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A conditional equity risk model for regulatory assessment

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Background : Solvency II regulation frame

The 2008 crisis prompted a worldwide trend of reinforcing capital requirement and harmonization of calculations.

Pillar 1 solvency capital requirement (SCR)

- VaR 99.5% over a one-year period of the basic own funds.
- Two methods of assessment : standard formula or approved Internal Model.

Pillar 2 own risk solvency assessment (ORSA)

- Need of an Internal Model (not requiring certification of the regulator)
- The use of different valuation bases from those required for the SCR calculation are allowed

Internal Model requirements : an accurate assessment of the risk profile avoiding excessive complexity.

Accurate assessment of the risk profile

Aim : avoid financial management disruptions.

Well-known unintended pro-cyclical effect of the raw standard formula : selling equity in periods of distressed markets although market risk typically rises in high periods and falls after market shocks.
This effect leads to poor assessment of required own funds.

Mitigating pro-cyclicality may in principle be achieved by :

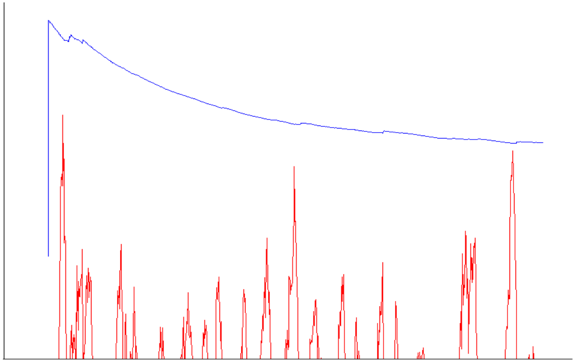
- The use of a *dampener* mechanism, explicitly allowed by the regulator.
- The use of an internal model for a better assessment of either the Pillar 1 SCR or the Pillar 2 ORSA.

Avoiding excessive complexity

A host of elaborate models has been proposed to reflect closely the nature of equities markets.
However, their complexity typically translates into calibration issues that render their use difficult : the impact of estimation inaccuracies on the assessment of the risk profile is hard to assess.

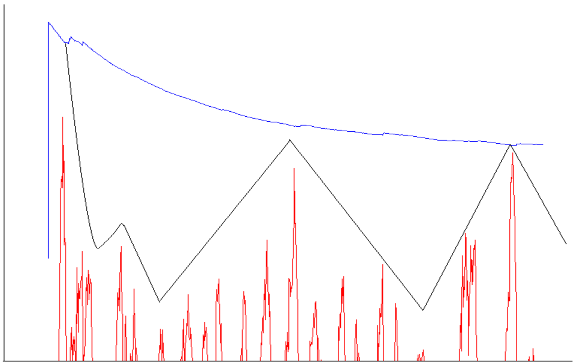
We adopt a radically different approach : instead of trying to model equities in a faithful, and thus, complex way, we concentrate on a minimal model whose sole ambition is to allow for a fine assessment of required own funds.

Illustration



S&P 500 losses between December 1927 and December 2014 (red) and 99.5% VaR for geometric Brownian motion (blue).
Required own funds are overestimated after market downs.

Illustration



S&P 500 losses between December 1927 and December 2014 (red), 99.5% VaR for geometric Brownian motion (blue) and for a hypothetical model that would minimize required own funds while remaining prudent (black).

Basic idea

Our basic idea in view of designing a model that would give VaRs resembling to a certain extent to the black curve above, is to refine the dampener mechanism, that was proposed by the regulator to lighten the burden put on financial companies in periods of market downs.

We examine the net effect of this mechanism in terms of modeling assumptions. This allows us to recast it in a “continuous frame”, that proves both more robust and more efficient as far as the estimation of required own funds is concerned.

The Brownian Continuous Dampener (BCD) model

Notations

- C_i = price at month i.
- R_i = arithmetic return at month i,

$$R_{i+1} = \frac{C_{i+1} - C_i}{C_i}.$$

- $MA_i(T)$ = moving average computed at month i over the T last months.

