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PRICING ASIAN OPTIONS: CONVERGENCE OF GRAM-CHARLIER SERIES

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ABSTRACT. We study the theoretical and numerical convergence of Gram-Charlier series applied to the pricing of Asian options. The distribution of the logarithm of the average is represented as a Gram-Charlier expansion. Fairly precise results about the convergence of Gram-Charlier series are proved in the case where the underlying security is modelled as a geometric Brownian motion. Two other cases are studied, in one the log-price is modelled as a variance gamma process, and the other is the Heston stochastic volatility model. We show that convergence of the Gram-Charlier series holds in the geometric Browian motion case (under a specified condition), but that it is unlikely in the other two cases. Numerical examples are given; in some cases the series gives good results, while in others it does not do well.

1. INTRODUCTION

Average (or Asian) options have been studied for more than two decades, but there is still no efficient, accurate method to price them. We summarize the previous literature. Approximating the average by a single lognormal distribution, a classical technique borrowed from engineering, was the first technique suggested (Turnbull and Wakeman [22], Levy [16]). Simulation was proposed by Kemna and Vorst [13], who also noted that the geometric average can be used both as a lower bound for calls and as a control variate. Vazquez-Abad and Dufresne [23] and subsequent authors studied the use of a change of measure in simulations. In the case of a continuous averaging Rogers and Shi [20] expressed the problem in terms of a partial differential differential equation and derived sharp bounds for option prices. That partial differential equation may also be obtained by time reveral (Dufresne [5], Linetsky [17]). There is no simple analytical solution to this partial differential equation, it may be solved numerically or using series.

For the continuous average there is an explicit Laplace transform for Asian option prices (Geman and Yor [9]). Shaw [21] and others have successfully inverted the transform. Dufresne [7] showed that Asian option prices can be expressed a infinite series involving the Laguerre polynomials and the moments of the reciprocal average.

Closer to the topic of this paper, Edgeworth series for the distribution of the average were used to approximate Asian option prices by Turnbull and Wakeman [22]; those series were more precisely generalized Edgeworth series, with the lognormal acting as base distribution. Those series, that were conceived of at least as far back as Cramér [4], have virtually no established mathematical properties and have been shown to fail numerically in the case of Asian options (Lemieux [15]). The convergence of those Edgeworth series has never been proved, and their theoretical convergence is highly unlikely.

This paper focuses on another expansion, the Gram-Charlier, which will be shown to converge for some sets of parameters (see Sections 2 and 3 below), when (i) the underlying is

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a geometric Brownian motion and (ii) the Gram-Charlier expansion is applied to the logarithm of the average (not to the average itself). Theoretical convergence is a great advantage, as is apparent in the numerical performance of the technique; the downside of this technique is that the essential ingreadients are the moments of the logarithm of the average price, and those are not available in closed form. Those moments thus need to be estimated by simulation. It is nevertheless interesting to study those series, as subsequent research may yield quicker ways of computing the moments of the logarithm of the average; there are asymptotic formulas for those moments in Dufresne [7], that could possibly be used as approximations, though this is not attempted in this paper.

These Gram-Charlier expansions for Asian options were first proposed by Popovic & Goldsman [19], where interesting numerical examples are given but no study of convergence is performed. Our contribution lies in (a) studying the convergence of the Gram-Charlier series and proving convergence in some cases; (b) proving explicit formulas for the approximate option prices, and (c) questioning the usefulness of the method in cases where the underlying is not modelled as geometric Brownian motion (exponential of variance gamma process in Section 4, Heston stochastic volatility model in Section 5). For (a) we prove apparently new results regarding the domain of existence of the moment-generating function of $\log^2 A$ and give explicit formulas for the moments of the logarithm of a continuous average.

Gram-Charlier expansions have been used in different contexts in finance before, as an improvement on the normal distribution for log-returns in the Black-Scholes model. Corrado & Su [2] and Jondeau & Rockinger [12] are only two of the several papers written on this topic, a more recent one is Chateau & Dufresne [1], where a more exhaustive list of references may be found.

2. Gram-Charlier series

2.1. **Gram-Charlier series.** Gram-Charlier series have a long history, going back to Laplace at the beginning of the 19th century. The names of Thiele, Hermite Chebyshev and Cramér are associated with the more modern theory of Gram-Charlier series (specific historical details are given in Chateau & Dufresne [1]). Define the Hermite polynomials

$$\mathrm{He}_k(x) \;=\; (-1)^k \frac{d}{dx^k} e^{-\frac{x^2}{2}}, \quad k \;=\; 0, 1, 2, \ldots.$$

The first few are:

$$\mathrm{He}_0(x) = 1,\ \mathrm{He}_1(x) = x,\ \mathrm{He}_2(x) = x^2 - 1,\ \mathrm{He}_3(x) = x^3 - 3x.$$

The Hermite polynomials are orthogonal (with respect to the weight function $e^{-\frac{x^2}{2}}$) and have been used to express functions as series, *e.g.*

$$h(x) \;=\; \sum_{k=0}^{\infty} c_k \mathrm{He}_k(x).$$

Gram-Charlier series are expansions for probability density functions (PDFs) in terms of the normal PDF and Hermite polynomials:

$$g(x) \;= \phi(x) \sum_{k=0}^{\infty} c_k \mathrm{He}_k(x), \tag{1}$$

where

$$\phi(x) \;=\; \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

A Gram-Charlier series is thus a Hermite series times the standard normal density. Cramér [3] (see also [4]) proved the following result:

Theorem 1. *Suppose a function $g(\cdot)$ is of finite variation in every finte interval $[a,b]$, and satisfies the condition*

$$\int_{-\infty}^{\infty} |g(x)| e^{\frac{x^2}{4}}\, dx \;<\; \infty. \tag{2}$$

(Piecewise differentiable functions have finite variation.) Then, for every $x \in \mathbb{R}$,

$$\frac{1}{2}(g(x+) + g(x-)) \;=\; \phi(x) \sum_{k=0}^{\infty} c_k \mathrm{He}_k(x),$$

where

$$c_k \;=\; \frac{1}{k!} \int_{-\infty}^{\infty} g(x) \mathrm{He}_k(x)\, dx, \quad k = 0, 1, \ldots$$

In this paper the function $g(\cdot)$ is the PDF of some variable Y, so condition (2) is rewritten as

$$\mathbb{E} e^{\frac{Y^2}{4}} < \infty.$$

This sufficient condition cannot be improved upon in general, since there are cases where the Gram-Charlier series defined above diverges, even though $\mathbb{E}e^{pY^2} < \infty$ for all $p < \frac{1}{4}$ (one such case is the normal distribution with mean 0 and variance 2). The coefficients c_k are combinations of moments of Y:

$$c_k \;=\; \frac{1}{k!} \mathbb{E}\mathrm{He}_k(Y), \quad k = 0, 1, \ldots \tag{3}$$

(so $c_0 = 1$.) The finiteness of all those coefficients is no guarantee that the Gram-Charlier series converges.

Using Gram-Charlier series numerically requires truncating the series. The result is not necessarily a probability distribution, as it may become negative over some intervals. However, any truncated Gram-Charlier series

$$\phi(x) \sum_{k=0}^{N} c_k \mathrm{He}_k(x) \tag{4}$$

integrates to 1, since $c_0 = 1$ and

$$\int_{-\infty}^{\infty} \phi(x) \mathrm{He}_k(x)\, dx \;=\; 0, \quad k = 1, 2, \ldots$$

Chateau & Dufresne [1] study the *Gram-Charlier distributions*, which, for finite N, consist of the true probability distributions of the form (4). In this paper, by contrast, we consider the approximations obtained when a (hopefully convergent) Gram-Charlier series is truncated.

2.2. **Use of truncated Gram-Charlier expansions in pricing Asian options.** If log-returns have a normal distribution, as in the Black-Scholes model, it is natural to imagine that the average price of the stock has a distribution that is approximately lognormal. This is the same as saying that the logarithm of the average approximately has a normal distribution. This "lognormal approximation" of sums of lognormally distributed variables has a long history, notably in engineering. Ever since the lognormal approximation has been used to price Asian options it has been noted that the approximation is occasionally excellent but otherwise not always very good. Dufresne [8] studied this question in detail. Intuitively one may then think that the distribution of $\log A$ is not so different from a normal, and that multiplying the normal distribution by a polynomial would be an improvement. Popovic

& Goldsman [19] have looked at this case, as we do in Section 3, but the same idea may be tested in cases where log-returns do not have a normal distribution.

