Assessing the effectiveness of local and global quadratic hedging under GARCH models

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Joint work with

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Motivation

- Delta hedging: most popular hedging approach
  \[ \frac{\partial \text{Option}}{\partial S} \cdot \Delta S \approx \Delta \text{Option} \]

- Groundbreaking work due in great part to Black, Merton and Scholes

- Very powerful theoretical concept based on the paradigm of market completeness and perfect replication

- Continuous-time
Motivation

- In real life, delta hedging results in an imperfect solution
- Impossibility of trading in continuous time
- Sudden price jumps
- Market frictions
- Incomplete markets
Motivation

- Alternatives to delta hedging

- **Local risk-minimization** (Schweizer, 1988, 1991)

- **Global risk-minimization** (Schweizer, 1995)

- Perfect replication is dropped in favor of a more realistic objective: *minimizing hedging costs*

- **Quadratic criterion**

- **Discrete time setting**
Objective

Objective: Investigate the empirical and practical relevance of local and global risk-minimization

Three main questions:

1. Value added of global VS local quadratic hedging?
2. Choice of measure: $\mathbb{P}$ VS $\mathbb{Q}$?
3. How is hedging effectiveness impacted by model risk?
Definition of the financial market

- **Discrete time**: trading occurs at \( \{0, 1, \ldots, T\} \)

- Two traded assets: one **risky stock** \( \{S_t\} \), and one **risk-free bond** \( \{B_t\} \), where \( B_t = \exp(rt) \)

- **Incomplete market**

- \( \mathbb{P} \): Real-world (physical, observed) probability measure

- \( \mathbb{Q} \): Equivalent martingale measure (risk-neutral measure)
Choice of model for the stock asset

- Implementation: **GARCH models**

- Ubiquitous in the econometrics literature due to their strength in explaining *volatility dynamics*

- Have also been shown to perform well as *option pricing* models (Christoffersen et al., 2010)

- **Easy to estimate** and manipulate (in contrast to stochastic volatility jump-diffusion models)

- Allows us to relate to Badescu et al. (2014); Ortega (2012); Rémillard and Rubenthaler (2013)
Risky asset \( \{S_t\}_{t=0}^T \) follows GJR-GARCH(1,1):

\[
\log \left( \frac{S_t}{S_{t-1}} \right) = \mu + \sigma_t \epsilon \\
\sigma_{t+1}^2 = \omega + \alpha \sigma_t^2 \left( |\epsilon_t| - \gamma \epsilon_t \right)^2 + \beta \sigma_t^2
\]

where \( \epsilon_t \) is a standardized Gaussian white noise under \( \mathbb{P} \)

**Note:** the methodology presented in the paper can be applied to any other GARCH(1,1) specification
Contingent claim

Our numerical analysis focuses on call options:

\[ H = \max(0, S_T - K) \]

General problem: risk management in an incomplete market of this derivative position with a dynamic hedging strategy and initial capital of \( V_0 \)
Hedging portfolio

Trading strategy: \( \theta = \{(\theta^{(B)}_t, \theta^{(S)}_t)\}_{t=0}^T \)

\( \theta^{(B)}_t \): number of bond asset shares to be held over the time period \([t, t+1)\)

\( \theta^{(S)}_t \): number of stock asset shares held over the time period \((t-1, t]\), i.e., \( \theta^{(S)}_t \) is determined at the previous time step

Value of hedging portfolio:

\[ V_t^\theta = \theta^{(B)}_t B_t + \theta^{(S)}_t S_t \]
Local quadratic hedging

- Hedging strategy that is **not self-financing**

- Computed through a **series of local optimizations** having for objective to minimize the incremental cost incurred at the next trading period