Basic idea : the corrected VaRs given by the dampener mechanisms can in effect be seen as plain VaRs corresponding to a modified model.

QIS5 implementation of dampener yields the modified Value at Risk :

$$\mathrm{VaR}_D = \mathrm{VaR} + \frac{C_{i+12} - MA_{i+12}(36)}{MA_{i+12}(36)},$$

where VaR = VaR given by the geometric Brownian model (gBm).

This amounts to replacing the Gaussian variable describing the one-year return in the original gBm model by an "apparent" model where the return follows a Gaussian law with same variance but where $\frac{C_{i+12} - MA_{i+12}(36)}{MA_{i+12}(36)}$ is added to the mean (a quantity which is known at the end of the period).

Our (simplified) formula for prices evolution reads :

$$R_{i+j} = Z_{i+j-1} + \frac{C_{i+j-1} - MA_{i+j-1}(36)}{MA_{i+j-1}(36)}$$

for $j = 1, \ldots, 12$. This is similar to the QIS5 dampener except that a correction is made at each time step : for every $i + j$, the original gBm model is replaced by an apparent one where the drift is $\frac{C_{i+j-1} - MA_{i+j-1}(36)}{MA_{i+j-1}(36)}$.

The dampening adjustments are made "continuously" all along the path instead of just once at the end of the period. The net correction now depends on the whole path since time i, while the EIOPA dampener depends only on the final price and the moving average of the prices in the considered period.
This fact is the core reason why our model behaves in a more robust way.

Discrete version

$$R_{i+1} = \exp(Z_i) - 1 + \frac{1}{12}\left(1 - \frac{C_i}{S_i}\right)^{+},$$

$$S_i = 2MA_i(84) - MA_i(36)$$

This model substitutes an "apparent" return for the "true" one. The "apparent" return is larger when markets are down. As a consequence, the "apparent" VaR is smaller.

No adjustment is made when the price is higher than the average

Not claimed to be an adequate representation of reality but tests show that one-year VaRs are evaluated in an accurate way that greatly reduces pro-cyclical effects

Continuous version

The stochastic process that implements this “continuous dampener” can be represented as the unique solution of the following stochastic functional differential equation (SFDE) :

$$dC_t = \left(F(\overline{C_{t,T_1}}) + \mu + \frac{\sigma^2}{2} \right) C_t \; dt + \sigma C_t dB_t,$$

where

$$F(\overline{C_{t,T_1}}) = \left(1 - \frac{C_t}{S_t} \right)^+$$

$$\overline{C_{t,T_1}} = \{ C(t-s), 0 \le s \le T_1 \}$$

$$A_{t,T} = \frac{1}{T} \int_{t-T}^{t} C(u) \; du, \quad S_t = 2A_{t,T_1} - A_{t,T_2}.$$

Theoretical analysis

Denote E the set of continuous functions from $[-T_1, 0]$ to $\mathbb{R}$
For a function G from E to $\mathbb{R}$, one considers the following SFDE :

$$dC(t) = G(\overline{C_{t,T_1}})dt + \sigma C_t dB(t)$$

$$\overline{C_{0,T_1}} = \xi := \{\xi(s), -T_1 \leq s \leq 0\}$$

Theorem

If there exists $K > 0$ such that

1. $|G(\varphi) - G(\psi)| \leq K||\varphi - \psi||$,
2. $G(\varphi)^2 \leq K(1 + ||\varphi||)^2$,

then there exists a unique solution C for any given initial condition.

Theoretical analysis

Unfortunately, with $G(\overline{C_{t,T_1}}) = \left(F(\overline{C_{t,T_1}}) + \frac{\sigma^2}{2}\right) C_t$, the conditions above are not verified.

The problem is of course that S_t that appears in the denominator of F is arbitrarily close to 0 with positive probability.

