied statistics tools. Karl Pearson, one of the founders of statistics, entitled his most famous book, The Grammar of Science. Statistics is the grammar or structure of science. After you master these three tools, are you done? No. Because, new tools are coming and will continue to come. Anytime we have data to analyze, that is the business of statistics. Since, as actuaries we analyze data, then for good or for ill, we are involved with statistics.

MR. EDWARD L. ROBBINS: I was asked to speak on the use of statistical tools for auditing with an emphasis on sampling and modeling because I am employed by an accounting firm and am familiar with the subject. I emphasize familiarity and not expertise for two reasons. The first reason is that I have not been at my company that long and, secondly, we have a division exclusively devoted to the statistical applications of the audit function. They are called Statistical Audit Specialists and they belong to the audit division of the firm. They are professionals like the other accountants and actuaries of the firm. They are required to undergo continuing education like the other professionals of the firm and they basically perform four functions: (1) give general advice to the auditors, (2) assist on the audits, (3) review in advance any statistical audit plans, and (4) perform post-reviews of any statistical audit results.

The fact that we have a statistical audit division benefits us in several, less obvious, ways. For example, they come out with standardized forms. These forms serve the audit staff so that reasonably low level staff can actually complete the input for the audit sample. The input is then used by a computer program and the output is an audit selection set of instructions. The other by-product that comes out of having a formalized sample selection procedure is that it reduces the exposure to litigation substantially. You minimize the possibility of an auditor being on the witness stand and having an attorney ask him: "You selected a sample of 30 in this case, yet in exactly the same situation two years ago, you selected a sample of 100. Why?" We are in a situation where we are exposed to an awful lot of potential litigation over time, so that if by establishing a standardized sample basis we have minimized this potential litigation factor, that is extremely important.

The reason I am going into our firm's organizational chart to some extent is that I am emphasizing that whenever you audit a large population of independent elements, there are some very real improvements that you can realize in precision and in cost from proper sampling technique. The major accounting firms recognize this. If the actuary knows how to sample well, the results should be good estimates at a relatively low cost. There can easily be 40 or 50 times better precision by going from a poorly drawn sample to a well chosen one at about the same sample size.

The important thing for the actuary to recognize is that these situations arise frequently in the life insurance environment and these techniques are not used to their capacity. Take the example of the home office actuary on February 26, who is about to finish all of his assembling and filing of the annual statement data and is about to send it out to the states and sign the actuarial opinion. There is really very little time to do a review of any depth at that time. What a typical actuary might do at that time is simply arrange for a summer project to spot check his reserves in some manner. His basic motivation is "How do I get this spot check out of the way quickly, and with the least possible cost and yet discover that which should be discovered?" It is a tough question and there are techniques for improving your ability to do this. In our firm, it is not the actuaries that do the job of sample selection. However, in many other instances the independent consulting actuary or the home office actuary may have to organize the sample selection and testing process himself. So it is a good idea to have a little background in some reasonably easy statistical techniques that can be applied in this setting.

Just a word about the different types of tests during audits. Audit tests are generally divided into two types. Attributes testing is the most common. That is the "yes-no" type of test. Is there an error or not? The other is the variables test. For example, you have a new computer run and you want to estimate the total on that computer run. The computer run is 80 pages long and you cannot do a tab total on it. So you do a sampling. You take the total sum from the sample and you gross that up to an estimate of the total population size. Another example would be if you have discovered an error of principle in the deferred premiums. You can determine the total error in the entire population based on a sampling of some of those deferred premiums. Thus attributes testing and variables testing are the basis of the two major types of formulas that have been worked upon by the Statistical Audit Division of our firm.

Leaving the audit function for a moment, let's take a traditional actuarial management tool, modeling and model offices. They have been around for a long time. Modeling is really nothing more than an attempt to get a representative sample in order to estimate a number or an array of numbers. Put into statistical terminology, modeling is an attempt to get the expected value of something given an extremely stratified sample. What is a stratified sample? Leaving precise definitions aside, it is a sample where pains are taken to be sure that the sample is representative of each major category in the population. Typically, when an actuary does modeling, he makes an effort to select sample elements so that the sample is drawn proportionately from the major population categories. Efforts are then made to gross up the model into total portfolio numbers. If, for example, the model is to be used for projections or for GAAP reserves it is important for the actuary to have a high comfort level with the model. By tying into portfolio totals, such as reconciling with statutory reserves, the actuary can achieve that comfort level.

I have discussed auditing and modeling as two prime applications of good statistical sampling techniques. Let's jump into the techniques themselves. There has been little quantifying of the statistical sampling processes employed by most actuaries. The sampling has mostly been designed by feel instead of by more formal techniques. Now this is not all bad. Research is expensive and an actuary who is well grounded in some of the principles involved may have gut feelings as to the sample selection which may give pretty good results. It certainly saves a lot of time and there are many concepts which defy quantification. Otherwise you might be doing too much research for the accuracy you get.

Certain rules of thumb of good sampling are the following: (1) Cost and time should be minimized while representativeness of the sample should be maximized. This is something like smoothness and fit. They tend to pull in opposite directions. (2) Once an element is being sampled, as many attributes of that element as possible should be tested. In other words, once you have an application folder out, test as many relevant things in that application folder as would be useful to you. (3) Over-emphasize your sampling effort in categories or blocks where the degree of internal variation is likely to be high. For example, if you are doing an actuarial model of two blocks where one block of business has a issue year range of 20 years and the other has a issue year range of 5 years, then you want to sample the first of the two blocks more.

There are some basic mathematical formulas that bear out these precepts and that also give you an actual quantification of these three precepts. First, let me define stratified sampling; it is breaking your population into two or more components, each of which is more homogeneous than the population at large. A proportionate stratified sample is the selection from each component of a number of sample elements in the same proportion that the component bears to the total population. For example, you have a stratum A and a stratum B that make up the entire population. If the population consists 60% of stratum A and 40% of stratum B, then your sample should also be chosen in that proportion.

Generally, the degree of improvement going from a random sample of the population to a stratified sample can be very significant if two things occur: (1) the strata have stratum means which are significantly apart from each other and (2) the sampling is random within each stratum and proportionate to the size of the population strata. Expressed mathematically, the degree of improvement in the variance of the estimate, moving from random sampling to stratified sampling is:

$$= \frac{1}{n}\left[E\left(\bar{x}_h^2\right) - \left(E\left(\bar{x}_h\right)\right)^2\right], \text{ where:}$$

$\bar{x}_h$ = Sample mean in stratum h.

n = Number of elements in entire sample.

E(x) = Expected value of x.

Thus the improvement is proportionate to the variance of the stratum sample means about the total sample mean. This expression is the variance of your stratum means around your estimate of the population mean multiplied by 1/n. I have attached the proof of this (Exhibit 1).

Now the question is, "How do you select your stratum size?" Proportionate stratification is not the best kind, let alone the only kind of stratification. It can be proved (Exhibit 2) that the following expression is the optimum method of choosing your stratum size.

The formula that selects the optimum number of elements to be drawn from a stratum once you know the total sample size, is:

$$n_h = n \cdot \frac{N_h S_h}{\sum_h N_h S_h}, \text{ where:}$$

$S_h = \sqrt{Var\left({}_h x_i\right)}$

n = Total sample size

n_h = Sample size of stratum h. Thus, $\sum_h n_h = n$.

N = Total population

N_h = Total size of stratum h. Thus, $\sum_h N_h = N$.

What this shows is that your stratum sample size varies with the number of the population in your stratum, "N_h". It also varies with the standard deviation of an element in a particular stratum, stratum h. Your total sample size, "n", times that weighting factor, gives your stratum sample size "n_h".

You could go a step further and say, "Some things cost me more than others to sample." One example is a complicated reserve vs. a simple reserve in the audit selection situation. You want to minimize your total cost and you also want to maximize your precision. Then how do you calculate your optimum stratum sample size? Suppose that your total cost is both an out-of-pocket cost and an implicit cost. Then

$$\text{Total Cost} = a + \sum_h c_h n_h + \text{Var(est.)}$$

where a = Fixed cost
c_h = Marginal cost of sampling stratum h
Var(est.) = Variance of the estimator of the implicit cost

If this is your cost function, a neat expression for your optimum stratum sample size falls out. You add one additional term to the formula above.

Total cost is minimized when

$$n_h = \left(\frac{N_h S_h}{\sqrt{c_h}} \div \sum_h \frac{N_h S_h}{\sqrt{c_h}} \right)$$

where $S_h = \sqrt{\text{Var}\left({}_h x_i\right)}$

c_h = Cost of sampling an element in stratum h.

Thus, n_h is proportionate to $\frac{N_h S_h}{\sqrt{c_h}}$ and

$N_h S_h \div \sqrt{c_h}$ is the weighting factor. See Exhibit 3 for the proof.

So now you can see that the larger the stratum size, the larger the variation within the stratum, and/or the smaller the cost, the more you want to sample the stratum. Therefore your stratum size is proportionate to that weighting factor.

The next question is how to obtain the total sample size. You can get your stratum size once you have your total sample size, but how do you get your total sample size? You can derive the method for getting your total sample size. You start with either one assumption or the other. You can start with how much you have to spend or with the maximum variability you can allow in your estimator. That is the A and B below respectively.

Thus, to obtain the total sample size n, start with one of two assumptions:

A. If total out-of-pocket cost is given:

$$C = a + \sum_h c_h n \frac{N_h S_h}{\sqrt{c_h}} \div \sum_h \frac{N_h S_h}{\sqrt{c_h}}$$

$$\text{Thus } n = (C - a) \frac{\sum_h \frac{N_h S_h}{\sqrt{c_h}}}{\sum_h N_h S_h \sqrt{c_h}}$$

B. If the variance of the estimator is given:

$$n = \frac{\sum_h S_h \frac{N_h}{N} \sqrt{C_h} \sum_h \frac{N_h}{N} \frac{S_h}{\sqrt{C_h}}}{\text{Var(est.)} + \sum_h \frac{N_h}{N^2} S_h^2}$$

See Exhibit 4 for the proof.

You can obtain the variance using the out-of-pocket cost equations for a given value of "n" and vice versa. The variance and costs obviously vary inversely with respect to each other. You can try differing values of "n" and iterate to get the most desirable blend of the estimated variance and the total cost. In other words, you can get your total sample size by working through these two formulas. In the real world you may get your total sample size from method A. Given the resulting out-of-pocket cost, you can plug that back in and get the degree of variance of your estimator and see if you like those two numbers. You iterate back and forth until you find your optimum sample size.

When an actuarial method is chosen by an actuary's intuition, it may be a pretty good method if he has some of these precepts regarding sampling choices in mind. However, it is nice to be able to support our impressions with demonstrations. The frequent problem with impressions is that they give you a direction without giving you a value. By using these methods we give you a means of getting that value.

I am going to conclude with two practical examples. Let's say an error of principle was found in the deferred premiums. The typical manner by which an actuary might go through the deferred premiums to determine the total amount of error might be to take every n^{th} policy of the in force. This method by its very nature gives a stratified sample, a proportionate stratified sample, so it is going to give better results than a random sample. But, you can improve that tremendously by something that actuaries have intuitively done. For example, all deferred premiums over $5,000 might be tested 100%, or a lot more intensively than by simply sampling every 50th policy or so. In this way, you have done better than proportionate stratification. Implicitly the variance among the larger amounts of deferred premiums is expected to be larger. This bears out one of the precepts that we have been talking about earlier in this discussion.

In the last practical example, let's take a typical modeling process, in which the actuary has gone through his in force and decided what central points are representative of plans, issue ages and issue years. Then he does a few more things to "true" this model up. Using the example of a moment ago, the actuary might take more elements from those blocks of business with greater issue year ranges. He will also reclassify the non-model plans into the model plans. In other words, a plan that has six policies in force might be classified into a similar plan that is in the model. In doing that, he is going to eventually recreate the population in force. His final step is to try to get as close as possible to the total in force, statutory reserve, and annualized premiums. Finally he will want to see that the model, when properly grossed up, reproduces all three of these things in a reasonable manner. What has he done in this process? He has made sure that every plan is represented. That is pretty good stratification. He has made sure that plans that encompass a wider category are better represented by more plan groups and he has eliminated an early bias by reproducing the aggregate in force, reserve, and annualized premium figures.

I have a feeling that I have not told many of you anything new. I have been redundant but I wanted to tie some of these tried and true traditional techniques to specific formulas, because a lot of actuarial feel is born out by these formulas. In any case, whatever the application, certain items are really not going to be worth the trouble to precisely pin down, such as the total cost function or the implicit cost of a larger variance of your estimator. So, sometimes it is not worth the trouble to be too precise as long as you know you are going in the right direction. I will give you one additional reference, if you are interested. It is George C. Campbell's, "Problems With Sampling Procedures for Reserve Valuation", Journal of the American Statistical Association, Volume 43, 1948.

MR. ROBERT P. CLANCY: I will admit that it is pretty ironic that I am here speaking to you today about actuarial applications of Operations Research topics that are on the Part 3 syllabus. It is really all my wife's fault. A few years ago when Operations Research was placed on the syllabus, my wife was in charge of recruiting instructors for the actuarial classes sponsored by the Actuaries Club of Boston. At the time, she was having a very difficult time trying to find any instructors who were both qualified and interested in teaching. About that time I made a foolish slip of the tongue that ten years earlier I had taken an Operations Research course in college. The next thing I knew, I had been recruited and was struggling to come up with some course materials. Finally I had to learn it well enough to teach. Little did I realize that this was going to turn out to be one of the better moves of my life. Since that time, I have moved into the investment operation of my company; investment strategy provides a number of applications for Operations Research techniques. I firmly believe that such investment strategy applications of Operations Research techniques are going to become quite important as our industry becomes more involved in the development and risk management of interest sensitive products. At any rate, I hope that you will see something here today which piques your interest so that you will want to go back and get a recent Part 3 student to investigate it in greater detail.

Let me start with a simple definition of Operations Research. Operations Research is using quantitative techniques in the decision making process. I hope this sounds natural for actuaries. The topics currently on the Part 3 syllabus include decision analysis, linear programming, dynamic programming, project scheduling, queuing theory and simulation. I will try to touch on most of these topics by giving examples of applications.

Since we are talking about making decisions, perhaps the best place to start is the subject of decision analysis. I think one of the most under-utilized and useful Operations Research techniques is the decision tree. If it has ever been true that a picture is worth a thousand words, then a decision tree is it. Here is an example of how a decision tree can be used in the underwriting process. (See Figure 1.)

In this diagram, the rectangles represent decisions to be made. In this simple example, they involve the decision whether or not to get an inspection report and whether to accept or reject an insurance application. The circles in the diagram represent chance events that can occur. In this example, they represent the possibilities of a favorable or an unfavorable inspection report and the possibilities of ending up with a normal risk or a bad risk.

The decision tree shows that we can decide to get an inspection report and the result can be either favorable or unfavorable. Depending on the result that comes back, we can then make the decision to accept the application or reject it. Should we choose to accept it, we will be insuring what will eventually be either a normal risk or a bad risk. Now for each combination of decisions and chance events that can occur, we can compute an outcome. In this simple example, the outcomes represent the profits or losses from insuring a normal or bad risk, adjusted for the cost of the inspection report. I find a decision tree allows management to quickly get a feel for a problem, or to see the big picture, and to understand how combinations of their decisions and chance events can conspire to produce good or bad results.

Typically one would enter the appropriate probabilities into the decision tree taking into account preceding events. For example, presumably the probability of getting a normal risk given that a favorable inspection report was received is greater than the probability of ending up with a normal risk given that an unfavorable report was received. By factoring in the appropriate probabilities and costs, one can derive an optimal decision strategy for a problem like this.

Underwriting research departments have effectively been using this technique for years although their decision making process may not be done in quite this over-simplified way. For anyone interested in a more detailed discussion of this application of the decision tree, I refer you to a discussion by Don Jones in the Transactions, Volume 22, part 2, starting on page 440.

Another extremely useful application of decision trees is the analysis of tax strategies. The various decisions in the diagram could include whether or not to set up a subsidiary, whether or not to involve the national office of the IRS, whether or not to appeal a ruling, etc. The various outcomes could range from gaining some desired tax savings to losing your shirt. Now, lawyers rarely give you probabilities for any of these events. So, it will be difficult, if not impossible to actually quantify the optimal decision strategy. Nevertheless, I have found such diagrams to be very useful, even without the probabilities, in allowing management to quickly see the big picture and to cut through some of the legalese. Therefore management can understand certain potential ramifications of their decisions.

Figure 1

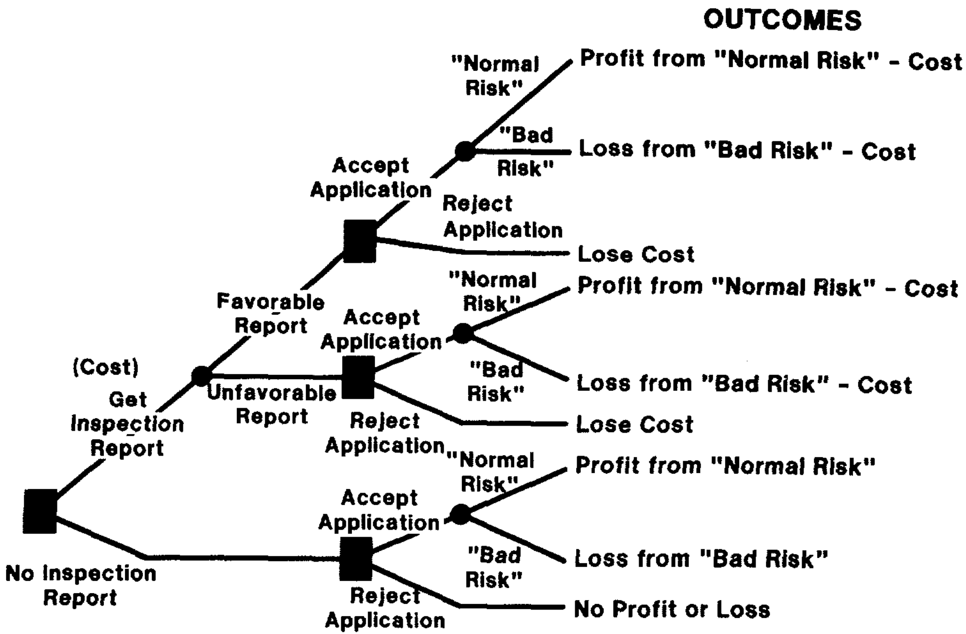
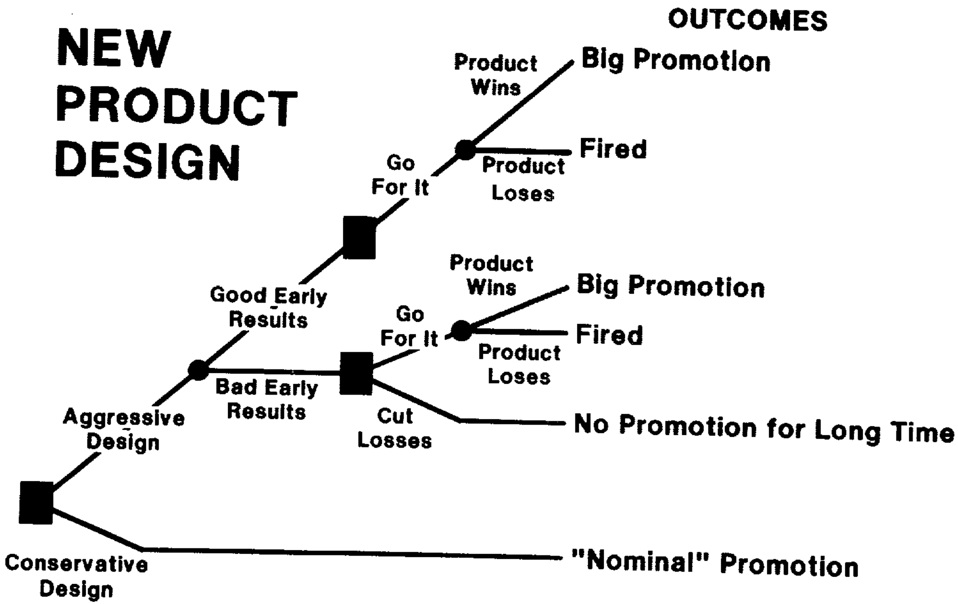