- Impose $V_T^\theta = H$. At time $t = T, T - 1, \ldots$, find the positions
  
  $$(\theta_t^{(B)}, \theta_t^{(S)})$$

  such that

  $$\mathbb{E}^{\mathbb{P}}[(C_t^{\theta} - C_{t-1}^{\theta})^2 | \mathcal{F}_{t-1}]$$

  is minimized under $\mathbb{P}$.
The solution to the local quadratic hedging problem is fully determined by the following backward recursive scheme initiated at $t = T$ and $\bar{H}_T = H$:

$$
\theta^{(S)}_t = \bar{\alpha}_t \\
\theta^{(B)}_{t-1} = B_{t-1}^{-1} (\bar{H}_{t-1} - \bar{\alpha}_t S_{t-1})
$$

where

$$
\Delta_t = S_t e^{-r} - S_{t-1} \\
\bar{\alpha}_t = \frac{\text{Cov} [e^{-r} \bar{H}_t, \Delta_t | \mathcal{F}_{t-1}]}{\text{Var} [\Delta_t | \mathcal{F}_{t-1}]} \\
\bar{H}_{t-1} = e^{-r} \mathbb{E} [\bar{H}_t | \mathcal{F}_{t-1}] - \bar{\alpha}_t \mathbb{E} [\Delta_t | \mathcal{F}_{t-1}]$

Hedging portfolio

- **Gains process** (discounted):
  \[ G_0^\theta = 0 \]
  \[ G_t^\theta = \sum_{n=1}^{t} \theta_n^{(S)} (B_n^{-1} S_n - B_{n-1}^{-1} S_{n-1}) \]

- **Cost process** (discounted):
  \[ C_0^\theta = V_0^\theta \]
  \[ C_t^\theta = B_t^{-1} V_t^\theta - G_t^\theta \]

- **Self-financing hedging strategy**: constant cost
Global quadratic hedging

- Self-financing hedging strategies
- Minimize terminal squared hedging error under \( \mathbb{P} \):
  \[
  \arg\min_{(V_0, \theta) \in \mathbb{R} \times \Theta} \mathbb{E}_\mathbb{P} \left[ (H - V_T^\theta)^2 \right]
  \]
- Variance-optimal hedging, mean-variance hedging, global or total risk-minimization
Global quadratic hedging

Theorem (Global quadratic hedging)

The solution to the global quadratic hedging problem is fully determined by $V_0 = H_0$ and the following backward recursive scheme initiated at $t = T$, $H_T = H$ and $\nu_{T+1} = 1$:

$$\theta_t^{(S)} = \alpha_t - V_{t-1}^\theta b_t / a_t$$

where,

$$\Delta_t = S_t e^{-r} - S_{t-1}$$
$$a_t = E[\Delta_t^2 \nu_{t+1} | \mathcal{F}_{t-1}]$$
$$b_t = E[\Delta_t \nu_{t+1} | \mathcal{F}_{t-1}]$$
$$d_t = e^{-r} E[H_t \Delta_t \nu_{t+1} | \mathcal{F}_{t-1}]$$
$$\alpha_t = d_t / a_t$$
$$\nu_t = E[(1 - \Delta_t b_t / a_t) \nu_{t+1} | \mathcal{F}_{t-1}]$$
$$H_{t-1} = \frac{e^{-r} E[H_t (1 - \Delta_t b_t / a_t) \nu_{t+1} | \mathcal{F}_{t-1}]}{\nu_t}$$
Global quadratic hedging

▶ It turns out that:

Local under $\mathbb{P} \neq$ Global under $\mathbb{P}$

but

Local under $\mathbb{Q} \iff$ Global under $\mathbb{Q}$

▶ Given current state variables, the position in the stock asset obtained with the local approach is independent of the initial capital $V_0$, and it is also independent of previous hedging costs.