A minimal modification consists in regularizing F so as to avoid blow-up :

$$F(\overline{C_{t,T_1}}) = \frac{(S_t - C_t)^+}{S_t} \mathbb{1}(S_t > \varepsilon) + S_t \frac{(\varepsilon - C_t)^+}{\varepsilon^2} \mathbb{1}(0 \leq S_t \leq \varepsilon)$$

Theoretical analysis : existence and uniqueness

Theorem : Let

$$G(\overline{C_{t,T_1}}) = \left(\frac{(S_t - C_t)^+}{S_t}\mathbb{1}(S_t > \varepsilon) + S_t\frac{(\varepsilon - C_t)^+}{\varepsilon^2}\mathbb{1}(0 \leq S_t \leq \varepsilon) + \frac{\sigma^2}{2}\right) C_t$$

Then for any $T > 0$, the equation

$$dC_t = G(\overline{C_{t,T_1}})C_t\ dt + \sigma C_t dB_t$$

with arbitrary admissible initial condition has a unique solution C on $[-T_1, T]$. Furthermore, $\mathbb{E}\left(\int_{-T_1}^{T} C(t)^2 dt\right) < \infty$.

Theoretical analysis : stability

What happens if T_1, T_2, σ and ε are estimated in an inaccurate way ?

Proposition Let $\left(T_1^{(n)}, T_2^{(n)}, \sigma^{(n)}\right)_n$ be a sequence of strictly positive elements of $\mathbb{R}^3$, with $T_1^{(n)} > T_2^{(n)}$ for all n, converging to (T_1, T_2, σ) where $T_1 > T_2 > 0, \sigma > 0$. Fix $\varepsilon > 0$. Consider, for $n \in \mathbb{N}$, the SFDE

$$dC_t^{(n)} = G^{(n)}\left(\overline{C^{(n)}_{t,T_1^{(n)}}}\right) dt + \sigma^{(n)} C_t^{(n)} dB_t, \quad \text{where}$$

$$G^{(n)}(\overline{C^{(n)}_{t,T_1^{(n)}}}) = \left(\frac{(S_t^{(n)} - C_t^{(n)})^+}{S_t^{(n)}} \mathbb{1} I(S_t^{(n)} > \varepsilon) + S_t^{(n)} \frac{(\varepsilon^{(n)} - C_t^{(n)})^+}{(\varepsilon)^2} \mathbb{1} I(0 \leq S_t^{(n)} \leq \varepsilon)\right.$$

Then, as n tends to infinity, the unique solution $C^{(n)}$ converges in the following sense

$$\lim_{n\to\infty} \mathbb{E}\left(\sup_{0\leq t\leq T} \left|C^{(n)}(t) - C(t)\right|^2\right) = 0.$$

Statistical analysis

Introduction

Aim : verify that the model reproduces prudential aspects of observed returns.
Increments are **not** iid.
$\Rightarrow$ tailored goodness-of-fit tests must be constructed.

Building blocks

Bernoulli random variables detecting the violations of either historical or simulated returns :

$$Y_T^p = \mathbb{1}(R_T \leq F_{\widehat{R_T}}^{-1}(p))$$

$$\widehat{Y}_T^p = \mathbb{1}(\widehat{R_T} \leq F_{\widehat{R_T}}^{-1}(p)).$$

Two "naive" hypotheses :
$\mathcal{H}_0$: *the random variables* Y_T^p *are iid Bernoulli with parameter* p,
$\widehat{\mathcal{H}}_0$: *the random variables* $\widehat{Y}_T^p$ *are iid Bernoulli with parameter* p.

Obviously, independence does not hold, and $\mathcal{H}_0$ and $\widehat{\mathcal{H}}_0$ are rejected.

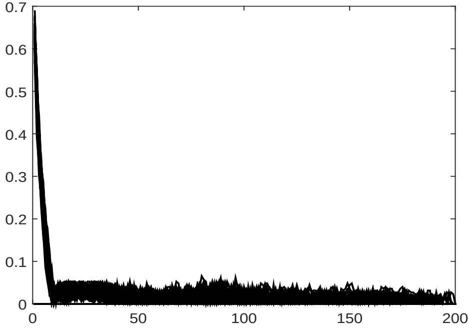
m-dependence

We are however able to verify experimentally that the correlations between the random variables Y_t^p and $Y_{t'}^p$ disappear when $|t - t'|$ is large enough, that is, the sequence $(Y_t^p)_t$ is m-dependent.