In all the models we consider, the distribution of the underlying security under the pricing (risk-neutral) measure $\mathbb{Q}$ is specified, and so the price of a European-style call on the average A with maturity T and strike K is

$$C_0 \;=\; e^{-rT}\mathbb{E}(A-K)_+,$$

where r is the risk-free rate of interest. (To simplify the notation, '"$\mathbb{E}$" denotes expectation under the risk-neutral measure, often written "$\mathbb{E}^{\mathbb{Q}}$" in the literature, since in this paper we do not need to refer to the physical measure). In each case the distribution of the logarithm of A is approximated by a truncated Gram-Charlier series; the Gram-Charlier distribution is given a location parameter a and scale parameter b, meaning that

$$X \;=\; \log A, \quad Y \;=\; \frac{X-a}{b}.$$

The PDF of Y is approximated by a truncated Gram-Charlier series:

$$\hat{f}_Y(y) \;=\; \phi(y)\sum_{k=0}^{N} c_k \mathrm{He}_k(y). \tag{5}$$

The approximated call price is then

$$\hat{C}_0 \;=\; \int_{\mathbb{R}} (e^{a+by}-K)_+ \hat{f}_Y(y)\,dy.$$

The advantage of the Gram-Charlier approximation is that this integral can be evaluated explicitly. The following computations are adapted from Chateau & Dufresne [1].

Theorem 2. *Under the assumptions above,*

$$\hat{C}_0 \;=\; e^{a+\frac{b^2}{2}-rT}\left(\tilde{c}_0\Phi(d_1)+\phi(d_1)\sum_{j=1}^{N}\tilde{c}_j\mathrm{He}_{j-1}(-d_1)\right)-Ke^{-rT}\left(\Phi(d_2)+\phi(d_2)\sum_{k=1}^{N}c_k\mathrm{He}_k(-d_2)\right),$$

where

$$\Phi(x) \;=\; \int_{-\infty}^{x}\phi(y)\,dy, \quad d_2 \;=\; \frac{a-\log K}{b}, \quad d_1 = d_2+b, \quad \tilde{c}_j \;=\; \sum_{k=j}^{N} c_k b^{k-j}\binom{k}{j}.$$

Proof. The no-arbitrage price is $C_0 = e^{-rT}\mathbb{E}(A-K)_+$, in which we substitute the truncated Gram-Charlier series (5) for the PDF of $(\log A - a)/b$:

$$\begin{aligned}\hat{C}_0 \;&=\; e^{-rT}\int_{\mathbb{R}}(e^{a+by}-K)_+\hat{f}_Y(y)\,dy\\ &=\; e^{-rT}\int_{-d_2}^{\infty}e^{a+by}\hat{f}_Y(y)\,dy - Ke^{-rT}\int_{-d_2}^{\infty}\hat{f}_Y(y)\,dy.\end{aligned}$$

The last integral is easier to simplify, one only needs to note that $(\phi(x)\mathrm{He}_{k-1}(x))' \;=\; -\phi(x)\mathrm{He}_k(x)$:

$$\int_{-d_2}^{\infty}\phi(x)\sum_{k=0}^{N}c_k\mathrm{He}_k(y)\,dy \;=\; \Phi(d_2)+\phi(d_2)\sum_{k=1}^{N}c_k\mathrm{He}_k(-d_2).$$

For the first part of the approximate price, we use a change of measure. Let

$$\nu(x) \;=\; \int_{-\infty}^{x}\phi(x)\sum_{k=0}^{N}c_k\mathrm{He}_k(y)\,dy, \quad \mu(x) \;=\; \int_{-\infty}^{x}e^{by}\,d\nu(y).$$

Use the bilateral Laplace transform to represent μ as a Gram-Charlier expansion:

$$\int_{\mathbb{R}} e^{sx}\,d\mu(x) \;=\; \int_{\mathbb{R}} e^{(b+s)x}\,d\nu(x) \;=\; \sum_{k=0}^{N} c_k \int_{\mathbb{R}} e^{(b+s)x}\phi(x)\mathrm{He}_k(x)\,dx.$$

Integrating by parts,

$$\int_{\mathbb{R}} e^{cx}\phi(x)\mathrm{He}_k(x)\,dx \;=\; c\int_{\mathbb{R}} e^{cx}\phi(x)\mathrm{He}_{k-1}(x)\,dx \;=\; (\cdots) \;=\; c^k e^{\frac{c^2}{2}},$$

which leads to

$$\int_{\mathbb{R}} e^{sx}\,d\mu(x) \;=\; \sum_{k=0}^{N} c_k(s+b)^k e^{\frac{(s+b)^2}{2}} \;=\; e^{\frac{b^2}{2}}\sum_{j=0}^{N}\tilde{c}_j s^j e^{bs+\frac{s^2}{2}}.$$

This says that

$$d\mu(x) \;=\; \phi(x-b)\sum_{j=0}^{k}\tilde{c}_j\mathrm{He}_j(x-b)\,dx,$$

implying

$$\int_{-d_2}^{\infty} e^{a+by}\hat{f}_Y(y)\,dy \;=\; e^{a+\frac{b^2}{2}}\Phi(d_1) + e^{a+\frac{b^2}{2}}\phi(d_1)\sum_{j=1}^{N}\tilde{c}_j\mathrm{He}_{j-1}(-d_1). \qquad \square$$

2.3. **Implementation.** If a function has a converging Gram-Charlier expansion

$$g(x) \;=\; \phi(x)\sum_{k=0}^{\infty} c_k\mathrm{He}_k(x) \tag{6}$$

then, using the orthogonality of the Hermite polynomials

$$\int_{-\infty}^{\infty} g(x)\mathrm{He}_j(x)\,dx \;=\; c_j\int_{-\infty}^{\infty}\phi(x)(\mathrm{He}_j(x))^2\,dx \;=\; c_j\,j! \quad j=0,1,\ldots.$$

(for more details, see Chateau & Dufresne [1] and Lebedev [14]). If $g(\cdot)$ is the PDF of a random variable Y then

$$c_j \;=\; \frac{1}{j!}\mathbb{E}\mathrm{He}_j(Y). \tag{7}$$

To approximate the distribution of $\log A$ we first need the parameters a and b, that represent centering and scale, respectively. We let

$$X \;=\; \log A, \quad a \;=\; \hat{\mathbb{E}}\log A, \quad b \;=\; \sqrt{\widehat{\mathbb{V}\mathrm{ar}}\log A}, \quad Y \;=\; \frac{X-a}{b}.$$

(The hat over the expectation or variance indicates values found by simulation.) The PDF of the random variable Y is assumed to have a Gram-Charlier expansion (6). The value of c_0 is always 1, and by definition it has expectation 0 and variance 1, so we set

$$c_1 \;=\; 0, \quad c_2 \;=\; 0,$$

in agreement with (7). The other c_j's are then defined as

$$c_j = \hat{\mathbb{E}}\mathrm{He}_j(Y), \quad j \;=\; 3,4,\ldots$$

The risk-free asset has return r, and the risky asset satisfies

$$dS_t \;=\; rS_t\,dt + \sigma S_t\,dW_t,$$

where W is a standard Brownian motion under the risk-neutral measure. Hence,

$$S_t \;=\; S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma \tilde{W}_t\right), \quad t \geq 0. \tag{8}$$

In all our numerical illustrations the payoff of the option is $(A-K)_+$, where

$$\frac{1}{m}\sum_{j=1}^{m} S_{t_j}, \quad t_j \;=\; \frac{jT}{m}.$$

3.1. **Convergence of the Gram-Charlier series for $\log \mathbf{A}$.** It is easily seen that

$$\mathbb{E}e^{pA^2} \;=\; \infty$$

for any $p > 0$, so the condition in Theorem 2 cannot be satisfied if Gram-Charlier series are applied to A itself. Moreover, the distribution of A is not determined by its moments, meaning that there is an infinite number of other distributions with the same moments as A. The Gram-Charlier series are based on moments; hence, when forming a Gram-Charlier expansion for the distribution of A we do not know which of those distribution the expansion would converge to, even if it did converge (the same comments apply to the generalized Edgeworth expansion described in Section 1). However, it is much more intuitive that the distribution of $\log A$ should be close to the normal.