OUTCOMES
"Normal Risk"
Profit from "Normal Risk" - Cost
"Bad Risk"
Loss from "Bad Risk" - Cost
Accept Application
Reject Application
Lose Cost
"Normal Risk"
Profit from "Normal Risk" - Cost
Favorable Report
Accept Application
(Cost)
"Bad Risk"
Loss from "Bad Risk" - Cost
Get Inspection Report
Unfavorable Report
Reject Application
Lose Cost
"Normal Risk"
Profit from "Normal Risk"
Accept Application
"Bad Risk"
Loss from "Bad Risk"
No Inspection Report
Reject Application
No Profit or Loss

Figure 2



NEW PRODUCT DESIGN
OUTCOMES
Aggressive Design
Conservative Design
Good Early Results
Bad Early Results
Go For It
Cut Losses
Product Wins
Product Loses
Big Promotion
Fired
No Promotion for Long Time
"Nominal" Promotion

One final comment on decision analysis relates to a subject called utility theory. Now, utility theory is very useful for quantifying results that are not easily quantifiable. For example, consider an ambitious young actuary who has been assigned to develop a new product. The actuary is debating with an aggressive or conservative product design. The actuary might construct a decision tree to choose whether to go with an aggressive product design assuming they would monitor the early results of promoting of the product. (See Figure 2.)

If the early results come back looking bad for the aggressive product then the company can decide at that point to cut their losses. Or they could decide to "go for it" by marketing the product extensively, and hope that it turns out to be a winner as opposed to a loser. If the early results are good, chances are the actuary would decide to "go for it" and hope that the product turns out to be a winner.

Note that the actuary has computed the outcomes, not so much in terms of profitability for the company, but in terms of career ramifications. In this case, the career ramifications are getting a big promotion, getting fired, getting no promotion for a very long time, or getting a nominal promotion. Now these outcomes may not seem very quantifiable but they are quantifiable relative to one another. Utility theory achieves this by asking a series of questions such as: would you rather have a job in which you have a guarantee of getting a nominal promotion or would you rather have a high risk job that gives you a 50% chance of getting a big promotion and a 50% chance of getting fired? By playing around with the probability of getting fired, utility theory can allow an individual to quantify his feeling about a nominal promotion relative to the other possible outcomes. Once these outcomes are quantified relative to one another they can be entered into the decision tree and an optimal decision strategy can be derived.

In short, utility theory can be combined with decision analysis to determine optimal decisions even when the results of such decisions are not easily quantifiable. Some of the potential wide range of decisions for which this technique could be used include the use by doctors to help people decide whether or not to undergo certain medical procedures such as surgery. Now clearly, one's feelings about potential surgical complications are not easy to quantify. This technique presumably would have plenty of potential applications in the insurance business where little emotion is involved.

Linear programming is another Operations Research topic with widespread applications especially for determining investment strategies. Consider an analysis for a new interest sensitive product. Here we consider three possible investments: a 5 year bond, a 15 year bond or a 30 year mortgage with three possible interest rate scenarios.

SURPLUS POSITION (30 YEARS)
PER $1 OF INITIAL INVESTMENT

	Initial Investment			
Scenario	5 Year Bond	15 Year Bond	30 Year Mortgage	81% 5 Yr. and 19% 15 Yr. Bonds
1	5.85	5.85	5.85	5.85
2	13.92	5.78	7.30	12.37
3	-.33	1.41	-2.20	0.00

This represents a simple type of C-3 risk analysis. For each combination of investment and scenario we compute the surplus position. In this case, it happens to be at the end of 30 years. All results are computed per dollar of initial investment.

Suppose that we would like to find a good combination of investments and we would like not to have to do a lot of trial and error analysis. In particular, we would like to investigate an initial investment strategy whereby we do not lose any money under any scenario. Since the surplus numbers are expressed per dollar of initial investment, an initial investment strategy which is a combination of the investments shown will be a linear combination of those surplus results. For example, an initial investment strategy that places 30% of the initial investment in five year bonds, 30% in 15 year bonds and 40% in 30 year mortgages will produce a surplus at the end of 30 years under scenario number two of 30% of $13.92 plus 30% of $5.78 plus 40% of $7.30. This totals to $8.83 per dollar of initial investment. This problem can be formulated as a linear programming problem. Note that the only scenario in which there is any chance of losing money is the third scenario.

Let X_5 = Fraction Invested in 5 Yr. Bond
X_{15} = Fraction Invested in 15 Yr. Bond
X_{30} = Fraction Invested in 30 Yr. Mortgage

Then we want:

(1) $X_5 + X_{15} + X_{30} = 1$

(2) Surplus Under Scenario # 3 $\geq$ 0, or
$-.33 \cdot X_5 + 1.41 \cdot X_{15} - 2.20 \cdot X_{30} \geq 0$

We can come up with some expressions to impose the realistic constraints and also the goals we are trying to achieve. For example, we want the sum of our investments to equal the whole which gives rise to the first constraint. We also want the surplus under each scenario to be equal to or greater than zero. As I pointed out, we only have to worry about that in scenario number three. So the second expression constrains our surplus under scenario number three to be equal to or greater than zero.

This begins to look like a linear programming problem.

GENERAL LINEAR PROGRAMMING PROBLEM

1. Find $X_1, X_2, \ldots X_N$ such that:

$$C_1 \cdot X_1 + C_2 \cdot X_2 + \ldots C_N \cdot X_N \text{ is Minimized}$$

Subject to a Set of Linear Constraints, Such as:

$$\begin{array}{l}
a_{1,1} \cdot X_1 + a_{1,2} \cdot X_2 + \ldots a_{1,N} \cdot X_N \geq b_1 \\
a_{2,1} \cdot X_1 + a_{2,2} \cdot X_2 + \ldots a_{2,N} \cdot X_N \geq b_2 \\
\vdots \\
a_{m,1} \cdot X_1 + a_{m,2} \cdot X_2 + \ldots a_{m,N} \cdot X_N \geq b_m
\end{array}$$

$$X_i \geq 0 \quad i = 1,2, \ldots N$$

2. In Equivalent Notation:

$$\text{Min } \overline{C} \cdot \overline{X}$$

$$\text{Subject to } A \cdot \overline{X} \geq \overline{b}$$

We have N decision variables here, X_1, X_2 to X_N, such that we want some linear combination of those decision variables to be minimized subject to a number of linear constraints. The linear expression that we are trying to minimize is called the objective function. The linear expressions with the inequalities are referred to as the constraints. For those of you who are more comfortable thinking in terms of matrix notation, a shorthand statement of the same problem is given as number 2 above.

Linear programming techniques allow us to solve these types of problems. The investment strategy problem that we are trying to solve begins to look like a simple linear programming problem with three decision variables. In the paper, "The Matching of Assets and Liabilities", in the 1980 Transactions, Jim Tilley presented the framework for solving these types of problems by formulating the objective function and the constraints as a linear programming problem. He also provided the software for solving the problem. Using the techniques from that paper, we find that there is an investment strategy that meets our goals. An initial investment strategy that places 81% of our initial investment in five year bonds and 19% in 15 year bonds will not lose money under any of the three scenarios and, in particular, under scenario number three.

Now, this problem may seem fairly simple and you could probably do it without some knowledge of Operations Research techniques. However, if the number of scenarios were more realistic and if the number of possible investments were much greater, the problem would not be a trivial one to do by hand and you would definitely want some software capabilities to handle it. For the record, if these scenarios were such that every investment strategy entailed a loss under at least one scenario then we could still use a linear programming approach where we would alter the objective function so as to use maxi-min criteria. All this means is that we could select the investment strategy which minimized our loss assuming that the worst possible scenario actually happened.

Another popular application of linear programming is the construction of dedicated portfolios. Dedicated portfolios are designed to match the cash flows from a set of assets against a specified set of liabilities. Pension plan sponsors have used dedicated portfolios to take care of projected annuity payments for a block of retired lives. Some insurance companies have used dedicated portfolios to match their assets with their GIC liabilities and/or structured settlements. Consider the following simple example where we have a one year bond and a two year bond. The one year par bond has a 12% annual coupon and the two year par bond has a 13% annual coupon. We know that for any amount that we invest in the one year bond we will get back 112% of our investment at the end of one year. We know that for any amount we choose to invest in the two year bond at the end of one year we will get back 13% of our initial investment and at the end of two years we will get back 113% of our initial investment.

Let:

X_1 = Amount to be Invested in 1 Year Bonds

X_2 = Amount to be Invested in 2 Year Bonds

Then,

Asset Flow in 1 Year $= 1.12 \cdot X_1 + .13 \cdot X_2$

Asset Flow in 2 Years $= 1.13 \cdot X_2$

Cost of Portfolio $= X_1 + X_2$

If we define X_1 and X_2 as the amounts to be invested in one and two year bonds, respectively, then we can compute expressions for the amount of money we will get back at the end of one year and also at the end of two years. Those expressions are above. We also know that the cost of our portfolio would be the sum of the two amounts invested. Well, suppose we want to match our asset flows from a portfolio against the following liability flows.

different. Hopefully the AIM triangle can agree on some graph that they feel represents a reasonable compromise and suits their objectives. That graph would represent some compromise of design and investment strategy features.

Another use of simulation arises with pension plan sponsor's asset allocation decisions. Simulation can be used to show the pension plan sponsor how effectively a synthetic put option strategy can be applied to portfolios of stocks and bonds. At least one large eastern company is marketing a synthetic put option strategy now on stock portfolios for pension funds. Simulation can also be used to help a pension plan sponsor decide how to allocate money between stocks, bonds, real estate, etc. The simulation can project a random scenario separately for each of several different types of investments while still allowing for appropriate levels of correlation among the returns. At least one consulting firm has marketed this service for pension funds in recent years.

I close my remarks by touching briefly on one more Operations Research topic. You may already be familiar with project scheduling. Any of you that have already worked on a big project involving many operating areas of the company have probably already seen a PERT chart. (See Figure 4.) In this PERT chart, the arrows represent activities that need to be performed and also the order in which certain activities need to be performed. Projects such as Whole Life policy enhancement programs, introduction of new products and EDP design and implementation schedules are ideal PERT chart applications. Also by using suggested techniques, one can determine how to efficiently speed up a project by scheduling overtime, extra personnel, etc.

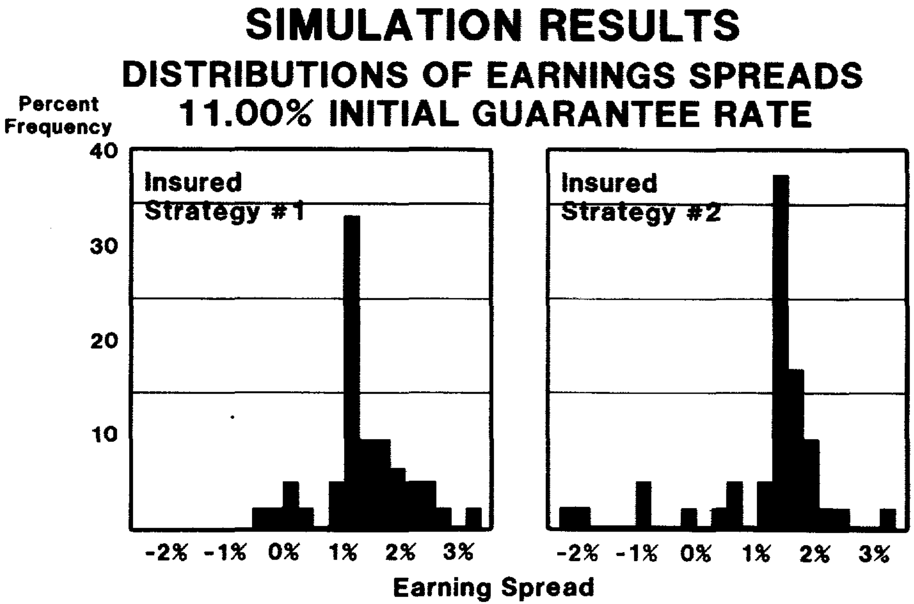
Well, I hope that I have mentioned something which aroused your curiosity. If I have mentioned some Operations Research application that you now want to ask your students to investigate in greater detail, then I have accomplished everything that I had hoped for.

MR. HOLLAND: We would like to take a few minutes now for questions and answers or for you to talk about practical applications of any of these topics in your practice.

MR. JOHN THOMPSON: I was interested in Mr. Robbins' mention of the application of sampling and modeling techniques in determining the aggregate deferred premium. It has been some time since I was involved in this kind of valuation problem, but, say thirty years ago, our approach was to classify annualized premiums with respect to anniversary month and mode of premium payment. That gives you directly the aggregate deferred premiums. Why should sampling techniques be necessary or desirable?

MR. ROBBINS: It really depends on the error found. If an error of principle has been found in the calculation of the deferred premiums, it can be any type of error. I think you are saying that as long as policies are classified as to anniversary and frequency and you have the annualized premium, you can get deferred premiums directly if the totals that you are working with are correct. But if they are not correct, you have to go back into the degree of incorrectness and the source of the incorrectness and determine the cause of the problem and make an estimate of the error.

Figure 3


SIMULATION RESULTS
DISTRIBUTIONS OF EARNINGS SPREADS
11.00% INITIAL GUARANTEE RATE
Percent Frequency
40
30
20
10
Insured Strategy #1
Insured Strategy #2
-2% -1% 0% 1% 2% 3%
-2% -1% 0% 1% 2% 3%
Earning Spread

Year	Liability Flow
1	\$1.38 Million
2	\$2.26 Million

Then we want: $\text{Min } X_1 + X_2$

Subject to: $1.12 \cdot X_1 + .13 \cdot X_2 \geq 1{,}380{,}000$

$$1.13 \cdot X_2 \geq 2{,}260{,}000$$

Then:

$$X_2 = \$2{,}000{,}000$$
$$X_1 = \$1{,}000{,}000$$

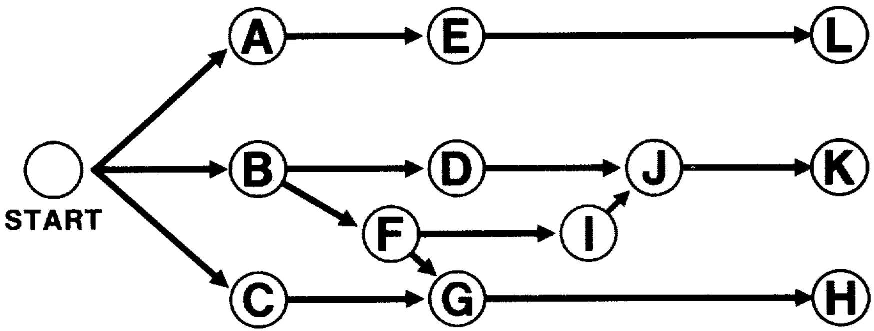
We would like to be able to pay off a liability of \$1.38 million at the end of one year and \$2.26 million at the end of two years. We would like to find the cheapest portfolio which will generate at least \$1.38 million of cash at the end of one year and \$2.26 million of cash at the end of two years. Well, that gives rise to the linear programming problem shown above. In solving this problem, we find that a \$2 million investment in two year bonds and a \$1 million investment in one year bonds does the trick.

Now this is an extremely simple problem and you may have been able to do it in your heads. But if we were trying to match a large number of liability flows and if we were looking at a large number of possible investments then the problem is no longer trivial. Again, you would want a linear programming software package to handle it. Even for a more complicated problem the format would be exactly the same. We would want to find the portfolio that has the least cost which will generate cash flows of at least the specified amounts at each specified point in time.

Another topic of growing interest in this era of interest sensitive products is the use of simulation techniques. Many problems, including actuarial ones, do not have readily available analytical solutions. Simulation models can help provide a solution in such circumstances. In addition, rapid advances in computer science in recent years have further enhanced the usefulness of simulation models. At the New York meeting last spring, we heard time and time again that simulation was an important part of the product pricing, product design and investment strategy process for developing interest sensitive products. Simulation provides important input to the AIM triangle where AIM stands for Actuarial, Investment and Marketing. In order to do a simulation for an interest sensitive product, we will want an interest rate model. One will probably want to use some time series or regression analysis similar to those described here to estimate the form and parameters of the model. The results of a set of simulations can be summarized in chart or graph form. (See Figure 3.)

Shown in Figure 3 are the summarized results for two synthetic option strategies supporting a single premium deferred annuity product. I do not want to get into synthetic option strategies, but let me say that if we were to change the investment strategy, or change the interest guarantee, or change some of the product design features we would end up with graphs that look

PERT CHART



MR. HOLLAND: I think a possible situation would be if a plan was misclassified. It was listed as an annual frequency and it actually could be a monthly premium. You want to find the error and straighten it out.

MR. BARRY SAVAGE: Again, a question for Mr. Robbins. It seems the best possible sample is a 100% sample. Ultimately, by setting up our systems to run on modern electronic systems, will all our data be such that we cannot always sample 100%? Moreover, the cost in extending the sample is essentially negligible. Why, then, would we want sampling techniques?

MR. ROBBINS: No, I agree with you, as long as machine time is fast enough and you have the proper data base available to you. Our firm has programs that use a company's in force and its reserve factors; we run these programs independently to get a reserve on 100% basis. Once you have evolved to that stage a 100% sample is the best.

I want to give you a slight aside here. I recently was involved with a client that had a $4 million additional reserve for his health policies. This client's health portfolio happens to be a closed block of business. In trying to do a reserve recomputation for tax purposes for this client, this $4 million block is comparable to a $400 million life reserve. This relatively little additional reserve consists of thousands of plan codes, a small yet typically complex health portfolio. Do you really go into an exact calculation according to the tax act specification or do you do something simpler on a sampling basis? A 100% sample is hard to do in a situation like that. I am suggesting that we do a model. We want to get a ratio of the tax reserve to the statutory reserve from the model and then gross it up to the population.

MR. HICKMAN: It may be that sampling will play a somewhat different role in the future. Suppose you have the audit job. Now, if you have one of these complete record keeping systems the audit is probably going to be achieved by having a "sample deck" of oddball cases to see how the system works. You don't worry about multiplication and you don't worry about the addition. What you are worried about is whether the system is really doing the job it is supposed to do. The sample does not look at the transactions that have occurred, but instead is a "sample deck" that includes all the strange cases that you can imagine. You use the sample to audit how the system really works. The design of that sample or test "deck" has many of the characteristics of the sampling schemes that Ed talked about, but the purpose is somewhat different. It is not so much to verify a number as it is to test the system. I think that the auditor could not possibly duplicate all of the possible transactions of your company. Rather, the auditor would have a sample that he hopes will plumb the depths of what the system should be able to do. It would be impossible in any good size company for the auditor to duplicate all the systems design work. All that he can do is use a test "deck" and the results from that test "deck" will largely determine how good the system is. I think that sampling will still have a role, but somewhat differently than what was done at your company back in the '40's. We no longer do it that way because we now can do a complete inventory at the same cost that it used to take for a sample, although we still will have to do the system check.

MR. HOLLAND: Another comment is that there is also an element of professional responsibility. We say that it is good to do a 100% sample. But a 100% sample may not be feasible for the individual who signs his name certifying that the reserves are good and sufficient or for a consulting actuary who comes in to look over the work that has been done. It may not be economically feasible for the individual based on the time constraints to do a 100% sample of all the reserve calculations. A chief actuary in a large company may have hundreds of people involved in putting the reserves together. He needs to be sure that based on his professional responsibility that the reserves have not only been calculated correctly, but that they are the correct reserves. I think that you can discover a lot of things by sampling or by using various other techniques such as regression analysis to see how a reserve would be expected to progress for a certain block. Then, as senior actuary, you can look at the results of the sample or of the regression and say "This is really unusual", or "This is more of a deviation than I would have expected. So let's go in and audit this particular block very carefully." There are a lot of statistical tools that will help you even if all of the work is done on a 100% basis.

MR. HICKMAN: Let me make a historical comment. Bob gave you a little of the history of matching and immunization ideas and one of his illustrations involved the use of linear programming. For those of you who were intrigued by what Bob said, you might want to look at the Transactions of the Faculty of Actuaries. Two men named Haynes and Kirton had substantial ideas on using linear programming for investment matching, but they don't receive much credit for them. (Haynes, A. T. and Kirton, R. J., "The Structures of a Life Office", Transactions of the Faculty of Actuaries, Vol. 21 (1933)). It is a fascinating paper.

FROM THE FLOOR: As a professor from the University of Massachusetts, I am fascinated by this from an educational point of view. I am trying to develop an actuarial science program at the University and I have just recently gone through the Part 3 syllabus. I found it very mathematical and similar to the things I saw in graduate school many years ago. Have there been any efforts made, or are there plans to develop some study notes pointing out all these applications to the students who are studying that Part 3 material? I think that would be a dramatic improvement, if it hasn't already been done.

MR. HOLLAND: That is an excellent suggestion. Speaking on behalf of the Education and Examination Committee, we really do want to communicate and find a way to put potential applications into the syllabus. I hope that today's record will be a part of it. In fact, communicating the practicality of the topics was part of the planning process when we upgraded or revised the Operations Research portion of the syllabus and added Applied Statistics. We have had a group of people working on a study note of practical actuarial applications. We have got a big job ahead of us in terms of communicating the real importance these topics can have to actuaries and we certainly plan to pursue it.

MR. HICKMAN: I would never want to argue against pointing out applications. As a matter of fact, I will argue in favor of it. But I want to make sure that you understand that when you take students that have never worked in an insurance company, as many Part 2 level students have not, the issues of immunization and of the estimation of deferred premiums don't grab them. To us, those are important matters. To them, those words don't mean much, yet. This is a hard chore and I would never deter the Education and Examination Committee to undertake it. It is worth doing, but it is hard. It is hard for people who have only been in college, who haven't had the experience, to understand some of those practical issues that pique our interest.

MR. HOLLAND: I agree. I think that we want to talk about where on the syllabus the practical aspects belong. The syllabus, to some extent, is devoted to the theory in the early exams, particularly Associateship exams, and to practice related issues in the Fellowship exams. We want to bring our actuaries along in their development. It may be that the applications will appear in a later exam than the theory.

MR. DAVE WILLIAMS: I think we are all familiar with the widely used econometric indicators. Considering the leading indicators which are supposed to foreshadow changes in the economy I am surprised that there has apparently been no statistical study to determine how valid they are, how relatively important they are, or the precision with which they do foretell what is going to take place in the economy in the next 6 to 12 months. Are you aware of any OR or statistical studies which show just how valid the leading economic indicators are?

MR. HICKMAN: Yes, there have been such studies. In the United States, the Department of Commerce regularly publishes an index of leading indicators. Now, what is a leading indicator? What you are looking for in time series terms are current values at time t that will help you forecast those at time t + 1. Victor Zurnowitz from the University of Chicago, while at the Commerce Department, was largely responsible for the selection and development of that set of leading indicators. There have been many studies of their effectiveness. Levels and trends are fairly easy to forecast, but, the timing of turning points is very difficult. There have been evaluations of the index of leading indicators. The Bureau of the Department of Commerce does have a continuous project to not only monitor how well that index does, but also to look for other elements that might become one of the indicators, and perhaps to delete some of the indicators that are already there.

MR. HOLLAND: I have a question that has to do with practical applications of queuing theory. When we set the Part 3 exam, we have recently been trying to come up with realistic scenarios. We understand that our multiple choice testing format does not test Applied Statistics the way you might if you had a lab and computers available. There are interesting questions that we can ask. For instance, in analysis of variance we frequently ask, "Is there a difference between your northeast sales production and southeast, mid-west, or western states?" You are developing a portfolio and you have an agency that produces 100% nonsmokers and you ask, "are they awfully efficient in finding nonsmokers?" Do you want to investigate that? But, on the other hand, you might find that a particular region is developing 75% nonsmokers or 80% or 85%. Where do you investigate? When do you want to determine if there is misrepresentation of some of the pertinent underwriting information that is being collected? So, we can ask practical questions on the syllabus on analysis of variance. We also can come up with fairly interesting Operations Research questions from time to time that are not formulas. But, when it comes to queuing theory, we always ask about car washes and that is a rather strange actuarial application and I was wondering how queuing theory can be applied in the insurance area, Bob?

MR. CLANCY: I did, in the interest of time, leave out examples of some Operations Research topics. One of those is queuing theory. Queuing theory is everything that you ever wanted to know about lines. One application for queuing theory would be in deciding on the number of clerks or underwriters, given that you do not want the number of applications building up above a certain level. One other application that I left out entirely is dynamic programming. That has a very wide range of applications also, but the examples tend to get fairly complicated. I left it out because of the complicated examples, not because it was an inappropriate topic for discussion.

MR. HICKMAN: This is perhaps the kind of thing that is difficult to bring up in an exam, but we all should recognize it. There is no intellectual difference between queuing theory and collective risk theory. If you replace the random number of claims in a fixed time period, with the random number of arrivals in a fixed time period and replace the service time with claim amount, you will see that they are exactly the same model. It is one of those amazing coincidences that early in the century, separated by the Skayerrak and Katteyat, Lundberg was doing collective risk theory and Erlang was doing queuing theory. Erlang was a telephone engineer and Lundberg was an actuary and they were really doing exactly the same thing. It took the rest of us 50 years to figure out that the two subjects were the same. The interrelationship between collective risk theory and queuing theory is an important one and should, I think, be used to reinforce each other.

Although there are some technical differences, if you start out at the end of the branches of a decision tree diagram and roll back by taking either the expected values or the maximizations, the process is essentially a dynamic programming technique. Perhaps the most important single application of dynamic programming in actuarial science is that it provides a systematic way to "solve" a tree.

MR. HOLLAND: Thank you for pointing out the bridge between queuing theory and risk theory. Our time is just about up. I would like to thank our panelists for their participation and thank you for your participation.

EXHIBIT 1

Comparison of
Variance of Sampling Estimates

Random Sample (n independent elements out of a population of N). Estimate = $\sum_{i=1}^{n} \frac{x_i}{n}$

vs.

Proportionate Stratified Sampling (h strata of n_h independent elements each ;
Estimate = $\sum_h \frac{n_h}{n} \sum_{i=1}^{n_h} \frac{x_i}{n_h} = E\,\bar{x}_h$

Theorem: In such case, the degree of improvement in the sampling estimate is proportionate to the variance of the stratum sample means about the total sample mean.

Standard error of Random Sampling estimate = $\frac{1}{n} Var(x_i)$ (1)

Standard error of Stratified Sampling estimate =

$$Var\left(\sum_h \bar{x}_h \frac{n_h}{n}\right)$$

$$= \sum_h Var\ \bar{x}_h \frac{n_h}{n} = \sum_h \left(\frac{n_h}{n}\right)^2 Var\left(\sum_{i=1}^{n_h} \frac{{}_hx_i}{n_h}\right)$$

$$= \sum_h \left(\frac{n_h}{n}\right)^2 \left(\frac{n_h}{n_h^2}\right) Var({}_hx_i) = \frac{1}{n} \sum_h \frac{n_h}{n} Var({}_hx_i) \quad (2)$$

$$(1) - (2) = \frac{1}{n} Var\ x_i - \frac{1}{n} \sum_h \frac{n_h}{n} Var({}_hx_i)$$

$$= \frac{1}{n}\left[E\,x_i^2 - (E\,x_i)^2 - \sum_h \frac{n_h}{n}\left(\sum_{i=1}^{n_h} \frac{{}_hx_i^2}{n_h} - \left(\sum_{i=1}^{n_h} \frac{{}_hx_i}{n_h}\right)^2\right)\right]$$

$$= \frac{1}{n}\left[E\,x_i^2 - (E\,x_i)^2 - \sum_{i=1}^{n} \frac{x_i^2}{n} + \sum_h \frac{n_h}{n}\left(\sum_{i=1}^{n_h} \frac{{}_hx_i}{n_h}\right)^2\right]$$

$$= \frac{1}{n}\left[\sum_h \frac{n_h}{n}\left(\sum_{i=1}^{n_h} \frac{{}_hx_i}{n_h}\right)^2 - (E\,\bar{x}_h)^2\right]$$

$$= \frac{1}{n}\left[E(\bar{x}_h^2) - (E\,\bar{x}_h)^2\right]. \qquad Q.E.D.$$

EXHIBIT 2

Theorem: Optimum sample size for stratum h (n_h) varies directly with stratum size and stratum variance.

Proof: Define Variance of the estimate as Var ($_n\bar{x}_{st}$). Where x_i independent.

and modify it via: - the Finite Population Correction $\left(\frac{N_h - n_h}{N_h}\right)$

- LaGrange Multiplier, λ, applied to a zero term

$$Var({}_n\bar{x}_{st}) = Var\left(\sum_h \frac{N_h}{N}\bar{x}_h\right) = \sum_h \left(\frac{N_h}{N}\right)^2 Var(\bar{x}_h)$$

$$= \sum_h \left(\frac{N_h}{N}\right)^2 \left(\frac{S_h^2}{n_h}\right)\left(\frac{N_h - n_h}{N_h}\right), \quad where \quad S_h^2 = Var({}_h x_i).$$

$$= \frac{1}{N^2} \sum_h N_h (N_h - n_h) \frac{S^2}{n_h}.$$

$$Thus, \; Var({}_n\bar{x}_{st}) = \frac{1}{N^2} \sum_h \left(\frac{N_h^2}{n_h} - N_h\right) S_h^2 + \lambda\left(\sum_h n_h - n\right).$$

$$\frac{d\,Var({}_n\bar{x}_{st})}{d\,n_h} = -\frac{N_h^2 \, S_h^2}{N^2 \, n_h^2} + \lambda = 0. \qquad \therefore n_h = \frac{N_h \, S_h}{N\sqrt{\lambda}}$$

$$But \quad \sum_h n_h = n = \sum_h \frac{N_h \, S_h}{N\sqrt{\lambda}} \qquad \therefore \lambda = \sum_h \frac{N_h \, S_h}{n \, N}$$

$$\therefore n_h = \frac{N_h \, S_h}{N \sum_h \frac{N_h \, S_h}{n \, N}} = n\left[\frac{N_h \, S_h}{\sum_h N_h \, S_h}\right]. \qquad Q.E.D.$$

Source: "Sampling Techniques," W. A. Cochran; J. Wiley & Sons, Inc. Copyright 1953.

EXHIBIT 3

$$\text{Cost Function: } CF = Var({}_n\bar{x}_{st}) + \lambda\left(a + \sum_h c_h n_h\right).$$

$$CF = \frac{1}{N^2}\sum_h\left(\frac{N_h^2}{n_h} - N_h\right)S_h^2 + \lambda\left(a + \sum_h c_h n_h\right).$$

$$\frac{dCF}{dn_h} = 0 = -\frac{1}{N^2}\frac{N_h^2 S_h^2}{n_h^2} + \lambda c_h. \quad \therefore n_h = \frac{N_h S_h}{N\sqrt{\lambda c_h}}$$

$$\text{But } n = \sum_h n_h = \frac{1}{N}\sum_h \frac{N_h S_h}{\sqrt{\lambda c_h}}. \qquad \therefore \sqrt{\lambda} = \frac{1}{n}\sum_h \frac{N_h S_h}{N\sqrt{c_h}}$$

$$\therefore n_h = n\left[\frac{\dfrac{N_h S_h}{\sqrt{c_h}}}{\sum_h \dfrac{N_h S_h}{\sqrt{c_h}}}\right]$$

and n_h is proportionate to $\dfrac{N_h S_h}{\sqrt{C_h}}$.

Ibid, pp. 75, 76.

EXHIBIT 4

A. Obtaining Sample Size, n, if you know your total Out-of-pocket cost.

Total Out-of-Pocket Cost = C = a + $\sum C_h n_h$.

$$C = a + \left[\sum_h C_h n \frac{N_h S_h}{\sqrt{C_h}} \div \sum_h \frac{N_h S_h}{\sqrt{C_h}} \right]$$

$$\therefore n = (C - a) \sum_h \frac{N_h S_h}{\sqrt{C_h}} \div \sum_h \sqrt{C_h}\, N_h S_h$$

B. Obtaining Sample Size n if the tolerable Variance of your estimator is a given item:

$$Var({}_n\bar{x}_{sr}) = \frac{1}{N^2} \sum_h N_h (N_h - n_h) \frac{S_h^2}{n_h}$$

$$= \frac{1}{N^2} \sum_h N_h \left(N_h - n \left[\frac{\frac{N_h S_h}{\sqrt{C_h}}}{\sum_h \frac{N_h S_h}{\sqrt{C_h}}} \right] \right) \cdot \frac{S_h^2}{n} \frac{\sum_h \frac{N_h S_h}{\sqrt{C_h}}}{\frac{N_h S_h}{\sqrt{C_h}}}$$

$$= \sum_h \frac{S_h^2 N_h^2 \sqrt{C_h} \sum_h \frac{N_h S_h}{\sqrt{C_h}}}{n N_h S_h N^2} - \frac{N_h S_h^2}{N^2}$$

$$= \frac{1}{n} \sum_h \frac{S_h N_h \sqrt{C_h}}{N^2} \sum_h \frac{N_h S_h}{\sqrt{C_h}} - \frac{N_h S_h^2}{N^2}$$

$$\therefore \quad n = \frac{\frac{1}{N^2} \sum_h S_h N_h \sqrt{C_h} \sum_h \frac{N_h S_h}{\sqrt{C_h}}}{Var({}_n\bar{x}_{sr}) + \frac{N_h}{N^2} S_h^2}$$

Ibid, p. 76.