A sequence (X_j) of random variables is said to be $m-$dependent if for all couple (A, B) of subsets of $\mathbb{N}$ such that $d(A, B) > m$, the sets $\{X_i, i \in A\}$ and $\{X_i, i \in B\}$ are independent, where

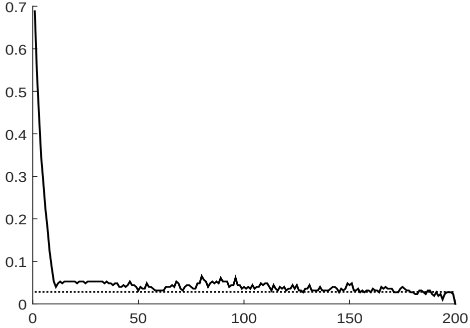
$$d(A, B) = \inf\{|i - j|, i \in A, j \in B\}.$$

m-dependence



Autocorrelations between the $\widehat{Y}_t^p$ for various starting times, lags between 0 and 200 and $p = 0.05$.

m-dependence



Maximum over all starting times of the autocorrelations between the $\widehat{Y}_t^p$ for lags between 0 and 200 and $p = 0.05$

Theorem [Hoeffding-Robbins]

Let $(X_i)_i$ be a sequence of $m-$dependent random variables such that, for all i, $\mathbb{E}(X_i) = 0$ and $\mathbb{E}\left(|X_i|^3\right) < \infty$. Set :

$$A_i = \mathbb{E}\left(X_{i+m}^2\right) + 2\sum_{j=1}^{m} \mathbb{E}\left(X_{i+m-j}X_{i+m}\right).$$

Then, if, for all i,

$$\lim_{p\to\infty} p^{-1} \sum_{k=1}^{p} A_{i+k} =: A$$

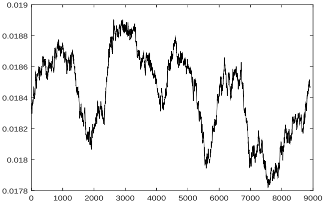
exists and is independent of i, the random variable $n^{-1/2}\left(X_1 + \ldots + X_n\right)$ tends in law to a centred Gaussian random variable with variance A when n tends to infinity.

Application

$$X_i = \mathbb{1}\left(\widehat{R}_i \le F_{\widehat{R}_i}^{-1}(p)\right) - \mathbb{E}\left(\mathbb{1}\left(\widehat{R}_i \le F_{\widehat{R}_i}^{-1}(p)\right)\right)$$

$$var\left[n^{-1/2}\left(X_1 + \ldots + X_n\right)\right] \cong 0.0187$$

Plot of $p^{-1}\sum_{k=1}^{p} A_{i+k}$



$\Rightarrow$ the limit A exists and is independent of i.

Statistical test

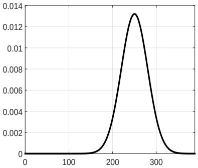
We define two statistics :

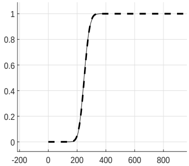
- $T_p = \dfrac{1}{n}\sum_{t=1}^{n} \mathbb{1}(R_t \leq F_{\widehat{R_t}}^{-1}(p))$, which counts the percentage of violations of historical returns, *i.e.* the percentage of historical returns up to time t that fall below the quantile of confidence at level p of the distribution $\widehat{R_t}$.
- $\widehat{T_p} = \dfrac{1}{n}\sum_{t=1}^{n} \mathbb{1}(\widehat{R_t} \leq F_{\widehat{R_t}}^{-1}(p))$, which counts the percentage of violations generated by the model.