If the average consists of just $m = 1$ averaging point then

$$Y \;=\; \frac{\log A - \mathbb{E}\log A}{\sqrt{\mathbb{V}\text{ar}\log A}}$$

has a standard normal distribution, so the condition $\mathbb{E}e^{\frac{Y^2}{4}} < \infty$ is satisfied and (Theorem 1) the Gram-Charlier converges. The following results give theoretical justifications for trusting those series when the log-returns have a normal distribution. Given a standard Brownian motion $\mathbf{W}$, we consider general discrete and continuous weighted averages

$$A \;=\; \sum_{j=1}^{N} w_j e^{\sigma W_{t_j}}, \quad 0 < t_1 < \cdots < t_N = T,\; w_j, \sigma \;>0\;\forall j$$

$$\overline{A} \;=\; \int_0^T \overline{w}_s e^{\sigma W_s}\,ds, \quad \sigma, T > 0,\; \overline{w}_s \geq 0\;\;\forall\, s,\; \overline{w}_s \;\geq\; \epsilon \;>\; 0\;\;\forall\, 0 \;\leq\; t^* \;\leq s \;<\; T.$$

Define

$$p^*(X) \;=\; \sup\{p \,|\, \mathbb{E}e^X \;<\; \infty\}.$$

(This is the "abscissa of convergence" of the moment generating function of X.)

Theorem 3. *(a) The distribution of A is not determined by its moments. If $\overline{w}_t = e^{\rho t}$ for some real number ρ then the distribution of $\overline{A}$ is not determined by its moments.*

(b) $\mathbb{E}A^p \;<\; \infty$ (resp. $\mathbb{E}\overline{A}^p \;<\; \infty$) for all $p \in \mathbb{R}$.

(c) The distribution of $\log A$ *(resp.* $\log \overline{A}$*) is determined by its moments.*

(d) $p^*(\log^2 A) \;=\; \frac{1}{2\sigma^2 T}$ (resp. $p^*(\log^2 \overline{A}) \;=\; \frac{1}{2\sigma^2 T}$).

Proof. (a) The variable A is a weighted sum of products of the independent lognormal variables:

$$L_j \;=\; e^{\sigma \Delta W_{t_j}}, \quad \Delta W_{t_j} \;=\; W_{t_{j+1}} - W_{t_j}, \quad A \;=\; f(L_1, \ldots, L_n).$$

It is well known that the lognormal distribution is not determined by its moments. If one of the L_j's, say L_1, is replaced with another variable, denote it $\tilde{L}_1$, that does not have a lognormal distribution but has the same moments as L_1, and is independent of $L_2, \ldots, L_n$, then the variable $f(\tilde{L}_1, L_2, \ldots, L_n)$ has the same moments as A but has a different distribution.

The same result concerning $\overline{A}$ is more involved, and is proved by Hörfelt [11], in the case where $\overline{w}_s$ is an exponential function.

(b) Let the running maximum of Brownian motion up to time t be $\overline{W}_t$, and the running minimum $\underline{W}_t$. Then

$$e^{\sigma \underline{W}_T} \sum_{j=1}^{n} w_j \;\leq\; A \;\leq\; e^{\sigma \overline{W}_T} \sum_{j=1}^{n} w_j. \tag{9}$$

The moments of arbitrary positive or negative order of the variables on the extreme left and right are all finite. Same proof for $\overline{A}$.

(c) A sufficient condition for the distribution of a variable X to be determined by its moments is that $\mathbb{E}e^{\rho X}$ be finite for $\rho \in (-\epsilon, \epsilon)$, for some $\epsilon > 0$ (since any distribution with the same moments as X then has the same moment generating function as X, implying that it is the same as the distribution of X). This condition is statisfied for $\log A$ and $\log \overline{A}$, from part (b).

(d) From (9),

$$|\log A| \;\leq \min(C_1 - \sigma \underline{W}_T, C_2 + \sigma \overline{W}_T), \quad C_1, C_2 \text{ constants.}$$

The distributions of $-\underline{W}_T$ and $\overline{W}_T$ are known to be the same as that of $|W_T|$. Hence, $\mathbb{E} \exp\left(p \log^2 A\right) < \infty$ for any $p < 1/(2\sigma^2 T)$. We also have

$$A \;\geq\; w_N e^{\sigma W_T} \quad \Longrightarrow \quad \log A \;\geq\; \log w_N + \sigma W_T.$$

This implies that $\mathbb{E} \exp\left(p \log^2 A\right) \;=\; \infty$ for any $p > 1/(2\sigma^2 T)$. We conclude that $p^*(\log^2 A) \;=\; 1/(2\sigma^2 T)$.

The case of continuous averages is very similar. The continuous counterpart of Eq. (9) implies that $p^*(\log^2 \overline{A}) \;\geq\; 1/(2\sigma^2 T)$. To prove the reverse inequality, choose $\delta \in T - t^*$, so that that

$$A \;\geq\; \epsilon \int_{T-\delta}^{T} e^{\sigma W_s}\, ds.$$

By the (continuous) arithmetic-geometric inequality,

$$\int_{T-\delta}^{T} e^{\sigma W_s}\, ds \;\geq\; \delta \exp\left(\frac{1}{\delta} \int_{T-\delta}^{T} W_s\, ds\right).$$

Now

$$\frac{1}{\delta} \int_{T-\delta}^{T} W_s\, ds \;=\; W_{T-\delta} + \widetilde{W},$$

where

$$\widetilde{W} \;=\; \frac{1}{\delta} \int_{T-\delta}^{T} (W_s - W_{T-\delta})\, ds \;\sim\; \mathbf{N}\left(0, \frac{\delta}{3}\right)$$

is independent of the Brownian motion up to time $T-\delta$. Hence,

$$\log \overline{A} \;\geq\; \log(\delta\epsilon) + W_{T-\delta} + \widetilde{W} \;\sim\; \mathbf{N}\left(\log(\delta\epsilon), T - \frac{2\delta}{3}\right).$$

This implies that $\mathbb{E}\exp(p\log^2 \overline{A}) \;=\; \infty$ for any $p \;\geq\; 1/(2\sigma^2(T-2\delta/3))$. This is valid for any $0<\delta<T-t^*$, so we conclude that $p^*(\log^2 \overline{A}) \;\leq\; 1/(2\sigma^2 T)$. $\square$

In order to apply Theorem 1 we need

$$p^*(\log^2 A) \;>\; \frac{1}{4\mathbb{V}\mathrm{ar}\log A} \tag{10}$$

in part (d) of Theorem 3. A simple formula for $\mathbb{V}\mathrm{ar}\log A$ is not known in general, but we study two special cases that are quite tractable.

3.2. **Example 1.** Consider a discrete average with $N=2$ terms, and, for simplicity, let $r=\frac{\sigma^2}{2}$ an $S_0=2$, implying that

$$A \;=\; e^{\sigma W_{t_1}} + e^{\sigma W_{t_2}},$$

or

$$\log A \;=\; \sigma W_{t_1} + \log(1+e^{\sigma \Delta W_{t_1}}).$$

Here $\Delta W_{t_1} = W_{t_2} - W_{t_1}$ is independent of W_{t_1}. Apply the formula for the moment-generating function of a non-central chi-square distribution:

$$\mathbb{E}e^{s(Z+c)^2} \;=\; \frac{1}{\sqrt{1-2s}}e^{sc^2/(1-2s)}, \quad Z\sim\mathbf{N}(0,1),\; s \;<\; \frac{1}{2},\; c\in\mathbb{R}.$$

Conditioning on ΔW_{t_1}, one finds (letting $\Delta t_1 = t_2 - t_1$)

$$\mathbb{E}\exp(p\log^2 A) \;=\; \frac{1}{\sqrt{1-2p\sigma^2 t_1}}\mathbb{E}\exp\left(\frac{p}{1-2p\sigma^2 t_1}\log^2(1+e^{\sigma\sqrt{\Delta_{t_1}}Z})\right)$$

if $2p\sigma^2 t_1 < 1$. Here Z as a standard normal distribution and the function

$$\log^2(1+e^{\sigma\sqrt{\Delta_{t_1}}z})$$

approaches 0 when z tends to minus infinity, while it is asymptoticaly equal to $\sigma^2\Delta t_1 z^2$ when z tends to plus infinity; hence the last expectation is finite if, and only if,

$$2p\sigma^2 t_1 \;<\; 1 \quad \text{and} \quad \frac{p\sigma^2\Delta t_1}{1-2p\sigma^2 t_1} \;<\; \frac{1}{2}.$$

These conditions are equivalent to $2p\sigma^2 t_2 < 1$, which confirms part (d) of Theorem 3.

In order to know whether (10) holds, we need the variance of $\log A$; that variable is the sum of two independent variables, the first a normal with variance $\sigma^2 t_1$, the other with a variance equal to

$$\mathbb{V}\mathrm{ar}\log(1+e^{\sigma\sqrt{\Delta t_1}Z}).$$

The function

$$g(q) \;=\; [\mathbb{V}\mathrm{ar}\log(1+e^{qZ})]/q^2$$

can be shown to tend to 0.25 as $q\to 0+$, and to have an asymptotic value of $(1-1/\pi)/2 \doteq 0.341$ as $q\to\infty$ (see Appendix). Moreover, numerical computations (not shown) indicate that $g(q)$ increases with q (we plotted it over the interval $(0,1000)$). Hence, we write

$$\mathbb{V}\mathrm{ar}\log A \;=\; \sigma^2 t_1 + \sigma^2\Delta t_1 g(\sigma\sqrt{\Delta t_1}),$$

and, using (10) and the value of p^* just obtained, we conclude that the Gram-Charlier series for $\log A$ will converge if

$$\frac{1}{2\sigma^2 t_2} \;>\; \frac{1}{4(\sigma^2 t_1 + \sigma^2 \Delta t_1 g(\sigma\sqrt{\Delta t_1}))}.$$

This condition is the same as

$$(1 - 2g(\sigma\sqrt{\Delta t_1}))\Delta t_1 \;<\; t_1, \tag{11}$$

which has some intuitive appeal. It says that Cramér's condition does not hold when Δt_1 is too large relative to t_1. It is interesting to note that the value of σ does not make a big difference in whether (11) holds or not; for relatively small values of $\sigma\sqrt{\Delta t_1}$, such as would be found in option pricing, condition (11) roughly says that convergence is certain to occur if t_2 is less than three times t_1.

3.3. **Example 2.** Consider a continuous average

$$M_t \;=\; \int_0^t e^{2W_s}\,ds.$$

The choice $\sigma = 2$ is temporary, it is there so we can make use of the remarkable *Bougerol identity*: if $(\mathbf{B}, \mathbf{W})$ is a pair of independent standard Brownian motions, then for each fixed $t > 0$

$$\int_0^t e^{W_s}\,dB_s \overset{\mathrm{d}}{=} \sinh(W_t),$$

where the symbol "$\overset{\mathrm{d}}{=}$" means "has the same distribution as". From the independence of $\mathbf{B}$ and $\mathbf{W}$ and the elementary properties of stochastic integrals the left hand side has the same distribution as $Z\sqrt{M_t}$, where Z has a standard normal distribution and is independent of M_t. Thus

$$|Z|\sqrt{M_t} \overset{\mathrm{d}}{=} \sinh|W_t|, \quad Z \;\sim\; \mathbf{N}(0,1).$$

From this representation we get

$$2\log|Z| + \log M_t \overset{\mathrm{d}}{=} 2\log\sinh|W_t|. \tag{12}$$

Taking expectations on each side, this yields

$$2\mathbb{E}\log|Z| + \mathbb{E}\log M_t \;=\; 2\mathbb{E}\log\sinh|W_t|. \tag{13}$$

Squaring (12) then yields

$$4\mathbb{E}\log^2|Z| + 4(\mathbb{E}\log M_t)(\mathbb{E}\log|Z|) + \mathbb{E}\log^2 M_t \;=\; 4\mathbb{E}\log^2\sinh|W_t|.$$

Subtracting the square of (13) from the last espression and rearranging yields

$$\mathbb{V}\mathrm{ar}\log M_t \;=\; 4\mathbb{V}\mathrm{ar}\log\sinh|W_t| - 4\mathbb{V}\mathrm{ar}\log|Z| + 8\mathbb{E}\log|Z|(\mathbb{E}\log|Z| - \mathbb{E}\sinh|W_t|).$$

This can be computed numerically and used to determine whether condition (10) holds; from Theorem 3 the latter is the same as

$$\mathbb{V}\mathrm{ar}\log M_t \;>\; t.$$

The two sides of this inequality are plotted in Figure 1. It can be seen that Cramér's condition for convergence is satisfied for every $t \le 20$ (one needs to check that $\mathbb{V}\mathrm{ar}\log M_t > t$, $t = \sigma^2 T/4$, which we have done for $t \le 20$).

Now, the scaling property of Brownian motion may be used to extend this result to arbitrary $\sigma > 0$:

$$\overline{A} \;=\; \int_0^T e^{\sigma W_s}\,ds \overset{\mathrm{d}}{=} \frac{4}{\sigma^2}\int_0^{\frac{\sigma^2 T}{4}} e^{2W_u}\,du \;=\; \frac{4}{\sigma^2} M_{\frac{\sigma^2 T}{4}}.$$

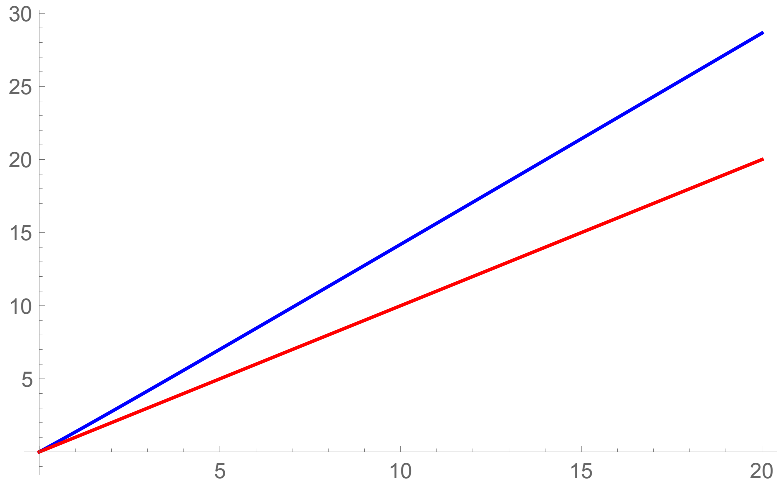

FIGURE 1. Var $\log A$ (top) versus the function $1/4p^*(\log \overline{A}) = t$.

Therefore, this example shows that the Gram-Charlier series converges in cases where the payoff is expressed in terms of a continuous average and $r = \sigma^2/2$.

3.4. **Performance of the Gram-Charlier approximations.**

We use the following parameters:

$$r = 0.05,\ \sigma = 1,\ S_0 = 1,\ m = 200,\ T = 1,\ \text{NumSim} = 10^6,$$

where NumSim is the number of iterations performed.

We compute approximate densities of orders four to ten. Figure 2 shows the results. The approximations are not always true densities, however, as they may not satisfy the non-negativity condition. For example, the degree 10 Gram-Charlier approximation of the density of $\log A$ is negative between $x = -3.898$ and $x = -2.754$, and the integral of the function between these two points is -5.148×10^{-7}. The approximate densities improve as the number of terms increases.

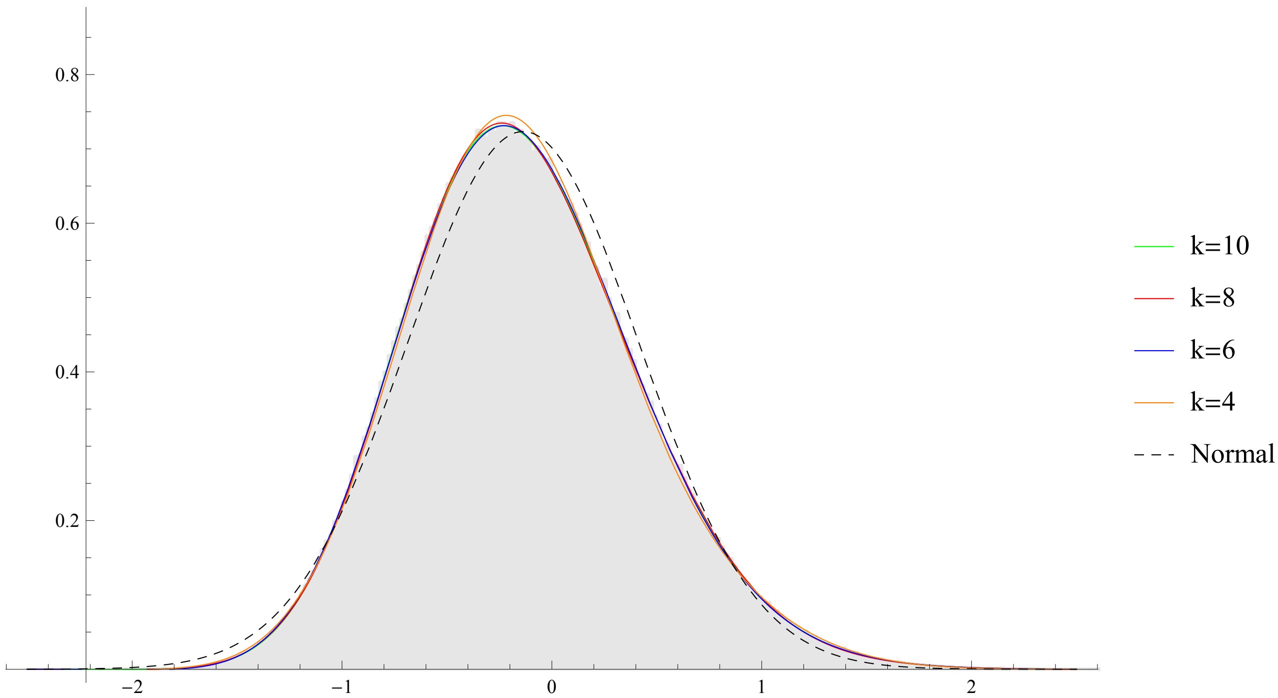

FIGURE 2. Simulated frequency distribution of $\log A$ vs Gram-Charlier approximations of degree k.

