An Overview of Probabilistic Fuzzy Systems
-- Some Preliminary Observations

Arnold F. Shapiro* and Dabuxilatu Wang**

*Smeal College of Business, Penn State University, University Park, PA 16802, USA
**Department. of Statistics, Guangzhou University, Guangzhou, China

Abstract

The two major sources of uncertainty are randomness and fuzziness, and they are complementary. This article extends this perspective to an integrated model where both types of uncertainty exist concurrently, and where each of the randomness and fuzziness components, while necessary, is not sufficient, in and of itself, to formulate the model. Many actuarial applications are of this sort.

Such integrated models have come to be known as probabilistic fuzzy systems (PFSs). Essentially, the PFS is a methodology that is built on a fuzzy inference system, which has been modified to accommodate a probabilistic fuzzy rule base. This provides a stochastic input-output mapping between the input fuzzy sets associated with the antecedent part of the rule base and the output fuzzy sets associated with the consequent part.

The purpose of this article is to present some preliminary observations with respect to PFSs, with the goal of introducing ARCH readers to the topic. To this end, some basic concepts that are pertinent to PFSs are presented, as is an introduction to PFSs, a discussion of their architecture, and a review the key features of their methodology. The article concludes with a commentary on PFSs and suggestions for further studies.

1 Introduction

The two major sources of uncertainty are randomness, or probabilistic uncertainty, and fuzziness, which embodies the imprecision on account of vagueness or lack of knowledge, and they are complementary. This article extends this perspective to an integrated model where both types of uncertainty exist concurrently, and where each of the randomness and fuzziness components, while necessary, is not sufficient, in and of itself, to formulate the model. In this way, fuzziness, which has been regarded as heuristic, has a clear connections to randomness. Many actuarial applications are of this sort.

Such integrated models have come to be known as probabilistic fuzzy systems (PFSs). Essentially, the PFS is a methodology that is built on a fuzzy inference system (FIS), which has been modified to accommodate a probabilistic fuzzy rule base. That is, a probabilistic framework is applied to an existing fuzzy model, under which the fuzzy and probabilistic uncertainties are simultaneously dealt with. This provides a stochastic input-output mapping between the input fuzzy sets associated with the antecedent part of the rule base and the output fuzzy sets associated with the consequent part.

The purpose of this article is to present some preliminary observations with respect to PFSs, with the goal of introducing ARCH readers to the topic. It begins with the Zadeh (1968) conceptualization of the original model. Then, before proceeding, some basic concepts that are pertinent to PFSs are discussed. Following that is an introduction to PFSs and a discussion of their architecture. Finally, we review the key features of their methodology. The article concludes with a commentary on PFSs and suggestions for further studies.

2 Basic concepts

Before proceeding, we present some basic concepts that are pertinent to the PFSs discussion. Topics covered include notation, the normalized MFs and k-th order MF centroids, the probability of a crisp event, fuzzy variables, the probability of a fuzzy event, the estimated probability of the fuzzy event, conditional fuzzy probability, the conjunction of fuzzy sets, and proper fuzzy partitions.

2.1 Notation

The following notation is used in this article:

$X \equiv$ a finite set, a continuous sample space
$A \equiv$ a compact subset of X (defines an event)
$\tilde{A} \equiv$ a fuzzy event
$x_1, x_2, ..., x_K \equiv$ a random sample on the domain X
$x \equiv$ a scalar variable
$f(x) \equiv$ the pdf of x

GOM ≡ grade of membership
J ≡ the number of MF in the consequent part of the rule base
K ≡ the number of MF in the antecedent part of the rule base
MF ≡ membership function
P(A) ≡ the probability of A
$\hat{P}(\tilde{A}) \equiv$ the estimated probability of the fuzzy event $\tilde{A}$
$P(\tilde{A} \mid \tilde{B}) \equiv$ the conditional fuzzy probability of $\tilde{A}$, given $\tilde{B}$
Q ≡ the number of rules in the rule base
$\chi_A(x) \equiv$ characteristic function
$\mu_{\tilde{A}}(x) \equiv$ a membership function, which gives the GOM of $x \in X$ in $\tilde{A}$