▶ Not true for the global approach.
Analysis

1. Idealized setting where there is no model risk:
   Market model = Hedging model

2. Model risk experiment:
   Market model ≠ Hedging model

3. Empirical test
Framework

- $N = 10,000$ paths of the risky asset are simulated
- Terminal hedging error, $H - V_T^\theta$, is computed for each path and strategy (same $V_0$) assuming a daily rebalancing of the hedging portfolio
- ATM call option: $S_0 = K = 100$
- $r = 2\%$
- $T = 3$ months or 3 years
- GARCH parameters based on S&P 500 returns (1987-2010)
Experiment *without* model risk

- **3-month** ATM call option (initial capital = 3.44)

<table>
<thead>
<tr>
<th>Model</th>
<th>RMSE</th>
<th>Semi-RMSE</th>
<th>95% VaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Global $P$</td>
<td>0.80</td>
<td>0.63</td>
<td>1.43</td>
</tr>
<tr>
<td>Local $P$</td>
<td>1%</td>
<td>1%</td>
<td>3%</td>
</tr>
<tr>
<td>Global/local $Q$</td>
<td>1%</td>
<td>1%</td>
<td>2%</td>
</tr>
<tr>
<td>Duan delta hedge</td>
<td>17%</td>
<td>14%</td>
<td>18%</td>
</tr>
</tbody>
</table>
Experiment without model risk

- 3-year ATM call option (initial capital = 15.04)

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<th>95% VaR</th>
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</thead>
<tbody>
<tr>
<td>Global P</td>
<td>1.73</td>
<td>1.41</td>
<td>2.71</td>
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<tr>
<td>Local P</td>
<td>11%</td>
<td>8%</td>
<td>22%</td>
</tr>
<tr>
<td>Global/local Q</td>
<td>11%</td>
<td>9%</td>
<td>23%</td>
</tr>
<tr>
<td>Duan delta hedge</td>
<td>41%</td>
<td>16%</td>
<td>32%</td>
</tr>
</tbody>
</table>
Experiment with model risk

- **3-year** ATM call option (initial capital = 15.04)
- **RMSE**

<table>
<thead>
<tr>
<th>Model</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Global P</td>
<td>1.61</td>
<td>2.47</td>
<td>2.85</td>
<td>3.89</td>
</tr>
<tr>
<td>Local P</td>
<td>16%</td>
<td>10%</td>
<td>12%</td>
<td>8%</td>
</tr>
<tr>
<td>Global/local Q</td>
<td>16%</td>
<td>11%</td>
<td>12%</td>
<td>8%</td>
</tr>
<tr>
<td>Duan delta hedge</td>
<td>52%</td>
<td>10%</td>
<td>38%</td>
<td>16%</td>
</tr>
</tbody>
</table>

- Model 1: Regime-switching GARCH with 2 states
- Model 2: EGARCH
- Model 3: Regime-switching with 4 states,
- Model 4: Stochastic volatility with jumps
Backtest

- Option contracts issued during 2008–2010
- Rolling-window S&P 500 returns
- Results for a 3-year ATM call option

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<th>95% VaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Global $\mathbb{P}$</td>
<td>1.49</td>
<td>1.47</td>
<td>4.02</td>
</tr>
<tr>
<td>Global/local $\mathbb{Q}$</td>
<td>2.62</td>
<td>2.59</td>
<td>6.33</td>
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<tr>
<td>Duan delta hedge</td>
<td>3.11</td>
<td>2.83</td>
<td>8.32</td>
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<tr>
<td>Local Heston</td>
<td>2.70</td>
<td>2.66</td>
<td>7.26</td>
</tr>
<tr>
<td>B-S delta hedge</td>
<td>2.43</td>
<td>1.82</td>
<td>3.47</td>
</tr>
</tbody>
</table>
Key message

1. Value added of global VS local quadratic hedging?
   - Long-term maturities
   - Value added for LEAPS, market-linked CDs, VAs

2. Choice of measure: $\mathbb{P}$ VS $\mathbb{Q}$?
   - Inconsequential for local approach
   - Significant impact for global approach (choose $\mathbb{P}$)

3. How is global hedging impacted by model risk?
   - Robust to model mis-specification
   - Pareto improvement at long-term maturities