A test of the Gram-Charlier approximations is the Kolmogorov-Smirnov test value

$$\max_x |\hat{f}_n(x) - f(x)|,$$

shown in Table 1. In place of the true density $f(\cdot)$ we use the frequency curve obtained by simulation.

	K-S statistic
GC degree 10	0.00118901
GC degree 8	0.00146130
GC degree 6	0.00220439
GC degree 4	0.00454614
Normal (degree 0)	0.02378388

TABLE 1. Kolomogorov-Smirnov test statistics

Another test we performed is the computation of moments of A using the Gram-Charlier approximations of oders up 10 (see Table 1, where sampling errors are presented for the case $\sigma = 1$). Integrating e^{nx} time the density of $\log A$ puts a lot of weight on higher values of x, and it is to be expected that errors increase with n. For small σ the approximate moments are remarkably good up to orders 8 or 10, but this is not so when σ is larger. The formulas used to calculate the moments are explained in the Appendix.

Table 3 shows approximate Asian call option prices. Prices for different Gram-Charlier degrees and volatility assumptions are presented in Table 2. All approximations of the option price fall within the 95% confidence interval of the sample price. Figure 3 compares the simulated option prices (dots) with the approximated ones (continuous curve) in the case where $\sigma = 1$ and a Gram-Charlier approximation of degree 10 is used.

4. SECOND APPLICATION: THE VARIANCE GAMMA PROCESS

The gamma distribution is infinitely divisible and may be used to define a Lévy process, which is positive and non-decreasing. The variance gamma (VG) process is obtained by replacing the time variable t in a Browian motion with a "stochastic clock", represented by a gamma process. The result is also a Lévy process. The variance gamma has been used to model log-returns of risky assets (Madan, Carr and Chang [18]). If $\mathbf{W}$ is a standard Brownian motion and

$$B_t = \theta t + \sigma W_t \tag{14}$$

then one defines

$$X_t = B_{T_t},$$

where $\mathbf{T}$ is a gamma process, indepedent of $\mathbf{W}$. The gamma process is given a single parameter ν, so $\mathbf{X}$ has three parameters (θ, σ, ν). The parameter ν is chosen so that $\mathbb{E}T_t = t$ and $\mathbb{V}\mathrm{ar}\, T_t = \nu t$.

The stock price process is defined as

$$S_t = S_0 \exp(rt + \omega t + X_t(\sigma, \nu, \theta)), \quad \omega = \frac{1}{\nu} \log\left(1 - \theta\nu - \frac{\sigma^2 \nu}{2}\right). \tag{15}$$

Writing down the characteristic function of the variance gamma process $\{X_t\}$ shows that it can be expressed as the difference of two gamma processes. Since

$$\mathbb{E}e^{pG^2} = \infty, \quad p > 0,$$

for any variable G with a gamma distribution, this also says that *the gamma process never satisfies the condition for convergence in Theorem 1.* Theoretical convergence of the Gram-Charlier series is then unlikely, even though the variance gamma process is a mixture of normal distributions. This does not preclude the possibility that a finite number of terms might still give a good approximation of the true density.

4.1. **Performance of Gram-Charlier approximations of log-average price.** For our simulations we initially assume the following parameters:

$$r = 0.05,\ \sigma = 0.5,\ \theta = -0.5,\ \nu = 0.3,$$
$$S_0 = 1,\ m = 200,\ T = 1,\ \text{NumSim} = 10^6.$$

Figure 4 shows the (dismal) results. The quality of the approximation improves if ν gets smaller, meaning that the variance gamma process is close to a Brownian motion (the variance of T_t is smaller), see Figure 5, where $\nu = 0.05$. Option prices (Table 4) show the same behaviour, the Gram-Charlier approximations are better for small values of ν. By and large we did not observe a large influence of the parameters θ and σ. We conclude that Gram-Charlier approximations are only useful for relatively small values of ν.

5. Third application: The Heston model

Heston [10] proposed using a square-root process to model squared volatility. This model has some tractability but there is no closed form expression for Asian option prices. The model is:

$$dS_t = \mu S_t\, dt + \sqrt{V_t} S_t\, dW_t^{(1)}$$
$$dv_t = \kappa(\theta - V_t)\, dt + \gamma\gamma\sqrt{v_t} S_t \mathrm{d}W_t^{(2)}.$$

There is correlation ρ between the two standard Brownian motions $\mathbf{W}^{(1)}, \mathbf{W}^{(2)}$.

The Laplace transform of the log-price is known for this model, it follows from results due to Feller in the 1950s. The formula for this Laplace transform shows that

$$\mathbb{E}(p \log S_t) \ = \ \infty$$

if p is larger than some p_0,which implies that $\mathbb{E}(p \log^2 S_t) = \infty$ for any $p > 0$. This means that *the log-price under the square-root volatility model never satisfies the condition for convergence in Theorem 1*, and so convergence of the Gram-Charlier series for the distribution of $\log A$ is unlikely.

5.1. **Performance of Gram-Charlier approximations of log-average price.** The base scenario is

$$r = 0.05,\ \gamma = 0.5,\ v_0 = 0.05,\ \theta = 0.05,\ \kappa = 2,\ \rho = 0,$$
$$S_0 = 1,\ m = 200,\ T = 1, \text{NumSim} = 10^6.$$

The performance of the approximation appears to depend heavily on the mean reversion parameter, κ, and the volatility of squared-volatility parameter, γ. The accuracy of the approximation seems to be independent of the long-run average volatility, θ, which has been tested up to $\theta = 1.5$ (with v_0 adjusted accordingly). With high θ the approximation deteriorates with higher maturities (T). In all cases tested, the Gram-Charlier approximations appeared to diverge, though the rate of divergence varied. The results are shown in Figures 6 and 7.

In cases where γ is relatively small, for example $\gamma = 0.1$, the accuracy of the approximation is less dependent on the parameter κ, than when γ is high, for example, $\gamma = 0.9$. When γ is small the approximation performs extremely well. The obvious explanation is that when γ is small the log-price is closer to a Brownian motion. When γ is relatively large, for example, $\gamma = 1$, the approximation is acceptable when κ is high. Prices for Asian call options are shown in Table 5 under various scenarios of γ and κ, with all other parameters as above. While the prices derived in the low γ cases are all within the 95% confidence interval of the sample price, the prices derived from the approximation in the high γ case fall further outside the confidence interval, the higher the order of the approximation.

The accuracy of the approximation appears to deteriorate quite rapidly as $|\rho|$ increases. The greater the magnitude of ρ, the faster the deterioration of the approximation with the order, as higher order approximations become meaningless more quickly. The case where $\rho = -0.3$ is shown in Figure 11.

6. Conclusion

We have shown explicit formulas for Asian option prices when the density of the logarithm of the average A is approximated by a Gram-Charlier truncated series. A significant drawback of the approximation method we used to price Asian options is that the moments of $\log A$ need to be obtained by simulation. The method is therefore not an improvement over direct simulation of Asian option prices, unless one is looking for an analytic formula for the density of A or for option prices. It is however not inconceivable that good approximations for the moments of $\log A$ might be found in future, and this is one of the aims of our research. We have tried to fit Gram-Charlier series using the moments of the A itself, but this was not successful.

We have shown apparently new results on the convergence of Gram-Charlier series for $\log A$ when the stock price S follows a geometric Brownian motion (the classical Black-Scholes model). In this case the method does show some promise, since the series can be shown to converge, and the moments of the logarithm of the average may have not too complicated expressions, as we have shown in the case where $r = \sigma^2/2$. In the other two models studied (variance gamma log-price, Heston) the method appears to have limited applicability, restricted to cases where the distribution of the log-price is not too different from a Brownian motion.