2.1.1 Normalized MFs and k-th order MF Centroids

For convenience, two types of normalized MFs will be used in what follows. The first, which is associated with the antecedent part of the rule base, will be denoted as $\overline{\mu}^{(1)}_{\tilde{A}_k}(x)$, and takes the form:

$$\overline{\mu}^{(1)}_{\tilde{A}_k}(x) \equiv \frac{\mu_{\tilde{A}_k}(x)}{\sum_{k=1}^{K} \mu_{\tilde{A}_k}(x)} \tag{1}$$

The second, which is associated with the consequent part of the rule base, will be denoted as $\overline{\mu}^{(2)}_{\tilde{C}_j}(y)$, and takes the form:

$$\overline{\mu}^{(2)}_{\tilde{C}_j}(y) \equiv \frac{\mu_{\tilde{C}_j}(y)}{\int_{-\infty}^{\infty} \mu_{\tilde{C}_j}(y)\, dy} \tag{2}$$

Moreover, the k-th order MF Centroid will be denoted as $E\left\{ y^k \mid \overline{\mu}^{(2)}_{\tilde{C}_j} \right\}$ and take the form:

$$E\left\{ y^k \mid \overline{\mu}^{(2)}_{\tilde{C}_j} \right\} \equiv \int_{-\infty}^{\infty} y^k\, \overline{\mu}^{(2)}_{\tilde{C}_j}(y)\, dy \tag{3}$$

2.2 Probability of a crisp event

The probability of a compact crisp subset A of X is: [van den Berg et al (2011, p. 5)]

$$\begin{aligned} P(A) &= \int_{x \in A} f(x)\,dx \\ &= \int_{-\infty}^{\infty} \chi_A(x) f(x)\,dx \end{aligned} \tag{4}$$

The probability of the crisp event A can be estimated as: [Viertl (2011, p. 43)]

$$\hat{P}(A) = \frac{1}{K} \sum_{k=1}^{K} \chi_A(x_k) \tag{5}$$

2.3 Fuzzy variables

Linguistic variables, which are the building blocks of fuzzy variables, may be defined (Zadeh, 1975, 1981) as variables whose values are expressed as words or sentences. Average future lifetime for a life aged x, for example, may be viewed both as a numerical value ranging over the interval $[0, \omega - x]$, where ω is the limiting age[1], and a linguistic variable that can take on values like short, medium, and long. Each of these linguistic values may be interpreted as a label of a fuzzy subset of the universe of discourse $[0, \omega - x]$, whose base variable is the generic numerical value future lifetime.

Fuzzy numbers, which are the focus for the current analysis, are numbers that have fuzzy properties, examples of which are the notions of “around 5 years” and “relatively short”. The general characteristic of a fuzzy number (Zadeh, 1975 and Dubois and Prade, 1980), referred to as its membership function, MF, and denoted by the symbol $\mu_{\tilde{A}}(x)$, where μ, $\tilde{A}$ and x denote a MF, fuzzy set and location, respectively, frequently is represented as shown in Figure 1.

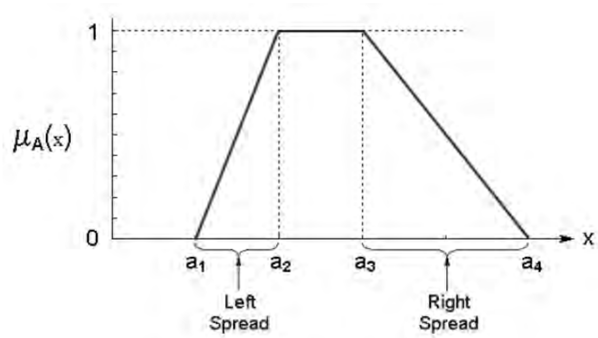

Figure 1: Trapezoidal Fuzzy Number

This shape of a fuzzy number is referred to as trapezoidal or “flat” and its MF often is denoted as (a_1,a_2,a_3,a_4) or $(a_1/a_2, a_3/a_4)$; when a_2 is equal to a_3, we get a triangular fuzzy number (TFN). The intervals between a_1 and a_2, and a_3 and a_4, are known as the left spread and right spread, respectively. When the two spreads are equal, the TFN is known as a symmetrical TFN (STFN). A fuzzy number is positive if $a_1 \geq 0$ and negative if $a_4 \leq 0$, and, as indicated, it usually is taken to

[1] The limiting age, ω, is the lowest age such that the probability of reaching that age, or older, is zero.

be a convex fuzzy subset of the real line, i.e.,

$$\mu_A(\lambda x_1 + (1-\lambda) x_2) \geq \min(\mu_A(x_1), \mu_A(x_2)),\ \lambda \in [0,1].$$

Other MF classes, such as the S-shaped and reverse-S-shaped, which are discussed below, can also serve as a fuzzy number, depending on the situation.

Finally, we note that a fuzzy measure [Dubois and Prade, (1980, p. 126)] is a monotonic non-additive set function g taking values in [0,1]. For fuzzy sets $\tilde{A}$ and $\tilde{B}$, it holds that:

$$g\left(\tilde{A} \cup \tilde{B}\right) \geq \max\left\{g(\tilde{A}), g(\tilde{B})\right\} \tag{6}$$

$$g\left(\tilde{A} \cap \tilde{B}\right) \leq \min\left\{g(\tilde{A}), g(\tilde{B})\right\}. \tag{7}$$

Obviously, a probability measure is a fuzzy measure, however, a fuzzy measure is not a probability measure.

2.4 Probability of a fuzzy event

Let $\tilde{A}$ be a fuzzy event and $\mu_{\tilde{A}}(x): X \to [0,1]$, the GOM of $x \in X$ in $\tilde{A}$.

Then, following Zadeh (1968, p. 424), the probability of the fuzzy set $\tilde{A}$ is[2]:

$$\begin{aligned} P(\tilde{A}) &= \int_{\tilde{A}} dP \\ &= \int_{\mathbb{R}^n} \mu_{\tilde{A}}(x)\, dP \\ &= \int_{-\infty}^{\infty} \mu_{\tilde{A}}(x)\ f(x)\, dx \\ &= E\left(\mu_{\tilde{A}}(x)\right) \end{aligned} \tag{8}$$

From the last line, we see that the probability of a fuzzy event is the expectation of its MF. When A is crisp, the usual probability for A is obtained.

By way of example, Figure 2[3] shows a representation of the essence of this idea, where the random variable is minor collision damage and the MF is with respect to low damage.

[2] There is not universal agreement on the appropriateness of (8). See, for example, Singpurwalla and Booker (2004, 870-71), who argue that $P(\tilde{A})$ is not a valid probability measure. Such concerns notwithstanding, we incorporate Zadeh's formulation of $P(\tilde{A})$ for the remainder of this article, noting that Dubois and Prade (1980, p. 141), and Nguyen and Wu (2006: 18), among others, do likewise.
[3] Adapted from van den Berg et al (2013) Fig. 1, Moura and Roisenberg (2015) Fig. 2, and Shapiro (2013) Fig. 11.

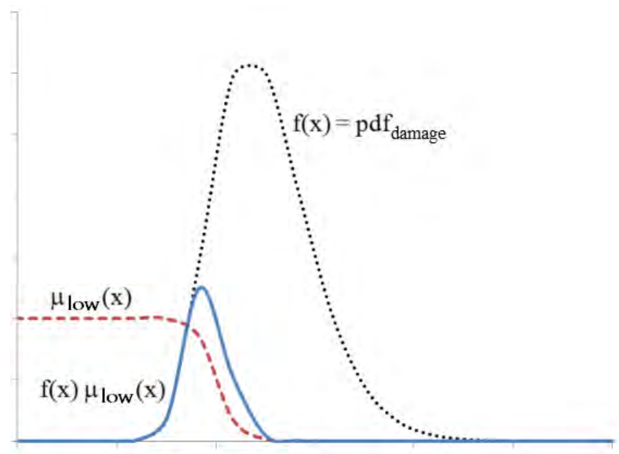

Figure 2: The PFS idea

The magnitude x of the collision damage is assumed to be the stochastic variable f(x) = pdf_{damage}, while the fuzzy notion of low damage is defined by the membership function $\mu_{low}(x)$. The product f(x) $\mu_{low}(x)$, which has been characterized by van den Berg et al (2013, p. 871) as a "fuzzy pdf", is used in (8) to calculate the probability that the collision damage is small. Thus, the probabilistic uncertainty is merged with the fuzzy uncertainty.

