A two-decrement model for the valuation and risk measurement of a guaranteed annuity option

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52nd Actuarial Research Conference
26 – 29 July 2017, Atlanta, Georgia, USA
Outline

1. Introduction
2. Modelling framework
3. Derivation of GAO prices
4. Risk measurement of GAO
5. Conclusion
1 Introduction
Research motivation

- **Financial innovations** – response to increased longevity and ageing population.

- Insurance market is becoming an investment hub.

- Interest and mortality risks – primary factors in valuation and risk management of longevity products. But, lapse risk is also very important.

- Lapse risk – possibility that policyholders terminate their policies early ... for various reasons.

- Dire consequences from policy lapses – huge losses and liquidity problem for insurance companies.
In current practice, lapse rate is assumed constant or deterministic in actuarial valuation.

Research advances on lapse risk modelling are rather slow, unlike those for interest and mortality dynamics.

Policyholders’ decision to surrender is directly affected by economic circumstances.
Objectives

- **Develop** an integrated approach that addresses simultaneously guaranteed annuity option (GAO)’s *pricing* and *capital requirement calculation*.

- **Construct** a two-decrement stochastic model in which *death* and policy lapse occurrences with their correlations to the financial risk are explicitly modelled.

- **Apply** series of *probability measure changes* resulting to forward, survival, and risk-endowment measures.

- **Determine** risk measures using *moment-based density method* and results benchmarked with the Monte-Carlo simulation. *Our formulation highlights the link between pricing and capital requirement.*
2 Modelling framework

- Interest rate model
- Mortality model
- Lapse rate model
- Valuation framework
We assume short-interest rate $r_t$ follows the Vasiček model via the SDE

$$dr_t = a(b - r_t)dt + \sigma dX_t,$$

where $a$, $b$, and $\sigma$ are positive constants and $X_t$ is a standard one-dimensional Brownian motion.
Interest rate model (cont’d)

Price $B(t, T)$ of a $T$-maturity zero-coupon bond at time $t < T$ is known to be

$$B(t, T) = E^Q[e^{-\int_t^T r_u du | F_t}] = e^{-A(t,T)r_t + D(t,T)},$$

where

$$A(t, T) = \frac{1 - e^{-a(T-t)}}{a}$$

and

$$D(t, T) = \left(b - \frac{\sigma^2}{2a^2}\right) [A(t, T) - (T - t)] - \frac{\sigma^2 A(t, T)^2}{4a}.$$
Mortality model

- The dynamics of the **force of mortality process** $\mu_t$ is given by

  \[
  d\mu_t = c\mu_t dt + \xi dY_t,
  \]

  where $c$ and $\xi$ are positive constants, and $Y_t$ is a standard Brownian motion correlated with $X_t$.

  \[
  dX_t dY_t = \rho_{12} dt.
  \]

- The **survival function** is defined by

  \[
  S(t, T) = \mathbb{E}^Q \left[ e^{-\int_t^T \mu_u du} \mid \mathcal{F}_t \right].
  \]
For the **lapse rate process** \( l_t \), we adopt the dynamics

\[
dl_t = h(m - l_t)\,dt + \zeta \,dZ_t,
\]

where \( h, \, m \) and \( \zeta \) are positive constants and \( Z_t \) is a standard BM **correlated with both** \( X_t \) and \( Y_t \).

In particular,

\[
dX_t \,dZ_t = \rho_{13} \,dt \quad \text{and} \quad dY_t \,dZ_t = \rho_{23} \,dt.
\]
Let $M^d(t, T)$ be the fair value at time $t$ of a pure endowment of $1$ at maturity $T$ when mortality is the only decrement, i.e.,
\[
M^d(t, T) = \mathbb{E}^Q \left[ e^{-\int_t^T r_u du} e^{-\int_t^T \mu_u du} \bigg| \mathcal{F}_t \right].
\]

Let $M^\tau(t, T)$ be the fair value at time $t$ of a $1$ pure endowment at maturity $T$ under a two-decrement model (both mortality and lapse rates are considered), i.e.,
\[
M^\tau(t, T) = \mathbb{E}^Q \left[ e^{-\int_t^T r_u du} e^{-\int_t^T \mu_u du} e^{-\int_t^T l_u du} \bigg| \mathcal{F}_t \right].
\]
Define $a_x(T)$ as the **annuity rate**. A life annuity is a contract that pays $1 to an insured annually conditional on his/her survival at the moment of payments.

That is,

$$a_x(T) = \sum_{n=0}^{\infty} E^Q \left[ e^{-\int_T^{T+n} r_u du} e^{-\int_T^{T+n} \mu_u du} \bigg| \mathcal{F}_T \right] = \sum_{n=0}^{\infty} M^d(T, T + n).$$
Valuation framework (cont’d)

- GAO is a contract that gives the policyholder the right to *convert a survival benefit into an annuity at a pre-specified guaranteed conversion rate* $g$.

- GAO’s **loss function** $L$ is the payoff ‘discounted’ by mortality and lapse factors, i.e.,

$$L = ge^{-\int_0^T \mu u du}e^{-\int_0^T l u du}(a_x(T) - K)^+,$$

where $K = 1/g$.

- The **fair value of GAO** at time 0, by risk-neutral pricing, is

$$P_{GAO} = g\mathbb{E}_Q^\mathcal{F}_0 \left[e^{-\int_0^T r u du}e^{-\int_0^T \mu u du}e^{-\int_0^T l u du}(a_x(T) - K)^+ \right].$$
Section Outline

Introduction
Modelling framework

Derivation of GAO prices
- The forward measure
- The survival measure
- The endowment-risk-adjusted measure
- Numerical implementation

Risk measurement of GAO
Conclusion

3 Derivation of GAO prices
- The forward measure
- The survival measure
- The endowment-risk-adjusted measure
- Numerical implementation
The forward measure

- The forward measure $\tilde{Q}$ is constructed with the aid of the Radon-Nikodým derivative and Girsanov’s theorem, and in particular
  \[
  \frac{d\tilde{Q}}{dQ}\bigg|_{\mathcal{F}_T} = \Lambda^\frac{1}{T} := \frac{e^{-\int_0^T r_u du} B(T, T)}{B(0, T)}.
  \]

- The dynamics of $\mu_t$ under $\tilde{Q}$ is given by
  \[
  d\mu_t = [-\rho_1 \sigma \xi A(t, T) + c\mu_t] dt + \xi d\tilde{Y}_t.
  \]
The forward measure (cont’d)

- The pure endowment under two-decrement model can be represented as

\[ M^T(t, T) = B(t, T) \mathbb{E}^{\tilde{Q}} \left[ e^{- \int_t^T \mu_u du} e^{- \int_t^T l_u du} \bigg| \mathcal{F}_t \right]. \]

- The pure endowment under one-decrement model can be expressed as

\[ M^d(t, T) = B(t, T) \mathbb{E}^{\tilde{Q}} \left[ e^{- \int_t^T \mu_u du} \bigg| \mathcal{F}_t \right]. \]
Given the dynamics of $\mu_t$ under $\tilde{Q}$, we have

$$M^d(t, T) = e^{D(t, T) + \tilde{H}(t, T) - A(t, T)r_t - \tilde{G}(t, T)\mu_t},$$

where

$$\tilde{G}(t, T) = \frac{e^{c(T-t)} - 1}{c}$$

and

$$\tilde{H}(t, T) = \left(\frac{\rho_{12}\sigma\xi}{ac} - \frac{\xi^2}{2c^2}\right)[\tilde{G}(t, T) - (T-t)] + \frac{\rho_{12}\sigma\xi}{ac}[A(t, T) - \phi(t, T)] + \frac{\xi^2}{4c} \tilde{G}(t, T)^2$$

with

$$\phi(t, T) = \frac{1 - e^{-(a-c)(T-t)}}{a-c}.$$
Hence, the **annuity rate** $a_x(T)$ can be expressed as

$$a_x(T) = \sum_{n=0}^{\infty} M^d(T, T + n) = \sum_{n=0}^{\infty} \beta^d(T, T + n) e^{-V^d(T, T+n)} ,$$

where

$$\beta^d(t, T) = e^{D(t,T)+\tilde{H}(t,T)}$$

and

$$V^d(t, T) = A(t, T)r_t + \tilde{G}(t, T)\mu_t.$$
The survival measure

- Define the (survival) measure $\tilde{Q}$ equivalent to $\tilde{Q}$ via

\[
\frac{d\tilde{Q}}{d\tilde{Q}}\bigg|_{\mathcal{F}_T} = \Lambda^2 := \frac{e^{-\int_0^T \mu_u du} S(T, T)}{S(0, T)}.
\]

- Thus,

\[
\mathbb{E}^{\tilde{Q}} \left[ e^{-\int_t^T \mu_u du} e^{-\int_t^T l_u du} \bigg| \mathcal{F}_t \right] = S(t, T) \mathbb{E}^{\tilde{Q}} \left[ e^{-\int_t^T l_u du} \bigg| \mathcal{F}_t \right].
\]

- We have

\[
M^T(t, T) = B(t, T) S(t, T) \mathbb{E}^{\tilde{Q}} \left[ e^{-\int_t^T l_u du} \bigg| \mathcal{F}_t \right].
\]

- The dynamics of $l_t$ under $\tilde{Q}$ is given by

\[
dl_t = (hm - \rho_{13} \sigma \zeta A(t, T) - \rho_{23} \xi \zeta \tilde{G}(t, T) - hl_t) dt + \zeta d\bar{Z}_t.
\]
The survival measure (cont’d)

We then have

\[ \mathbb{E}^Q \left[ e^{-\int_t^T l_u du} | \mathcal{F}_t \right] = e^{-lt \bar{I}(t,T) + \bar{J}(t,T)}, \]

where

\[ \bar{I}(t,T) = \frac{1 - e^{-h(T-t)}}{h}, \]

and

\[ \bar{J}(t,T) = \left( \frac{\rho_{23} \xi \zeta}{ch} - \frac{\rho_{13} \sigma \zeta}{ah} - \frac{\zeta^2}{2h^2} + m \right) \left[ \bar{I}(t,T) - (T-t) \right] + \frac{\rho_{13} \sigma \zeta}{ah} [A(t,T) - \psi(t,T)] + \frac{\rho_{23} \xi \zeta}{ch} [\tilde{G}(t,T) - \psi(t,T)] \]

\[ - \frac{\zeta^2}{4h} \bar{I}(t,T)^2 \]

with

\[ \psi(t,T) = \frac{1 - e^{-(h-c)(T-t)}}{h-c} \quad \text{and} \quad \vartheta(t,T) = \frac{1 - e^{-(a+h)(T-t)}}{a+h}. \]
We obtain the analytic solution

\[ M^\tau(t, T) = \beta^\tau(t, T) e^{-V^\tau(t, T)}, \]

where

\[ \beta^\tau(t, T) = e^{D(t, T) + \tilde{H}(t, T) + J(t, T)} \]

and

\[ V^\tau(t, T) = A(t, T) r_t + \tilde{G}(t, T) \mu_t + \tilde{I}(t, T) l_t. \]
The endowment-risk-adjusted measure $\hat{Q}$

- Define measure $\hat{Q}$ equivalent to $Q$ as

$$ \frac{d\hat{Q}}{dQ} \Bigg|_{\mathcal{F}_T} = \Lambda^3_T := \frac{e^{-\int_0^T r_u du} e^{-\int_0^T \mu_u du} e^{-\int_0^T l_u du} M^\tau(T,T)}{M^\tau(0,T)}. $$

- Consequently,

$$ P_{GAO} = gM^\tau(0,T)\mathbb{E}^{\hat{Q}}[(a_x(T) - K)^+ | \mathcal{F}_0] $$

$$ = gM^\tau(0,T)\mathbb{E}^{\hat{Q}} \left[ \left( \sum_{n=0}^\infty \beta^d(T,T+n) e^{-V^d(T,T+n)} - K \right)^+ | \mathcal{F}_0 \right]. $$
The stochastic dynamics of $r_t$, $\mu_t$ and $l_t$ under $\hat{Q}$ are

\[
    dr_t = (ab - \sigma^2 A(t, T) - \rho_{12} \sigma \xi \tilde{G}(t, T) - \rho_{13} \sigma \zeta \tilde{I}(t, T) - ar_t)dt + \sigma d\hat{X}_t, \\
    d\mu_t = (c\mu_t - \rho_{12} \sigma \xi A(t, T) - \xi^2 \tilde{G}(t, T) - \rho_{23} \xi \zeta \tilde{I}(t, T))dt + \xi d\hat{Y}_t, \\
\]

and

\[
    dl_t = (hm - \rho_{13} \sigma \zeta A(t, T) - \zeta^2 \tilde{I}(t, T) - \rho_{23} \xi \zeta \tilde{G}(t, T))dt + \xi d\hat{Z}_t, \\
\]

where $d\hat{X}_t d\hat{Y}_t = \rho_{12} dt$, $d\hat{X}_t d\hat{Z}_t = \rho_{13} dt$ and $d\hat{Y}_t d\hat{Z}_t = \rho_{23} dt$. 

Numerical implementation

We provide a numerical experiment using our proposed (i) **change-of-measure method** for pricing and (ii) **Monte-Carlo simulation** (benchmark).

### Table 1: Parameter values

<table>
<thead>
<tr>
<th>Contract specification</th>
<th>Interest rate model</th>
<th>Mortality model</th>
<th>Lapse rate model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g = 11.1%$</td>
<td>$a = 0.15$</td>
<td>$c = 0.1$</td>
<td>$h = 0.12$</td>
</tr>
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<td>$T = 15$</td>
<td>$b = 0.045$</td>
<td>$\xi = 0.0003$</td>
<td>$m = 0.02$</td>
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<tr>
<td>$n = 35$</td>
<td>$\sigma = 0.03$</td>
<td>$\mu_0 = -0.006$</td>
<td>$\zeta = 0.01$</td>
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<tr>
<td></td>
<td>$r_0 = 0.045$</td>
<td></td>
<td>$l_0 = 0.02$</td>
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</table>
Table 2: Comparison of GAO prices obtained using our proposed method and Monte-Carlo method

<table>
<thead>
<tr>
<th>$(\rho_{12}, \rho_{13}, \rho_{23})$</th>
<th>MC</th>
<th>Proposed</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-0.9, -0.9, 0.81)$</td>
<td>0.06012 (0.00024)</td>
<td>0.05942 (0.00019)</td>
</tr>
<tr>
<td>$(-0.6, -0.6, 0.36)$</td>
<td>0.06682 (0.00030)</td>
<td>0.06608 (0.00021)</td>
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<tr>
<td>$(-0.3, -0.3, 0.09)$</td>
<td>0.07407 (0.00036)</td>
<td>0.07414 (0.00023)</td>
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<tr>
<td>$(0.0, 0.0, 0.0)$</td>
<td>0.08270 (0.00045)</td>
<td>0.08272 (0.00025)</td>
</tr>
<tr>
<td>$(0.3, 0.3, 0.3)$</td>
<td>0.09444 (0.00054)</td>
<td>0.09396 (0.00028)</td>
</tr>
<tr>
<td>$(0.6, 0.6, 0.6)$</td>
<td>0.10758 (0.00069)</td>
<td>0.10650 (0.00032)</td>
</tr>
<tr>
<td>$(0.9, 0.9, 0.9)$</td>
<td>0.11993 (0.00081)</td>
<td>0.11954 (0.00035)</td>
</tr>
<tr>
<td>$(-0.9, 0.81, -0.9)$</td>
<td>0.07866 (0.00043)</td>
<td>0.07868 (0.00023)</td>
</tr>
<tr>
<td>$(-0.6, 0.36, -0.6)$</td>
<td>0.07773 (0.00041)</td>
<td>0.07710 (0.00023)</td>
</tr>
<tr>
<td>$(-0.3, 0.09, -0.3)$</td>
<td>0.07941 (0.00042)</td>
<td>0.07880 (0.00024)</td>
</tr>
<tr>
<td>$(0.81, -0.9, -0.9)$</td>
<td>0.07947 (0.00038)</td>
<td>0.07865 (0.00026)</td>
</tr>
<tr>
<td>$(0.36, -0.6, -0.6)$</td>
<td>0.07875 (0.00038)</td>
<td>0.07772 (0.00025)</td>
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<tr>
<td>$(0.09, -0.3, -0.3)$</td>
<td>0.07957 (0.00040)</td>
<td>0.07972 (0.00025)</td>
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</table>

average computing time |
213.82 secs | 0.14 secs
### Table 3: GAO prices under constant and stochastic lapse rates

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<th>$\rho_{12}$</th>
<th>Constant</th>
<th>$\rho_{13}$</th>
<th>-0.9</th>
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<th>0.5</th>
<th>0.9</th>
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</table>
Section Outline

Introduction
Modelling framework
Derivation of GAO prices
Risk measurement of GAO
Description of risk measures
Moment-based density approximation
Numerical implementation

4 Risk measurement of GAO

- Description of risk measures
- Moment-based density approximation
- Numerical implementation
We evaluate GAO’s capital requirements through several risk measures recommended by regulatory authorities.