7. Appendix

7.1. **The function $g(\cdot)$ in Example 1, Section 3.** The variable Z has a standard normal distribution in

$$g(q) \;=\; [\mathbb{V}\mathsf{ar} \log(1 + e^{qZ})]/q^2.$$

When $q \to \infty$, we use the inequality $\log(1 + y) \le y$ to justify

$$\frac{1}{q}\mathbb{E}(\log(1 + e^{qZ})\mathbf{1}_{\{Z<0\}}) \;\le\; \frac{1}{q}\mathbb{E}(e^{qZ})\mathbf{1}_{\{Z<0\}}) \;\to\; 0.$$

This implies

$$\frac{1}{q}\mathbb{E}(\log(1+e^{qZ})\mathbf{1}_{\{Z>0\}}) \;=\; \mathbb{E}(Z\mathbf{1}_{\{Z>0\}})+\frac{1}{q}\mathbb{E}(\log(1+e^{-qZ})\mathbf{1}_{\{Z>0\}}) \;\to\; \mathbb{E}(Z\mathbf{1}_{\{Z>0\}}) \;=\; \frac{1}{\sqrt{2\pi}}.$$

In the same way,

$$\frac{1}{q^2}\mathbb{E}(\log^2(1+e^{qZ})\mathbf{1}_{\{Z<0\}}) \;\to\; 0$$

$$\frac{1}{q^2}\mathbb{E}(\log^2(1+e^{qZ})\mathbf{1}_{\{Z>0\}}) \;\to\; \mathbb{E}(Z^2\mathbf{1}_{\{Z>0\}}) \;=\; \frac{1}{2}.$$

Hence,

$$g(q) \;\to\; \frac{1}{2}-\frac{1}{2\pi} \;\doteq\; .340845 \quad \text{as} \quad q \;\to\; \infty.$$

Now turn to the limit as $q \to 0$. Subtracting $\log 2$, we see that

$$\mathbb{V}\mathrm{ar}\log(1+e^{qZ}) \;=\; \mathbb{V}\mathrm{ar}\, h(qZ), \quad \text{where} \quad h(x) \;=\; \log\left(\frac{1+e^x}{2}\right).$$

The function $h(\cdot)$ is increasing, it has a bounded derivative for $x \in [-1,1]$, and $h(0) = 0$. We will use the following inequality, valid for $z > 0$:

$$\Phi(-z) \;=\; \int_z^\infty \phi(x)\,dx \;\le\; \frac{1}{z}\int_z^\infty x\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx \;=\; \frac{\phi(z)}{z}.$$

First,

$$\frac{1}{q}\mathbb{E}(h(qZ)\mathbf{1}_{\{qZ<-1\}}) \;\le\; |h(-1)|\Phi\left(-\frac{1}{q}\right) \;\to\; 0 \quad \text{as} \quad q \to 0+.$$

Next, writing $h(x) \;=\; \log\left(1+\dfrac{e^x-1}{2}\right)$,

$$\frac{1}{q}\mathbb{E}(h(qZ)\mathbf{1}_{\{qZ>1\}}) \;\le\; \frac{1}{q}\mathbb{E}\left(\frac{e^{qZ}-1}{2}\mathbf{1}_{\{qZ>1\}}\right) \;\to\; 0$$

$$\frac{1}{q}\mathbb{E}(h(qZ)\mathbf{1}_{\{|qZ|\le 1\}}) \;=\; \mathbb{E}\left(\frac{1}{q}\int_0^{qZ} h'(x)\,dx\mathbf{1}_{\{|qZ|\le 1\}}\right) \;\to\; h'(0)\mathbb{E}Z \;=\; 0.$$

The same ideas apply to the second moment: as $q \to 0+$,

$$\frac{1}{q^2}\mathbb{E}(h(qZ)^2\mathbf{1}_{\{qZ<-1\}}) \;\le\; h(-1)^2\Phi\left(-\frac{1}{q}\right) \;\to\; 0$$

$$\frac{1}{q^2}\mathbb{E}(h(qZ)^2\mathbf{1}_{\{qZ>1\}}) \;\le\; \frac{1}{q^2}\mathbb{E}\left(\frac{e^{qZ}-1}{2}\mathbf{1}_{\{qZ>1\}}\right)^2 \;\to\; 0$$

$$\frac{1}{q^2}\mathbb{E}(h(qZ)^2\mathbf{1}_{\{|qZ|\le 1\}}) \;\to\; h'(0)^2\mathbb{E}Z^2 \;=\; \frac{1}{4}.$$

We conclude that $g(0+) = 1/4$.

7.2. **Moments of the arithmetic average.** The following is taken from Dufresne [6]. The same method works any time log-returns have independent increments and averaging time points are evenly spaced. In the Black-Scholes model, let $\mathbf{S}$ be the stock price process, with

$$S_t = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma W_t}, \quad R_j = S_{t_j}/S_{t_{j-1}}, \quad j = 1,\ldots,m$$

$$A = \frac{1}{m}\sum_{i=1}^m S_{t_i} = \frac{S_0}{m}(R_1 + R_1R_2 + \ldots + R_1R_2\cdots R_m).$$

The variables $\{R_j\}$ are independent, and we assume $t_j = jT/m$, making them identically distributed.

Let $B_k = R_1 + R_1R_2 + \cdots + R_1R_2\cdots R_k$, $B_0 = 0$. Then

$$B_k = R_k(1 + B_{k-1}), \quad k = 1, \ldots, m,$$

where R_k, B_{k-1} are independent. This implies, for $n = 1, 2, \ldots,$

$$\mathbb{E}(B_k^n) = \mathbb{E}(R_k^n)\mathbb{E}(1 + B_{m-1})^n = \mathbb{E}(R_k^n)\sum_{j=0}^{n}\binom{n}{j}\mathbb{E}(B_{k-1}^j), \quad k = 1, \ldots, m.$$

The moments of R_k are obtained from the moment generating function of the normal distribution, and

$$\mathbb{E}(A^n) = \left(\frac{S_0}{m}\right)^n \mathbb{E}(B_m^n).$$

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Moments →	1	2	3	4	5	6	7	8	9	10
$\sigma = 0.1$										
GC degree 10	1.0256	1.05545	1.08988	1.12929	1.17413	1.22495	1.28237	1.34711	1.42001	1.50203
GC degree 8	1.0256	1.05545	1.08988	1.12929	1.17413	1.22495	1.28237	1.34711	1.42001	1.50203
GC degree 6	1.0256	1.05545	1.08988	1.12929	1.17413	1.22495	1.28237	1.34711	1.42001	1.50204
GC degree 4	1.0256	1.05545	1.08988	1.12929	1.17413	1.22495	1.28237	1.34711	1.42002	1.50204
Sample	1.0256	1.05545	1.08988	1.12929	1.17413	1.22495	1.28237	1.34711	1.42001	1.50203
True	1.02555	1.05534	1.0897	1.12903	1.1738	1.22453	1.28185	1.34648	1.41927	1.50116
$\sigma = 1$										
GC degree 10	1.0261	1.52062	3.49245	13.1958	83.221	836.937	12458.8	257655.	7.119×10^{6}	2.576×10^{8}
error	(0.002)	(0.011)	(0.105)	(1.669)	(32.363)	(700.83)	(1.717×10^{4})	(4.9×10^{5})	(1.674×10^{7})	(6.994×10^{8})
GC degree 8	1.0261	1.52056	3.48748	13.0145	78.2373	708.957	9016.4	153797.	3.445×10^{6}	1.006×10^{8}
error	(0.002)	(0.011)	(0.097)	(1.2)	(17.312)	(282.560)	(5353.29)	(1.212×10^{5})	(3.358×10^{6})	(1.16×10^{8})
GC degree 6	1.02609	1.51985	3.46473	12.5793	71.0543	588.196	6757.03	104201.	2.123×10^{6}	5.682×10^{7}
error	(0.002)	(0.011)	(0.078)	(0.703)	(7.238)	(87.205)	(1.266.33)	(2.272×10^{4})	(5.133×10^{5})	(1.482×10^{7})
GC degree 4	1.02605	1.51608	3.38688	11.4885	56.9081	392.957	3667.14	45564.1	749760.	1.633×10^{7}
error	(0.002)	(0.01)	(0.055)	(0.352)	(2.674)	(24.794)	(287.918)	(4264.77)	(8.161×10^{4})	(2.035×10^{6})
Sample	1.0261	1.52062	3.49179	13.1476	80.9017	744.335	9012.71	129254.	2.053×10^{6}	3.475×10^{7}
error	(0.002)	(0.011)	(0.108)	(1.929)	(45.403)	(1173.53)	(3.153×10^{4})	(8.654×10^{5})	(2.409×10^{7})	(6.775×10^{8})
True	1.02555	1.52142	3.53399	14.0893	106.106	1646.56	56349.	4.457×10^{6}	8.396×10^{8}	3.8434×10^{11}
$\sigma = 1.5$										
GC degree 10	1.02592	2.59143	21.4118	522.629	26228.8	2.216×10^{6}	2.998×10^{8}	6.507×10^{10}	2.301×10^{13}	1.347×10^{16}
GC degree 8	1.02591	2.58195	20.0209	396.315	14704.9	904144.	9.045×10^{7}	1.488×10^{10}	4.087×10^{12}	1.901×10^{15}
GC degree 6	1.02575	2.53966	17.4444	271.636	7739.8	373416.	3.014×10^{7}	4.098×10^{9}	9.489×10^{11}	3.779×10^{14}
GC degree 4	1.02468	2.4409	14.052	162.73	3338.25	117206.	7.048×10^{6}	7.323×10^{8}	1.327×10^{11}	4.222×10^{13}
Sample	1.02592	2.59209	21.5731	532.855	24659.9	1.523×10^{6}	1.077×10^{8}	8.178×10^{9}	6.474×10^{11}	5.267×10^{13}
True	1.02555	2.6337	26.1811	1517.07	637159.	2.099×10^{9}	5.614×10^{13}	1.247×10^{19}	2.336×10^{25}	3.743×10^{32}

