Pricing Risk through Simulation: Revisiting Tilley
Bundling and Least Squares Monte Carlo Methods
EXTERNAL FACTORS:

Regulatory Environment
Government and judicial context
Physical Environment
Economic and Social Environment
Industry and Business Environment
Shepheard (2006)
Regulatory Environment
Government and judicial context
Physical Environment
Economic and Social Environment
Industry and Business Environment
Sheperd (2006)
Professionalism
**Simulation Models for Derivatives with early exercise features**

"Monte Carlo simulation can only be used for European-style options." (Hull, 1993 as cited by Fu et al, 2001)

"The goal of this paper is to dispel the prevailing belief that American style options cannot be valued efficiently in a simulation model." (Tilley, 1993)

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**Now**

Risk Metrics
This presentation: Expected Positive Exposure, Potential Future Exposure through Tilley Bundling and LSM

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**Development of Monte Carlo techniques**

Longstaff and Shwartz develop a simple least-squares approach. (Longstaff and Shwartz, 2001)

"Monte Carlo simulation is well suited to valuing path-dependent options ..." (Hull, 2012)

Fu et al (2001) consider 3 approaches
Simulation Models for Derivatives with early exercise features

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Fu et al (2001) consider 3 approaches:
- mimic backwards induction algorithm
- parametrize early exercise curve
- finding efficient upper and lower bounds
- mimic backwards induction algorithm
- parametrize early exercise curve
- finding efficient upper and lower bounds
Risk Metrics
This presentation: Expected Positive Exposure, Potential Future Exposure through Tilley Bundling and LSM
- Basel II/III - Defines EPE, PFE
- Technological progress
- Convergence of financial institutions: risk still there when in the money
Expected

Expected Positive

97.7%
Euro-Call
So = K = 100
Volatility = 20%
Drift = 6%
No dividends
1 year
PFE, EPE?

PFE = 65.0075
EPE = 18.9229
\[ \int_{0}^{e^{0.06 PFE_{0.99} + 100}} \frac{1}{x \sigma \sqrt{2\pi}} \exp \left[ -\frac{\left( \log(x) - \log(100) - \left( \mu - \frac{\sigma^2}{2} \right) \right)^2}{2\sigma^2} \right] \, dx = 0.99 \]

\[ EPE = e^{-0.06} \left( \frac{\int_{100}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{\left( \log(x) - \log(100) - \left( \mu - \frac{\sigma^2}{2} \right) \right)^2}{2\sigma^2} \right] \, dx}{\int_{100}^{\infty} \frac{1}{x \sigma \sqrt{2\pi}} \exp \left[ -\frac{\left( \log(x) - \log(100) - \left( \mu - \frac{\sigma^2}{2} \right) \right)^2}{2\sigma^2} \right] \, dx} - 100 \right) \]
PFE = 65.0075
EPE = 18.9229
American Put Option
Strike = Initial Price = 100  drift = 0.06
Volatility 40%
Period: 1 year
100k paths
50 exercise pts/year
MINIMUM SPEED

Metrics for a European Option ending at time $t$
So = K = 100
Volatility = 40%
Drift = 6%
No dividends
5 year
50 exercise times/ year

\[
\sum_{\text{all times}} f_2(x, t, 100 - \text{PFE} \times 0.0125) = 0.01(51 - K)
\]

\[
\sum_{\text{all times}} \frac{0.00225 f_2(x, t, 100) (100 - f_2(x, t, 100))}{(K - x)} \sum_{\text{all times}} f_2(x, t, 100)
\]
So = K = 100
Volatility = 40%
Drift = 6%
No dividends
5 year
50 exercise times/ year
\[ F_2(x, t, a) = \int_0^a \frac{\exp \left[ -\frac{(\log(x) - \log(S_0) - (\mu - \frac{\sigma^2}{2}) \frac{t}{50})^2}{\sigma^2 \frac{t}{25}} \right]}{x \sigma \sqrt{\pi \frac{t}{25}}} \, dx \]

\[ F_3(x, t, a) = \int_0^a \frac{\exp \left[ -\frac{(\log(x) - \log(S_0) - (\mu - \frac{\sigma^2}{2}) \frac{t}{50})^2}{\sigma^2 \frac{t}{25}} \right]}{\sigma \sqrt{\pi \frac{t}{25}}} \, dx \]

\[ \sum_{t=k}^{50} F_2(x, t, 100 - PFE_{0.99}e^{0.0012t}) = 0.01(51 - k) \]

\[ \text{EPE}_{k^{th \ time}} = \frac{\sum_{t=k}^{50} e^{-0.0012t} F_2(x, t, 100) \left( 100 - \frac{F_3(x,t,100)}{F_2(x,t,100)} \right)}{(51 - k) \sum_{t=k}^{50} F_2(x, t, 100)} \]
LEAST SQUARES METHOD

Find paths that are in-the-money at time $t$.