For $0 < \alpha < 1$, value at risk (VaR) is defined as

$$\text{VaR}_{\alpha}(Z) = \inf \{z : P(Z \leq z) \geq \alpha \}.$$ 

For $0 < \alpha < 1$, conditional tail expectation (CTE) is defined as

$$\text{CTE}_{\alpha}(Z) = \mathbb{E}[Z | Z > \text{VaR}_{\alpha}(Z)].$$
Description of risk measures (cont’d)

- The **distortion risk measure** is defined as

\[ \zeta_\chi(z) = \int_0^\infty \chi(S_Z(z))dz, \]

where \( S_Z(z) \) is the survival function of the loss random variable (RV) \( Z \), and \( \chi(x) \) is the distortion function \( \chi: [0, 1] \rightarrow [0, 1] \), which is a non-decreasing function with \( \chi(0) = 0 \) and \( \chi(1) = 1 \).

- The **distortion function** can be

  (Proportional hazard transform) \( \chi(x) = x^\gamma \),

  (Wang transform) \( \chi(x) = \Phi(\Phi^{-1}(x) + \Phi^{-1}(\nu)) \),

  (Lookback transform) \( \chi(x) = x^\eta(1 - \eta \log(x)) \).
The spectral risk measure $\varphi$ is given by

$$\varphi_\omega = \int_0^1 \omega(\upsilon)q(\upsilon)d\upsilon,$$

where $\omega(\upsilon)$ is a weighting function such that $\int_0^1 \omega(\upsilon)d\upsilon = 1$ and $q(\upsilon)$ is a quantile function of a loss RV.

Two commonly-used weighting functions

$$\omega_E(\upsilon) = \frac{\kappa e^{-\kappa(1-\upsilon)}}{1 - e^{-\kappa}} \text{ (exponential function),}$$

$$\omega_P(\upsilon) = \delta \upsilon^{\delta-1} \text{ (power function).}$$
Moment-based density approximation

- **Underlying idea:** the exact density function with known first $n$ moments can be approximated by the product of
  (i) a base density, and
  (ii) a polynomial of degree $q$.

- **Define the ‘liability’ or loss RV**

$$L_p = ge^{-\int_0^T \mu_u du} e^{-\int_0^T l_u du} \left( \sum_{n=0}^{\infty} \beta^d(T, T + n) e^{-V^d(T, T+n) - K} \right).$$

- **Write**

$$L := \begin{cases} 
0 & \text{if } L_p \leq 0, \\
L_p & \text{if } L_p > 0. 
\end{cases}$$
Procedure for moment-based density approximation

- Choose the gamma distribution as the base function.

- Make the transformation $Z := L_p - u$, where $u$ is a relatively small value.

- Let the moments of the random variable $Z$ be $\mu_Z(i)$ for $i = 0, 1, \ldots, q$.

- Let the theoretical moments of the base function $\Psi(z)$ be $m_Z(i)$ for $i = 0, 1, \ldots, 2q$.

- The parameters $\alpha$ and $\theta$ of $\Psi(z)$ are determined by setting $\mu_Z(i) = m_Z(i)$ for $i = 1, 2$. 

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The parameters $\alpha$ and $\theta$ of $\Psi(z)$ are determined by setting $\mu_Z(i) = m_Z(i)$ for $i = 1, 2$.
The approximated density of $L_p$ is given by

$$f_{L_p}(l) = \frac{(l - u)^{\alpha - 1}}{\Gamma(\alpha)\theta^\alpha} e^{-\frac{(l - u)}{\theta}} \sum_{i=0}^{q} k_i (l - u)^i.$$ 

$k_0, k_1, \ldots, k_n$ are determined by

$$(k_0, k_1, \ldots, k_n)^\top = M^{-1}(\mu_Z(0), \mu_Z(1), \ldots, \mu_Z(q))^\top,$$

where $M$ is a $(q + 1) \times (q + 1)$ symmetric matrix whose $(i + 1)^{th}$ row is $(m_Z(i), m_Z(i + 1), \ldots, m_Z(i + q))$. 

A two-decrement model for the valuation and risk measurement of a guaranteed annuity option Yixing Zhao
Figure 1: Approximating the distribution of $L_p$
### Table 4: Risk measures of gross loss for GAO under different sample sizes

<table>
<thead>
<tr>
<th>Risk measures</th>
<th>$N = 10,000$</th>
<th>$N = 100,000$</th>
<th>$N = 1,000,000$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ECDF</td>
<td>MCDF</td>
<td>ECDF</td>
</tr>
<tr>
<td>VaR ($\alpha = 0.90$)</td>
<td>0.1659</td>
<td>0.1631</td>
<td>0.1655</td>
</tr>
<tr>
<td>VaR ($\alpha = 0.95$)</td>
<td>0.2115</td>
<td>0.2068</td>
<td>0.2114</td>
</tr>
<tr>
<td>VaR ($\alpha = 0.99$)</td>
<td>0.3127</td>
<td>0.3042</td>
<td>0.3190</td>
</tr>
<tr>
<td>CTE ($\alpha = 0.90$)</td>
<td>0.2300</td>
<td>0.2243</td>
<td>0.2328</td>
</tr>
<tr>
<td>CTE ($\alpha = 0.95$)</td>
<td>0.2739</td>
<td>0.2664</td>
<td>0.2798</td>
</tr>
<tr>
<td>CTE ($\alpha = 0.99$)</td>
<td>0.3757</td>
<td>0.3658</td>
<td>0.3964</td>
</tr>
<tr>
<td>WT ($\gamma = 0.90$)</td>
<td>0.1872</td>
<td>0.1818</td>
<td>0.1920</td>
</tr>
<tr>
<td>WT ($\gamma = 0.50$)</td>
<td>0.2340</td>
<td>0.2254</td>
<td>0.2426</td>
</tr>
<tr>
<td>WT ($\gamma = 0.10$)</td>
<td>0.3356</td>
<td>0.3156</td>
<td>0.3574</td>
</tr>
<tr>
<td>PH ($\iota = 0.10$)</td>
<td>0.0752</td>
<td>0.0736</td>
<td>0.0760</td>
</tr>
<tr>
<td>PH ($\iota = 0.05$)</td>
<td>0.1361</td>
<td>0.1317</td>
<td>0.1401</td>
</tr>
<tr>
<td>PH ($\iota = 0.01$)</td>
<td>0.4199</td>
<td>0.4076</td>
<td>0.4895</td>
</tr>
<tr>
<td>LB ($\eta = 0.90$)</td>
<td>0.1549</td>
<td>0.1496</td>
<td>0.1575</td>
</tr>
<tr>
<td>LB ($\eta = 0.50$)</td>
<td>0.2668</td>
<td>0.2500</td>
<td>0.2816</td>
</tr>
<tr>
<td>LB ($\eta = 0.10$)</td>
<td>0.5885</td>
<td>0.5670</td>
<td>0.7190</td>
</tr>
<tr>
<td>EWQRM ($\kappa = 1$)</td>
<td>0.0867</td>
<td>0.0852</td>
<td>0.0875</td>
</tr>
<tr>
<td>EWQRM ($\kappa = 20$)</td>
<td>0.2469</td>
<td>0.2444</td>
<td>0.2517</td>
</tr>
<tr>
<td>EWQRM ($\kappa = 100$)</td>
<td>0.3512</td>
<td>0.3564</td>
<td>0.3660</td>
</tr>
<tr>
<td>PWRM ($\delta = 1$)</td>
<td>0.0672</td>
<td>0.0659</td>
<td>0.0678</td>
</tr>
<tr>
<td>PWRM ($\delta = 20$)</td>
<td>0.2485</td>
<td>0.2460</td>
<td>0.2534</td>
</tr>
<tr>
<td>PWRM ($\delta = 100$)</td>
<td>0.3515</td>
<td>0.3567</td>
<td>0.3664</td>
</tr>
</tbody>
</table>
Figure 2: Variation of risk measures as a function of $\rho_{13}$ with a given $\rho_{12}$ and $\rho_{23} = \rho_{12}\rho_{13}$
Numerical implementation (cont’d)

Figure 3: Sensitivity of risk measures to various parameters
5 Conclusion

Conclusion
This work contributes to the development of an integrated modelling framework for GAO pricing and capital requirement determination.

Each of the three risk factors has an affine structure specification and their correlations with one another is fully described.

We employed iteratively the change of probability measure technique to efficiently and accurately compute GAO prices.

We further evaluated seven different risk measures for GAO through the empirical CDF and moment-based density approximation methods.
Our numerical results show efficiency and accuracy of our proposed methods GAO’s valuation and risk measurement.

Our results suggest that lapse rate’s stochastic behaviour must be captured accurately and taken into account when designing, pricing and monitoring insurance products.