TABLE 2. Approximate moments of A, Black-Scholes case.

Asian Call Prices	$\sigma = 0.1$	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 1.5$
GC degree 10	0.036620	0.123903	0.231135	0.330852
GC degree 8	0.036620	0.123910	0.231189	0.331501
GC degree 6	0.036619	0.123897	0.230957	0.329924
GC degree 4	0.036620	0.123787	0.230379	0.328730
Simulation	0.036621	0.123910	0.231101	0.330900
95% confidence	(0.0361815,	(0.121732,	(0.230008,	(0.329807,
interval	0.0370598)	0.126087)	0.232193)	0.331992)

TABLE 3. Asian call option prices with strike 1, Black-Scholes case.

Asian Call Prices	$\nu = 0.05$	$\nu = 0.1$	$\nu = 0.2$	$\nu = 0.3$
GC degree 10	0.037552	0.036729	0.035338	0.029753
GC degree 8	0.037592	0.036887	0.035649	0.033755
GC degree 6	0.037496	0.037002	0.036918	0.037521
GC degree 4	0.037605	0.036894	0.035754	0.034256
Simulation	0.037573	0.036846	0.035848	0.034635
95% CI upper	(0.037348,	(0.036621,	(0.035626,	(0.034416,
lower	0.037798)	0.037071)	0.036070)	0.034854)

TABLE 4. Asian option prices with strike 1, variance gamma case.

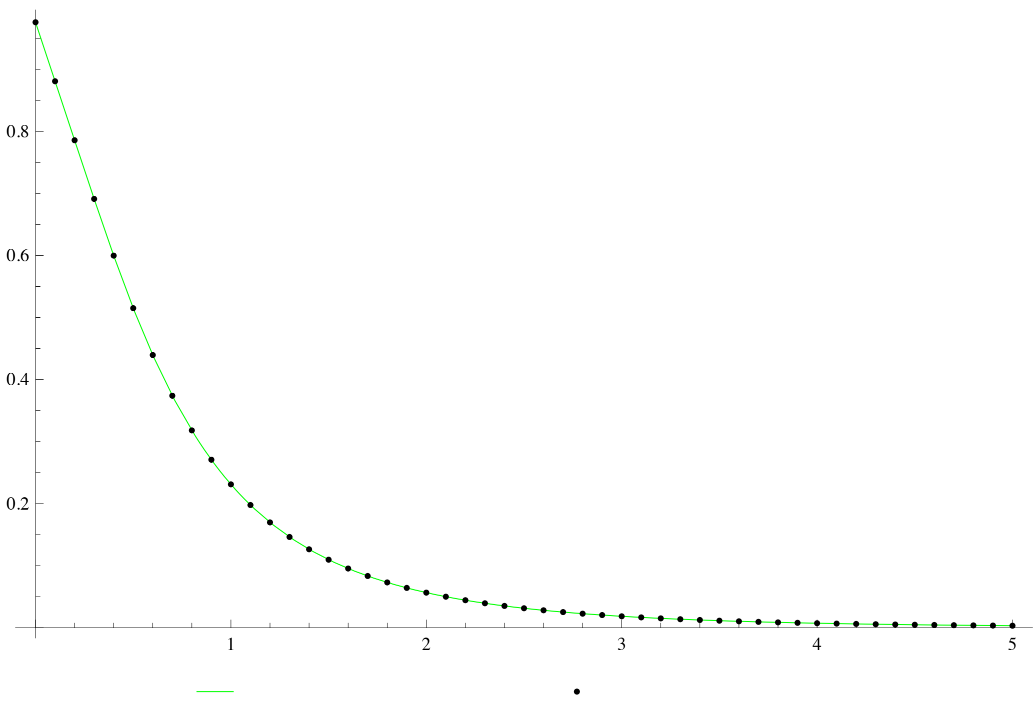

FIGURE 3. Asian Option call price with volatility 1, for varying strikes.

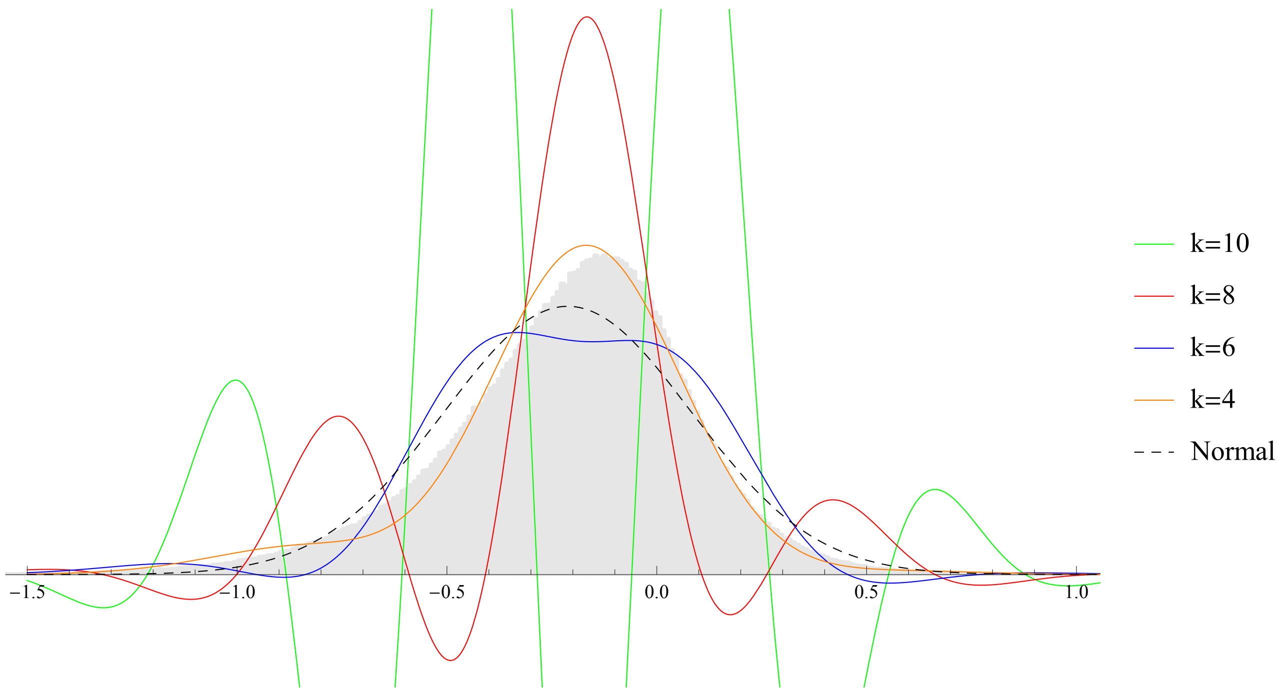

FIGURE 4. Approximating the log-average density with GC of varying degree (k), variance gamma with $\nu = 0.3$.