$t = t - 1$

Calculate $EPE_t$ and $PFE_t$ of $FV_t$ and discount by $\frac{1}{(1+r)^t}$

Obtain $e^{-dlT}FV_{t+1}$ for these paths

if $PV_t > EFV_t$, then $FV_t = PV_t$
else $FV_t = e^{-dlT}FV_{t+1}$
end

Build a regression function to determine $EFV_t$. 

$S_{25}$ $e^{-dlT}FV_{25}$
98.12 17.45
64.38 31.21
69.27 33.47
80.24 22.91
83.09 28.66
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<thead>
<tr>
<th>$S_{25}$</th>
<th>98.12</th>
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<tbody>
<tr>
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<td>144.72</td>
</tr>
<tr>
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<td>131.41</td>
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<td></td>
<td>113.38</td>
</tr>
<tr>
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<td>54.38</td>
</tr>
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</table>

Find paths that are in-the-money at time $t$.
Obtain $e^{\Delta t r} F_{V_{t+1}}$ for these paths

<table>
<thead>
<tr>
<th>$S_{25}$</th>
<th>$e^{-0.0012}$</th>
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<tbody>
<tr>
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<td>22.91</td>
<td></td>
</tr>
<tr>
<td>83.09</td>
<td>28.66</td>
<td></td>
</tr>
</tbody>
</table>
Build a regression function to determine $\text{EFV}_t$. 

$Y = 0.000009195S^3_{25} - 0.0135S^2_{25} - 0.3565S_{25} + 90.1722$
if $PV_t > EFV_t$
then $FV_t = PV_t$
else $FV_t = e^{-dtr} FV_{t+1}$
end

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<tr>
<th>$S_{25}$</th>
<th>$PV_{25}$</th>
<th>$EFV_{25}$</th>
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<td>54.38</td>
<td>45.62</td>
<td>45.49</td>
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</table>

$|PV_{25}| FV_{25}$

| 1.88 | 17.45 |
| 0.00 | 31.21 |
| 0.00 | 33.47 |
| 0.00 | 22.91 |
| 45.62| 45.62 |
Calculate $EPE_t$ and $PFE_t$ of $FV_t$ and discount by $+dlt$
Calculate $EPE_t$ and $PFE_t$ of $FV_t$ and discount by $LDA$.
Tilley Bundling

Calculate EPE and PFE of Z*PV discounted to PV

Order S_i in descending (ascending) order of value for a put (call) option

Find the earliest exercise point past + for each path

Z - matrix with at most one 1 for each path

if PV_i < PV_{\text{max}}
then FV = H_i
else FV = PV_i
end

if PV_i > H_i
then X_i = 1
else X_i = 0
end

Find Sharp Boundary
92,508th Value

H_i = Bundle mean of e^{-\alpha t_i} FV_i
H_i range: 0.0316 to 56.6057

Divide into bundles with equal paths. Ideally # of Bundles = # of paths
250 bundles of 400 paths
Order $S_t$ in descending (ascending) order of value for a put (call) option
Divide into bundles with equal paths. Ideally \# of Bundles = \# of paths

250 bundles of 400 paths
\[ H_t = \text{Bundle mean of } e^{-dtr} FV_t \]

\[ H_{25} \text{ range: 0.0316 to 56.6057} \]
if $PV_t > H_t$
then $X_t = 1$
else $X_t = 0$
end
Find Sharp Boundary

92,508th Value
if $PV_t < PV_{\text{sharp}}$
then $FV_t = H_t$
else $FV_t = PV_t$
end

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<th>Rank</th>
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<tr>
<td>92510</td>
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<td>33.6250</td>
</tr>
</tbody>
</table>
Find the earliest exercise point past $t$ for each path $Z$ - matrix with at most one 1 for each path
Calculate EPE and PFE of Z*PV discounted to PV
Initial Price = 100, Sigma = 0.40 and T = 1
Future Work:
Hybrids:
  Piecewise Regression
  Logistic Regression
Multi-Dimensional Sorting
  Tilley (K-NN)
Presenter:
Dominic Cortis

Research Co-Authors:
Rickard Branvall, Citigroup (London)
Juxi Li, University of Leicester (UK)
References:


