A two-decrement model for the valuation and risk measurement of a guaranteed annuity option

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Outline

Section Outline

Research motivation

- Financial innovations – response to increased longevity and ageing population.
- Insurance market is becoming an investment hub.
- Interest and mortality risks – primary factors in valuation and risk management of longevity products. But, lapse risk is also very important.
- Lapse risk – possibility that policyholders terminate their policies early ... for various reasons.
- Dire consequences from policy lapses – huge losses and liquidity problem for insurance companies.

Research motivation (cont'd)

- In current practice, lapse rate is assumed constant or deterministic in actuarial valuation.
- Research advances on lapse risk modelling are rather slow, unlike those for interest and mortality dynamics.
- Policyholders' decision to surrender is directly affected by economic circumstances.

Objectives

- **Develop** an integrated approach that addresses simultaneously guaranteed annuity option (GAO)'s *pricing* and *capital requirement calculation.*
- **Construct** a two-decrement stochastic model in which death and policy lapse occurrences with their correlations to the financial risk are explicitly modelled.
- **Apply** series of probability measure changes resulting to forward, survival, and risk-endowment measures.
- **Determine** risk measures using moment-based density method and results benchmarked with the Monte-Carlo simulation. *Our forumulation highlights the link between pricing and capital requirement.*

Section Outline

2 Modelling framework

- Interest rate model
- Mortality model
- Lapse rate model
- Valuation framework

Interest rate model

We assume short-interest rate r_t follows the Vasiček model via the SDE

$$\boxed{dr_t = a(b - r_t)dt + \sigma dX_t},$$

where a, b, and σ are positive constants and X_t is a standard one-dimensional Brownian motion.

Interest rate model (cont'd)

Price $B(t,T)$ of a T-maturity zero-coupon bond at time $t<T$ is known to be

$$B(t,T) = \mathrm{E}^{Q}[e^{-\int_t^T r_u du} | \mathcal{F}_t] = e^{-A(t,T)r_t + D(t,T)},$$

where

$$A(t,T) = \frac{1-e^{-a(T-t)}}{a}$$

and

$$D(t,T) = \left(b - \frac{\sigma^2}{2a^2}\right)[A(t,T) - (T-t)] - \frac{\sigma^2 A(t,T)^2}{4a}.$$

Mortality model

- The dynamics of the **force of mortality process** μ_t is given by

$$\boxed{d\mu_t = c\mu_t dt + \xi dY_t},$$

where c and ξ are positive constants, and Y_t is a standard Brownian motion **correlated** with X_t,

$$dX_t dY_t = \rho_{12} dt.$$

- The **survival function** is defined by

$$\boxed{S(t,T) = \mathbb{E}^Q \left[e^{-\int_t^T \mu_u du} \middle| \mathcal{F}_t \right]}.$$

Lapse rate framework

- For the **lapse rate process** l_t, we adopt the dynamics

$$\boxed{dl_t = h(m - l_t)dt + \zeta dZ_t},$$

where h, m and ζ are positive constants and Z_t is a standard BM **correlated with both** X_t and Y_t.

- In particular,

$$dX_t dZ_t = \rho_{13} dt \quad \text{and} \quad dY_t dZ_t = \rho_{23} dt.$$

Valuation framework

- Let $M^d(t,T)$ be the **fair value at time t of a pure endowment** of \$1 at maturity T when mortality is the only decrement, i.e.,

$$M^d(t,T) = \mathbb{E}^Q\left[e^{-\int_t^T r_u du} e^{-\int_t^T \mu_u du} \middle| \mathcal{F}_t\right].$$

- Let $M^\tau(t,T)$ be the **fair value at time t of a \$1 pure endowment** at maturity T under a two-decrement model (both mortality and lapse rates are considered), i.e.,

$$M^\tau(t,T) = \mathbb{E}^Q\left[e^{-\int_t^T r_u du} e^{-\int_t^T \mu_u du} e^{-\int_t^T l_u du} \middle| \mathcal{F}_t\right].$$

Valuation framework(cont'd)

Define $a_x(T)$ as the **annuity rate.** A life annuity is a contract that pays \$1 to an insured annually conditional on his/her survival at the moment of payments.

That is,

$$a_x(T) = \sum_{n=0}^{\infty} \mathbb{E}^Q \left[e^{-\int_T^{T+n} r_u du} e^{-\int_T^{T+n} \mu_u du} \middle| \mathcal{F}_T \right] = \sum_{n=0}^{\infty} M^d(T, T+n).$$

Valuation framework (cont'd)

- GAO is a contract that gives the policyholder the right to *convert a survival benefit into an annuity at a pre-specified guaranteed conversion rate* g.

- GAO's **loss function** L is the payoff 'discounted' by mortality and lapse factors, i.e.,

$$\boxed{L = g e^{-\int_0^T \mu_u du} e^{-\int_0^T l_u du} (a_x(T) - K)^+},$$

where $K = 1/g$.

- The **fair value of GAO** at time 0, by risk-neutral pricing, is

$$\boxed{P_{GAO} = g \mathbb{E}^Q \left[e^{-\int_0^T r_u du} e^{-\int_0^T \mu_u du} e^{-\int_0^T l_u du} (a_x(T) - K)^+ \middle| \mathcal{F}_0 \right]}.$$

Section Outline

3 Derivation of GAO prices

- The forward measure
- The survival measure
- The endowment-risk-adjusted measure
- Numerical implementation

The forward measure

- The forward measure $\widetilde{Q}$ is constructed with the aid of the Radon-Nikodŷm derivative and Girsanov's theorem, and in particular

$$\left.\frac{d\widetilde{Q}}{dQ}\right|_{\mathcal{F}_T} = \Lambda_T^1 := \frac{e^{-\int_0^T r_u du} B(T,T)}{B(0,T)}.$$

- The dynamics of μ_t under $\widetilde{Q}$ is given by

$$d\mu_t = [-\rho_{12}\sigma\xi A(t,T) + c\mu_t]dt + \xi d\widetilde{Y}_t.$$

The forward measure (cont'd)

- The *pure endowment under two-decrement model* can be represented as

$$M^{\tau}(t,T) = B(t,T)\mathbb{E}^{\widetilde{Q}}\left[e^{-\int_t^T \mu_u du} e^{-\int_t^T l_u du} \middle| \mathcal{F}_t \right].$$

- The *pure endowment under one-decrement model* can be expressed as

$$M^{d}(t,T) = B(t,T)\mathbb{E}^{\widetilde{Q}}\left[e^{-\int_t^T \mu_u du} \middle| \mathcal{F}_t \right].$$

The forward measure (cont'd)

Given the dynamics of μ_t under $\widetilde{Q}$, we have

$$\boxed{M^d(t,T) = e^{D(t,T)+\widetilde{H}(t,T)-A(t,T)r_t-\widetilde{G}(t,T)\mu_t}},$$

where

$$\widetilde{G}(t,T) = \frac{e^{c(T-t)}-1}{c}$$

and

$$\begin{aligned}\widetilde{H}(t,T) &= \left(\frac{\rho_{12}\sigma\xi}{ac} - \frac{\xi^2}{2c^2}\right)[\widetilde{G}(t,T)-(T-t)] + \\ &\quad \frac{\rho_{12}\sigma\xi}{ac}[A(t,T)-\phi(t,T)] + \frac{\xi^2}{4c}\widetilde{G}(t,T)^2\end{aligned}$$

with

$$\phi(t,T) = \frac{1-e^{-(a-c)(T-t)}}{a-c}.$$

The forward measure(cont'd)

Hence, the **annuity rate** $a_x(T)$ can be expressed as

$$a_x(T) = \sum_{n=0}^{\infty} M^d(T, T+n) = \sum_{n=0}^{\infty} \beta^d(T, T+n) e^{-V^d(T,T+n)},$$

where

$$\beta^d(t, T) = e^{D(t,T)+\widetilde{H}(t,T)}$$

and

$$V^d(t, T) = A(t, T) r_t + \widetilde{G}(t, T)\mu_t.$$

The survival measure

- *Define the (survival) measure $\bar{Q}$* equivalent to $\widetilde{Q}$ via

$$\left.\frac{d\bar{Q}}{d\widetilde{Q}}\right|_{\mathcal{F}_T} = \Lambda_T^2 := \frac{e^{-\int_0^T \mu_u du} S(T,T)}{S(0,T)}.$$

- Thus,

$$\mathbb{E}^{\widetilde{Q}}\left[e^{-\int_t^T \mu_u du} e^{-\int_t^T l_u du} \middle| \mathcal{F}_t \right] = S(t,T)\mathbb{E}^{\bar{Q}}\left[e^{-\int_t^T l_u du} \middle| \mathcal{F}_t \right].$$

- We have

$$M^{\tau}(t,T) = B(t,T)S(t,T)\mathbb{E}^{\bar{Q}}\left[e^{-\int_t^T l_u du} \middle| \mathcal{F}_t \right].$$

- The dynamics of l_t under $\bar{Q}$ is given by

$$dl_t = (hm - \rho_{13}\sigma\zeta A(t,T) - \rho_{23}\xi\zeta\widetilde{G}(t,T) - hl_t)dt + \zeta d\overline{Z}_t.$$

The survival measure (cont'd)

We then have

$$\boxed{\mathbb{E}^{\overline{Q}}[e^{-\int_t^T l_u du}|\mathcal{F}_t] = e^{-l_t \bar{I}(t,T)+\bar{J}(t,T)}},$$

where

$$\bar{I}(t,T) = \frac{1-e^{-h(T-t)}}{h},$$

and

$$\begin{aligned}\bar{J}(t,T) \quad = \quad & \left(\frac{\rho_{23}\xi\zeta}{ch} - \frac{\rho_{13}\sigma\zeta}{ah} - \frac{\zeta^2}{2h^2} + m\right)[\bar{I}(t,T)-(T-t)] + \\ & \frac{\rho_{13}\sigma\zeta}{ah}[A(t,T)-\vartheta(t,T)] + \frac{\rho_{23}\xi\zeta}{ch}[\widetilde{G}(t,T)-\psi(t,T)] \\ & -\frac{\zeta^2}{4h}\bar{I}(t,T)^2\end{aligned}$$

with

$$\psi(t,T) = \frac{1-e^{-(h-c)(T-t)}}{h-c} \quad \text{and} \quad \vartheta(t,T) = \frac{1-e^{-(a+h)(T-t)}}{a+h}.$$

The survival measure (cont'd)

We obtain the analytic solution

$$\boxed{M^{\tau}(t,T) = \beta^{\tau}(t,T)e^{-V^{\tau}(t,T)}},$$

where

$$\beta^{\tau}(t,T) = e^{D(t,T)+\widetilde{H}(t,T)+\bar{J}(t,T)}$$

and

$$V^{\tau}(t,T) = A(t,T)r_t + \widetilde{G}(t,T)\mu_t + \bar{I}(t,T)l_t.$$

The endowment-risk-adjusted measure $\widehat{Q}$

- Define measure $\widehat{Q}$ equivalent to Q as

$$\left.\frac{d\widehat{Q}}{dQ}\right|_{\mathcal{F}_T} = \Lambda_T^3 := \frac{e^{-\int_0^T r_u du} e^{-\int_0^T \mu_u du} e^{-\int_0^T l_u du} M^\tau(T,T)}{M^\tau(0,T)}.$$

- Consequently,

$$P_{GAO} = g M^\tau(0,T) \mathbb{E}^{\widehat{Q}}[(a_x(T) - K)^+ | \mathcal{F}_0]$$

$$\boxed{= g M^\tau(0,T) \mathbb{E}^{\widehat{Q}}\left[\left(\sum_{n=0}^{\infty} \beta^d(T, T+n) e^{-V^d(T,T+n)} - K\right)^+ \middle| \mathcal{F}_0\right]}.$$

The endowment-risk-adjusted measure (cont'd)

The stochastic dynamics of r_t, μ_t and I_t under $\widehat{Q}$ are

$$dr_t = (ab - \sigma^2 A(t,T) - \rho_{12}\sigma\xi\widetilde{G}(t,T) - \rho_{13}\sigma\zeta\bar{I}(t,T) - ar_t)dt + \sigma d\widehat{X}_t,$$

$$d\mu_t = (c\mu_t - \rho_{12}\sigma\xi A(t,T) - \xi^2\widetilde{G}(t,T) - \rho_{23}\xi\zeta\bar{I}(t,T))dt + \xi d\widehat{Y}_t,$$

and

$$dI_t = (hm - \rho_{13}\sigma\zeta A(t,T) - \zeta^2\bar{I}(t,T) - \rho_{23}\xi\zeta\widetilde{G}(t,T))dt + \xi d\widehat{Z}_t,$$

where $d\widehat{X}_t d\widehat{Y}_t = \rho_{12}dt$, $d\widehat{X}_t d\widehat{Z}_t = \rho_{13}dt$ and $d\widehat{Y}_t d\widehat{Z}_t = \rho_{23}dt$.

Numerical implementation

We provide a numerical experiment using our proposed (i) **change-of-measure method** for pricing and (ii) **Monte-Carlo simulation** (**benchmark**).

Table 1: Parameter values

Contract specification			
$g = 11.1\%$	$T = 15$	$n = 35$	
Interest rate model			
$a = 0.15$	$b = 0.045$	$\sigma = 0.03$	$r_0 = 0.045$
Mortality model			
$c = 0.1$	$\xi = 0.0003$	$\mu_0 = -0.006$	
Lapse rate model			
$h = 0.12$	$m = 0.02$	$\zeta = 0.01$	$l_0 = 0.02$

Numerical implementation (cont'd)

Table 2: **Comparison of GAO prices obtained using our proposed method and Monte-Carlo method**

$(\rho_{12},\ \rho_{13},\ \rho_{23})$	MC	Proposed
$(-0.9, -0.9, 0.81)$	0.06012 (0.00024)	0.05942 (0.00019)
$(-0.6, -0.6, 0.36)$	0.06682 (0.00030)	0.06608 (0.00021)
$(-0.3, -0.3, 0.09)$	0.07407 (0.00036)	0.07414 (0.00023)
$(0.0, 0.0, 0.0)$	0.08270 (0.00045)	0.08272 (0.00025)
$(0.3, 0.3, 0.3)$	0.09444 (0.00054)	0.09396 (0.00028)
$(0.6, 0.6, 0.6)$	0.10758 (0.00069)	0.10650 (0.00032)
$(0.9, 0.9, 0.9)$	0.11993 (0.00081)	0.11954 (0.00035)
$(-0.9, 0.81, -0.9)$	0.07866 (0.00043)	0.07868 (0.00023)
$(-0.6, 0.36, -0.6)$	0.07773 (0.00041)	0.07710 (0.00023)
$(-0.3, 0.09, -0.3)$	0.07941 (0.00042)	0.07880 (0.00024)
$(0.81, -0.9, -0.9)$	0.07947 (0.00038)	0.07865 (0.00026)
$(0.36, -0.6, -0.6)$	0.07875 (0.00038)	0.07772 (0.00025)
$(0.09, -0.3, -0.3)$	0.07957 (0.00040)	0.07972 (0.00025)
average computing time	213.82 secs	0.14 secs

Table 3: **GAO prices under constant and stochastic lapse rates**

ρ_{12}	Constant	ρ_{13}				
		-0.9	-0.5	0	0.5	0.9
-0.9	0.0679	0.0595	0.0635	0.0689	0.0747	0.0800
-0.8	0.0693	0.0603	0.0651	0.0704	0.0762	0.0817
-0.7	0.0712	0.0615	0.0660	0.0721	0.0777	0.0835
-0.6	0.0717	0.0631	0.0675	0.0731	0.0800	0.0859
-0.5	0.0738	0.0639	0.0685	0.0751	0.0819	0.0880
-0.4	0.0750	0.0652	0.0698	0.0768	0.0838	0.0899
-0.3	0.0765	0.0662	0.0712	0.0782	0.0854	0.0919
-0.2	0.0780	0.0677	0.0727	0.0800	0.0876	0.0944
-0.1	0.0808	0.0685	0.0739	0.0814	0.0892	0.0961
0.0	0.0810	0.0700	0.0754	0.0831	0.0911	0.0985
0.1	0.0836	0.0711	0.0769	0.0843	0.0933	0.1001
0.2	0.0842	0.0718	0.0780	0.0859	0.0948	0.1029
0.3	0.0863	0.0731	0.0799	0.0881	0.0975	0.1049
0.4	0.0883	0.0749	0.0812	0.0898	0.0993	0.1071
0.5	0.0896	0.0757	0.0823	0.0911	0.1013	0.1097
0.6	0.0913	0.0770	0.0839	0.0929	0.1035	0.1121
0.7	0.0930	0.0776	0.0854	0.0949	0.1057	0.1145
0.8	0.0948	0.0793	0.0865	0.0964	0.1074	0.1168
0.9	0.0964	0.0812	0.0884	0.0980	0.1094	0.1192

Section Outline

4. Risk measurement of GAO
   - Description of risk measures
   - Moment-based density approximation
   - Numerical implementation

Description of risk measures

- We evaluate GAO's capital requirements through several risk measures recommended by regulatory authorities.
- For $0 < \alpha < 1$, value at risk (VaR) is defined as

$$\mathrm{VaR}_\alpha(Z) = \inf\{z : P(Z \leq z) \geq \alpha\}.$$

- For $0 < \alpha < 1$, conditional tail expectation (CTE) is defined as

$$\mathrm{CTE}_\alpha(Z) = \mathbb{E}[Z|Z > \mathrm{VaR}_\alpha(Z)].$$

Description of risk measures (cont'd)

- The **distortion risk measure** is defined as

$$\zeta_{\chi}(z) = \int_0^{\infty} \chi(S_Z(z))dz,$$

where $S_Z(z)$ is the survival function of the loss random variable (RV) Z, and $\chi(x)$ is the distortion function $\chi\colon [0,1] \to [0,1]$, which is a non-decreasing function with $\chi(0) = 0$ and $\chi(1) = 1$.

- The **distortion function** can be

$$\text{(Proportional hazard transform)} \quad \chi(x) = x^{\gamma},$$

$$\text{(Wang transform)} \quad \chi(x) = \Phi(\Phi^{-1}(x) + \Phi^{-1}(\iota)),$$

$$\text{(Lookback transform)} \quad \chi(x) = x^{\eta}(1 - \eta \log(x)).$$

Description of risk measures(cont'd)

- The **spectral risk measure** φ is given by

$$\varphi_\omega = \int_0^1 \omega(\upsilon) q(\upsilon) d\upsilon,$$

where $\omega(\upsilon)$ is a weighting function such that $\int_0^1 \omega(\upsilon) d\upsilon = 1$ and $q(\upsilon)$ is a quantile function of a loss RV.

- Two commonly-used **weighting functions**

$$\omega_E(\upsilon) = \frac{\kappa e^{-\kappa(1-\upsilon)}}{1-e^{-\kappa}} \text{ (exponential function)},$$

$$\omega_P(\upsilon) = \delta \nu^{\delta-1} \text{ (power function)}.$$

- **Underlying idea:** the exact density function with known first n moments can be approximated by the product of
  (i) a base density, and
  (ii) a polynomial of degree q.

- Define the '**liability**' or loss RV

$$L_p = g e^{-\int_0^T \mu_u du} e^{-\int_0^T l_u du} \left(\sum_{n=0}^{\infty} \beta^d(T, T+n) e^{-V^d(T,T+n)} - K \right).$$

- Write

$$L := \begin{cases} 0 & \text{if } L_p \leq 0, \\ L_p & \text{if } L_p > 0. \end{cases}$$

Procedure for moment-based density approximation

- Choose the gamma distribution as the base function.
- Make the transformation $Z := L_p - u$, where u is a relatively small value.
- Let the moments of the random variable Z be $\mu_Z(i)$ for $i = 0, 1, \ldots, q$.
- Let the theoretical moments of the base function $\Psi(z)$ be $m_Z(i)$ for $i = 0, 1, \ldots, 2q$.
- The parameters α and θ of $\Psi(z)$ are determined by setting $\mu_Z(i) = m_Z(i)$ for $i = 1, 2$.

Procedure for moment-based density approximation (cont'd)

- The **approximated density** of L_p is given by

$$f_{L_p}(l) = \frac{(l-u)^{\alpha-1}}{\Gamma(\alpha)\theta^{\alpha}} e^{-(l-u)/\theta} \sum_{i=0}^{q} k_i (l-u)^i.$$

- $k_0, k_1, \ldots, k_n$ are determined by

$$(k_0, k_1, \ldots, k_n)^\top = \mathbf{M}^{-1}(\mu_Z(0), \mu_Z(1), \ldots, \mu_Z(q))^\top,$$

where $\mathbf{M}$ is a $(q+1)\times(q+1)$ symmetric matrix whose $(i+1)^{\text{th}}$ row is $(m_Z(i), m_Z(i+1), \ldots, m_Z(i+q))$.