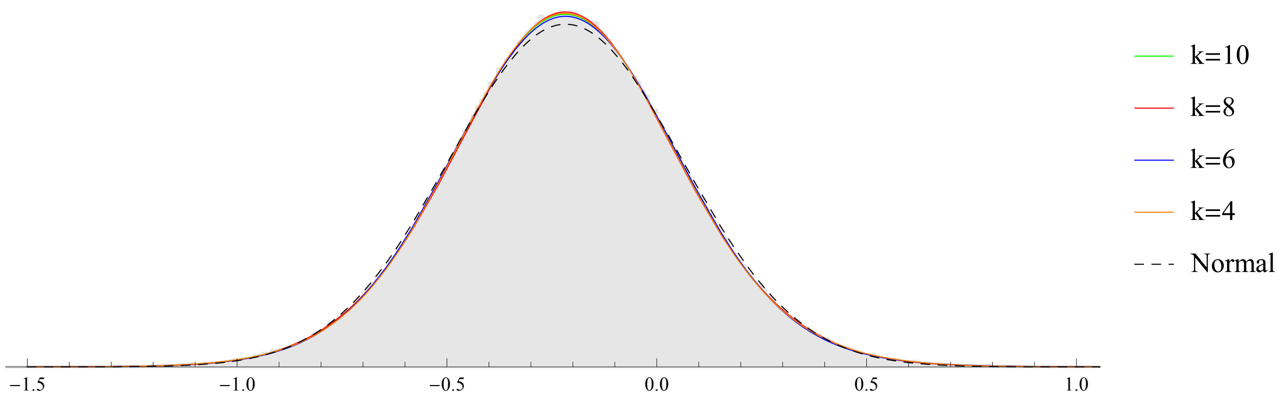

FIGURE 5. Approximating the log-average density with Gram-Charlier of varying degree (k), variance gamma with $\nu = 0.05$.

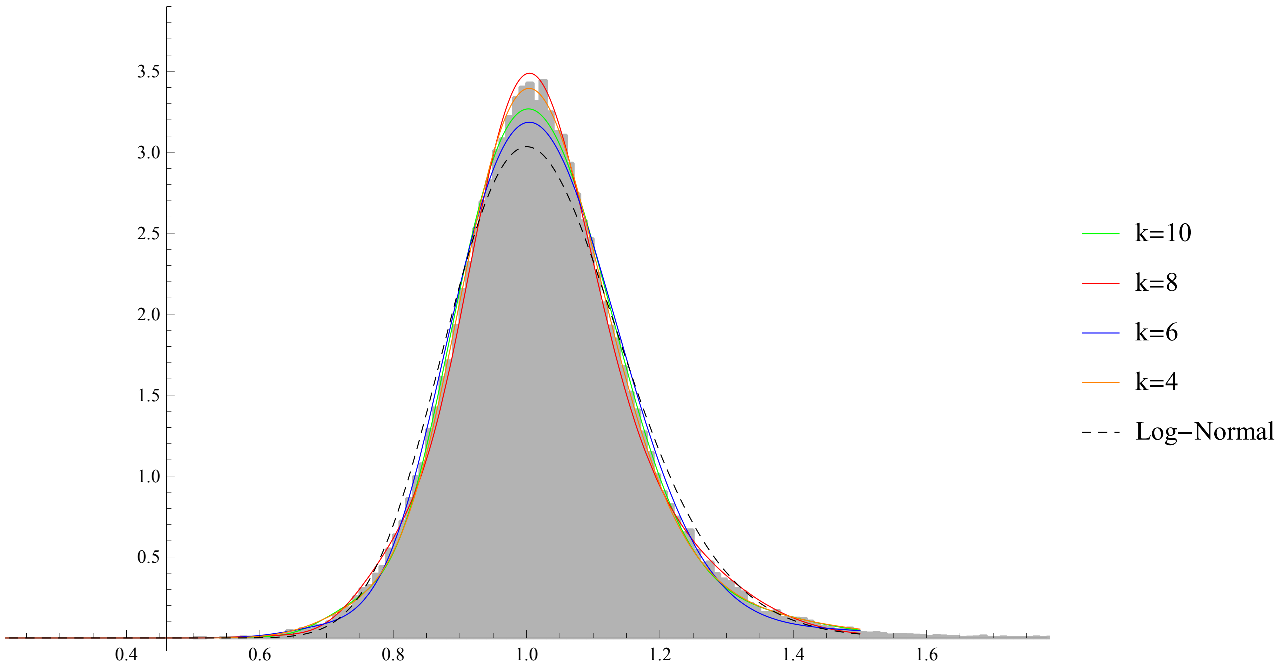

FIGURE 6. Gram-Charlier approximations for $\log A$, Heston model with $\rho = 0$.

Asian Call Prices $\gamma = 0.1$	$\kappa = 0.5$	$\kappa = 1$	$\kappa = 2$	$\kappa = 4$
GC degree 10	0.063417	0.063440	0.063471	0.063505
GC degree 8	0.063417	0.063439	0.063471	0.063505
GC degree 6	0.063404	0.063427	0.063458	0.063492
GC degree 4	0.063396	0.063420	0.063454	0.063490
Simulation	0.063366	0.063388	0.063419	0.063455
95% CI upper lower	(0.062769, 0.063963)	(0.062802, 0.063974)	(0.062834, 0.064005)	(0.062869, 0.064041)
$\gamma = 1$	$\kappa = 0.5$	$\kappa = 1$	$\kappa = 2$	$\kappa = 4$
GC degree 10	0.393768	0.222653	0.101475	0.065715
GC degree 8	0.007245	0.030785	0.051592	0.061008
GC degree 6	0.076491	0.072549	0.067928	0.064493
GC degree 4	0.060696	0.061024	0.061510	0.062125
Simulation	0.063769	0.063504	0.063138	0.062875
95% CI upper lower	(0.063074, 0.064465)	(0.062838, 0.064169)	(0.062507, 0.063770)	(0.062289, 0.063461)

TABLE 5. Asian option prices with strike 1, Heston model.

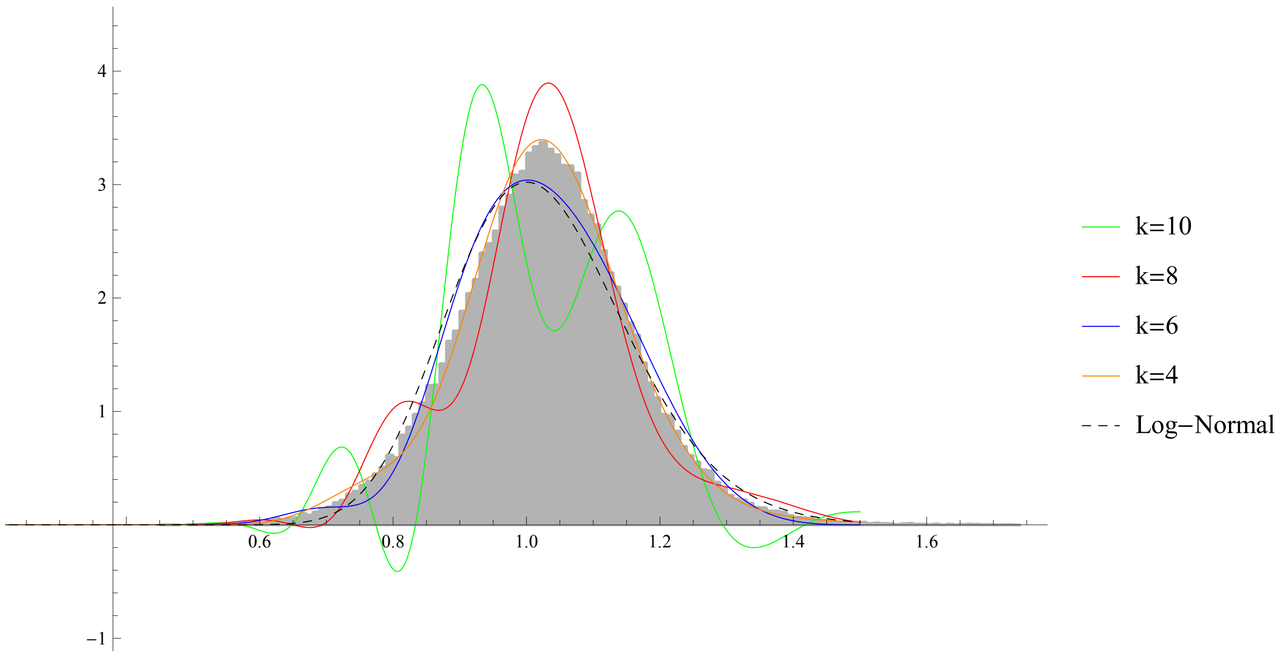

FIGURE 7. Gram-Charlier approximations for $\log A$, Heston model with $\rho = -0.3$.