Statistical test

Convergence :

$$\lim_{n\to\infty} \widehat{T_p} = \mathcal{N}(p, \sigma(p)).$$





σ_p is estimated empirically in two ways which yield the same result.

First-order stochastic dominance

The risk is assessed in a prudential way if :

$$\widehat{R_t} \leq R_t,$$

$$\mathbb{1}(R_t \leq F^{-1}_{\widehat{R_t}}(p)) \leq \mathbb{1}(\widehat{R_t} \leq F^{-1}_{\widehat{R_t}}(p)),$$

$$\mathcal{H}_0(p) : T_p \leq \widehat{T_p}$$

which means : on average, the model generates more violations than observed for historical returns.

$$p - value(p) = \mathbb{P}\left[T_p \leq \mathcal{N}(\mu(p), \sigma(p))\right].$$

Numerical experiments

Verification of the counter-cyclical property

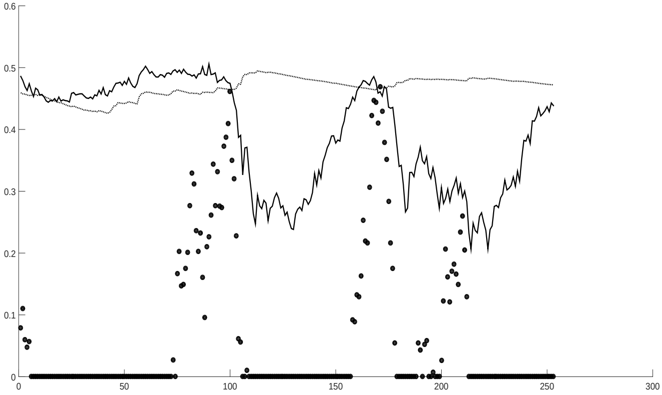
Comparison with three classical models :

- Geometric Brownian motion (gBm),
- GARCH(1,1),
- AR(1).

Criterion for assessing the models : a model is adequate if it meets the prudential requirements and requires a “small” amount of capital.

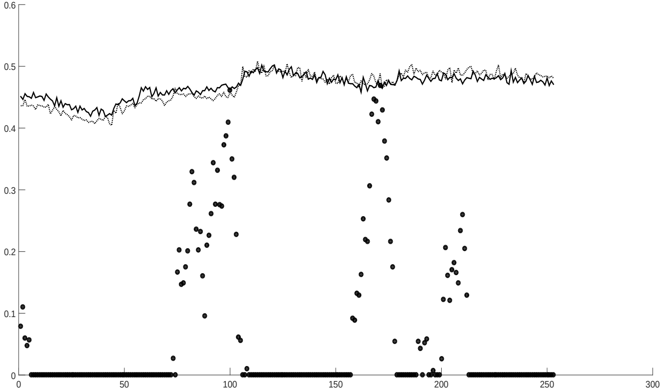
In order to allow for a fair comparison, we tune the “volatility” parameter in each model so that the proportion of violations is exactly equal to, or slightly smaller than, 0.5%.

Eurostoxx50 : historical losses (circles), gBm (dotted) and BCD (solid)



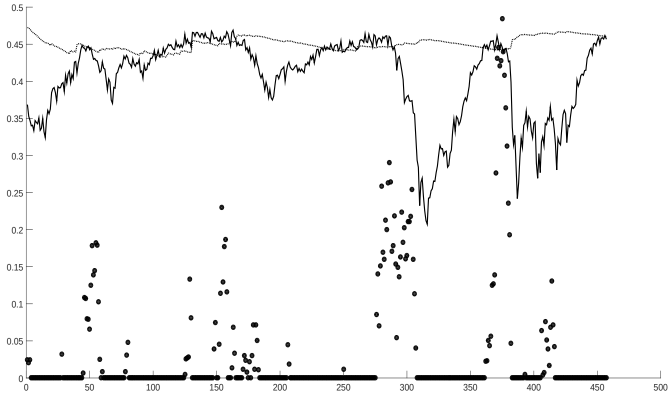
Period : December 1986 $\Rightarrow$ December 2014

Eurostoxx50 : GARCH(1,1) (dotted) and AR(1) (solid)



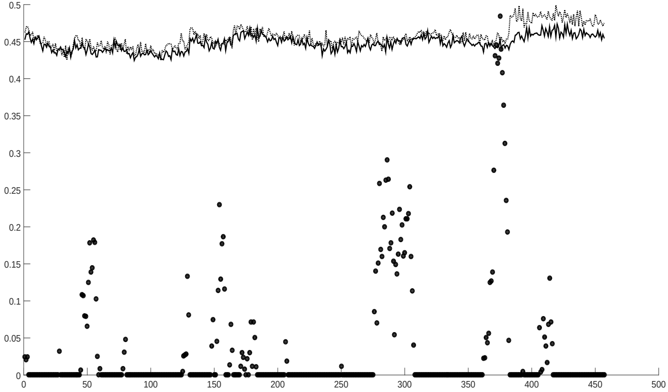
Period : December 1986 ⇒ December 2014

MSCI : gBm (dotted) and BCD (solid)



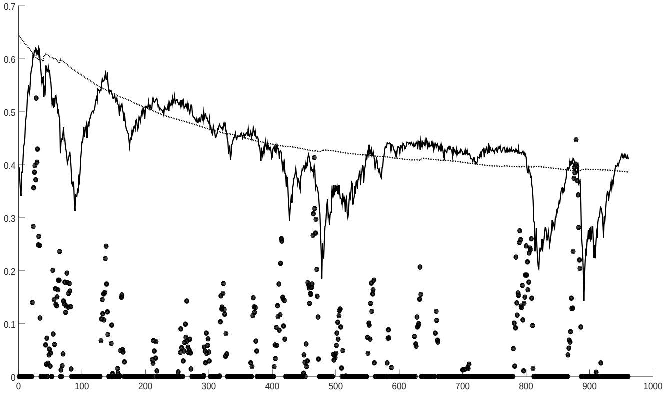
Period : December 1969 $\Rightarrow$ December 2014

MSCI : GARCH(1,1) (dotted) and AR(1) (solid)



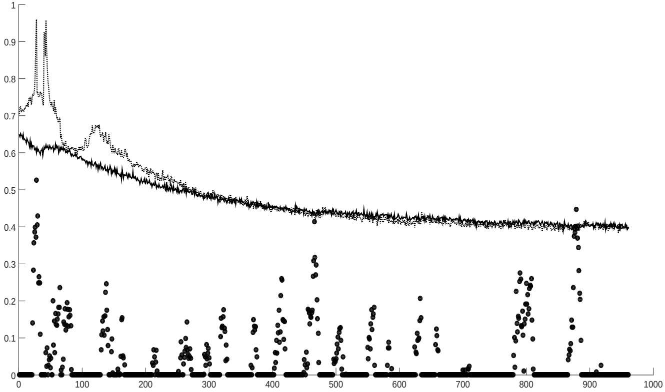
Period : December 1969 ⇒ December 2014

S&P 500 : gBm (dotted) and BCD (solid)



Period : December 1927 $\Rightarrow$ December 2014

S&P 500 : GARCH(1,1) (dotted) and AR(1) (solid)



Period : December 1927 ⇒ December 2014

Theoretical and empirical number of losses exceeding VaR

	MSCI	Eurostoxx50	S&P500
Theoretical	2	1	4
gBm	2	1	4
BCD	2	1	4
GARCH(1,1)	1	1	3
AR(1)	2	1	1

Areas under VaR

	MSCI	Eurostoxx50	S&P500
gBm	205	118	436
BCD	185	99	407
GARCH(1,1)	208	117	462
AR(1)	204	118	448

Verification of the first-order stochastic dominance

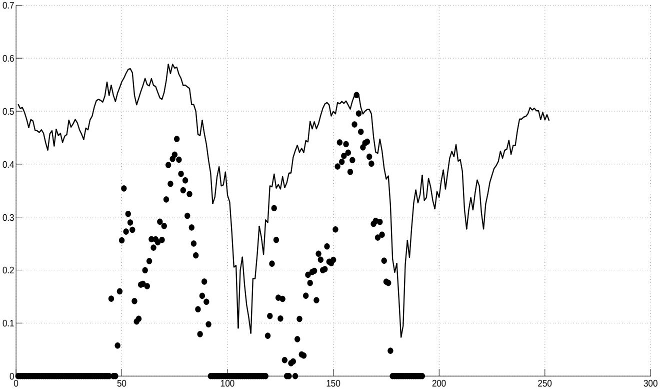
Confidence level	0.5%	5%
MSCI	52.96%	87.48%
Eurostoxx50	53.59%	11.9%
S&P500	56.61%	97.27%

$p-$ values.

Backtesting for time horizon greater than 1 year

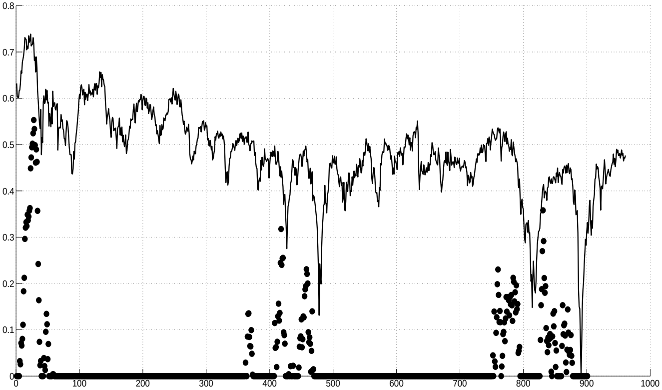
Experiments indicate that the model gives reliable results for a time horizon comprised between 6 months and 5 years.

Eurostoxx50 : 5-year VaR BCD



Period : December 1986 ⇒ December 2014

S&P 500 : 5-year VaR BCD



Period : December 1927 $\Rightarrow$ December 2014

Conclusion

Advantages of the BCD model :

- Mitigates pro-cyclicality.
- Equity risk not underestimated.
- Simple.
- Can be used for time horizons larger than 1 year.

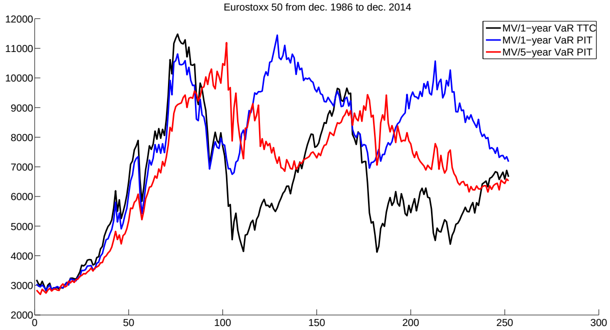
Point-in-Time and Through-the-Cycle

In a real-world projection two frameworks are generally considered :

- a *Point-in-Time* (PIT) estimate of the distribution consists in a forward-looking projection relevant to the given time horizon, it is the best estimate of the return distribution conditioned on the state of today's market
- a *Through-the-Cycle* (TTC) estimate is a projection based upon an unconditional estimate of the distribution in a given time horizon

Our conditional model is incorporated within a *Point-in-Time* framework.

Volatility of solvency ratio w.r.t. time horizon and model used



⇒ PIT model exhibits a higher volatility of the VaR than TTC model but it gives less sensitive solvency ratio than TTC model

⇒ Moreover the higher the time horizon is, the more stable the solvency ratio is

Conclusion

Conclusion

From our model we conclude that :

- our backtesting results show that PIT approach combined with market consistent valuation leads to less volatility of solvency ratio and so avoids asset management disruption
- an optimal capital requirement for long term investors should both be based on a time horizon which is consistent with the holding time of the portfolio (and so greater than 1 year) and take into account the market level at the valuation date (Point-In-Time approach)