Numerical implementation

Figure 1: **Approximating the distribution of L_p**

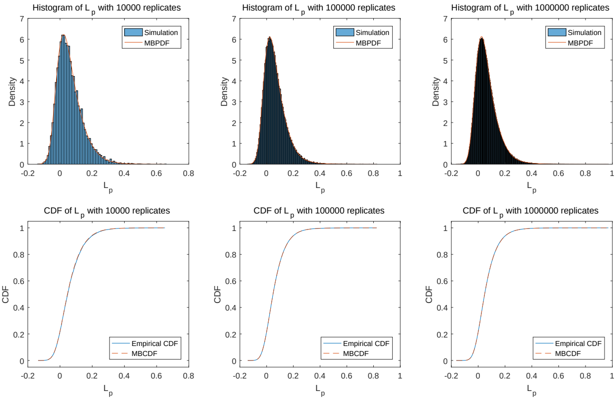


Table 4: **Risk measures of gross loss for GAO under different sample sizes**

Risk measures	$N = 10,000$		$N = 100,000$		$N = 1,000,000$	
	ECDF	MCDF	ECDF	MCDF	ECDF	MCDF
VaR ($\alpha = 0.90$)	0.1659	0.1631	0.1655	0.1658	0.1669	0.1681
VaR ($\alpha = 0.95$)	0.2115	0.2068	0.2114	0.2104	0.2139	0.2143
VaR ($\alpha = 0.99$)	0.3127	0.3042	0.3190	0.3211	0.3237	0.3271
CTE ($\alpha = 0.90$)	0.2300	0.2243	0.2328	0.2321	0.2353	0.2364
CTE ($\alpha = 0.95$)	0.2739	0.2664	0.2798	0.2787	0.2830	0.2842
CTE ($\alpha = 0.99$)	0.3757	0.3658	0.3964	0.3935	0.3970	0.4016
WT ($\gamma = 0.90$)	0.1872	0.1818	0.1920	0.1922	0.1935	0.1938
WT ($\gamma = 0.50$)	0.2340	0.2254	0.2426	0.2430	0.2444	0.2443
WT ($\gamma = 0.10$)	0.3356	0.3156	0.3574	0.3586	0.3614	0.3570
PH ($\iota = 0.10$)	0.0752	0.0736	0.0760	0.0760	0.0765	0.0767
PH ($\iota = 0.05$)	0.1361	0.1317	0.1401	0.1402	0.1413	0.1411
PH ($\iota = 0.01$)	0.4199	0.4076	0.4895	0.4908	0.5332	0.5345
LB ($\eta = 0.90$)	0.1549	0.1496	0.1575	0.1576	0.1587	0.1593
LB ($\eta = 0.50$)	0.2668	0.2500	0.2816	0.2826	0.2853	0.2822
LB ($\eta = 0.10$)	0.5885	0.5670	0.7190	0.7215	0.8148	0.8186
EWQRM ($\kappa = 1$)	0.0867	0.0852	0.0875	0.0875	0.0881	0.0884
EWQRM ($\kappa = 20$)	0.2469	0.2444	0.2517	0.2517	0.2542	0.2557
EWQRM ($\kappa = 100$)	0.3512	0.3564	0.3660	0.3664	0.3673	0.3714
PWRM ($\delta = 1$)	0.0672	0.0659	0.0678	0.0678	0.0683	0.0684
PWRM ($\delta = 20$)	0.2485	0.2460	0.2534	0.2534	0.2559	0.2574
PWRM ($\delta = 100$)	0.3515	0.3567	0.3664	0.3668	0.3676	0.3718

Numerical implementation (cont'd)

Figure 2: **Variation of risk measures as a function of ρ_{13} with a given ρ_{12} and $\rho_{23} = \rho_{12}\rho_{13}$**

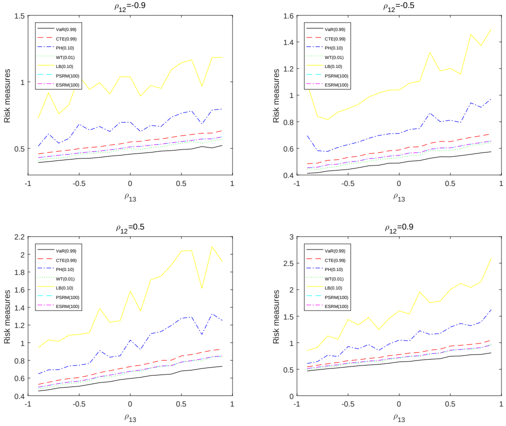
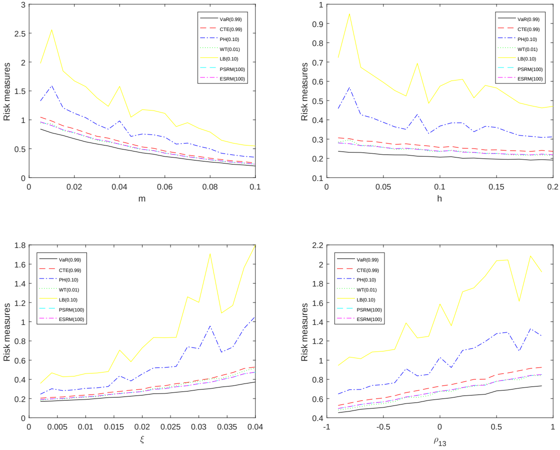


Figure 3: **Sensitivity of risk measures to various parameters**



Section Outline

5 Conclusion

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Conclusion

- This work contributes to the **development of an integrated modelling framework for GAO** pricing and capital requirement determination.
- Each of the three risk factors has an affine structure specification and their correlations with one another is fully described.
- We employed iteratively the change of probability measure technique to efficiently and accurately compute GAO prices.
- We further evaluated seven different risk measures for GAO through the empirical CDF and moment-based density approximation methods.

Conclusion (cont'd)

- Our numerical results show efficiency and accuracy of our proposed methods GAO's valuation and risk measurement.
- Our results suggest that lapse rate's stochastic behaviour must be captured accurately and taken into account when designing, pricing and monitoring insurance products.