2.4.1 Estimated probability of the fuzzy event

Let $\tilde{A}$ be a fuzzy event, x_k be a sample value, and K be the number of samples.

Then, the estimated probability of the fuzzy event $\tilde{A}$, $\hat{P}(\tilde{A})$, is given by: [van den Berg et al (2011, p. 7)]

$$\hat{P}(\tilde{A}) = \frac{1}{K}\sum_{k=1}^{K}\mu_{\tilde{A}}(x_k) \tag{9}$$

2.5 Conditional fuzzy probability

Let $\tilde{A}$ and $\tilde{B}$ be fuzzy events.

Then, the conditional fuzzy probability of $\tilde{A}$, given $\tilde{B}$, is: [van den Berg et al (2011, pp. 6-7)]

$$
\begin{aligned}
P(\tilde{A} \mid \tilde{B}) &= \frac{P(\tilde{A} \cap \tilde{B})}{P(\tilde{B})} \\
&= \frac{\int_{-\infty}^{\infty} \left[\mu_{\tilde{A}}(x) \cap \mu_{\tilde{B}}(x)\right] f(x)\, dx}{\int_{-\infty}^{\infty} \mu_{\tilde{B}}(x) f(x)\, dx} \\
&= \frac{\int_{-\infty}^{\infty} \left[\mu_{\tilde{A}}(x) \otimes \mu_{\tilde{B}}(x)\right] f(x)\, dx}{\int_{-\infty}^{\infty} \mu_{\tilde{B}}(x) f(x)\, dx}
\end{aligned}
\tag{10}
$$

The estimate of the conditional fuzzy probability of $\tilde{A}$, given $\tilde{B}$, is:

$$
\hat{P}(\tilde{A} \mid \tilde{B}) = \frac{\sum_{k=1}^{K} \mu_{\tilde{A}}(x_k) \otimes \mu_{\tilde{B}}(x_k)}{\sum_{k=1}^{K} \mu_{\tilde{B}}(x_k)} \tag{11}
$$

2.5.1 The conjunction of fuzzy sets A and B

In the foregoing, using the product operator, $\otimes$, to implement the conjunction operator, $\cap$, guarantees that the sum of the conditional probabilities for a given fuzzy event equals 1. [Kaymak et al (2003, p. 332)]

Figure 3 shows the conjunction of fuzzy sets $\tilde{A}$ and $\tilde{B}$, based on the intersection operator, $\min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$, and the product operator, $\mu_{\tilde{A}}(x) \otimes \mu_{\tilde{B}}(x)$.

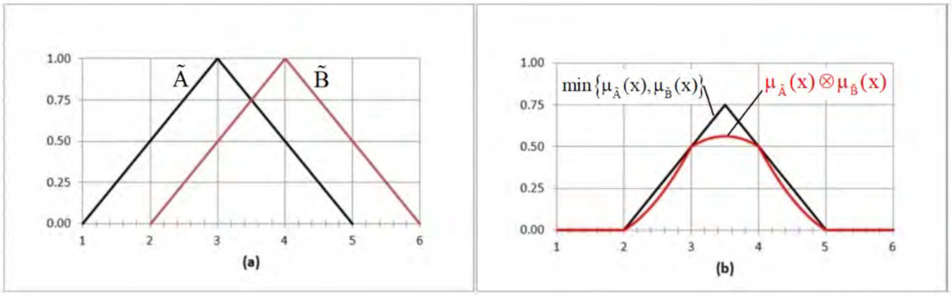

Figure 3: Conjunction of fuzzy sets $\tilde{A}$ and $\tilde{B}$

2.6 A proper fuzzy partition

An underlying assumption in this article is that if $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_K$ are fuzzy events in sample space X, then these fuzzy events form a proper fuzzy partition if, for all $x \in X$, [van den Berg et al (2011, p. 6)]

$$\sum_{k=1}^{K} \mu_{\tilde{A}_k}(x) = 1 \qquad (12)$$

This definition can be traced to Ruspini (1969, p. 29) and is what Dubois and Prade (1980, p. 13) refer to as the orthogonality condition.

A simple example of a proper fuzzy partition is shown in Figure 4, where at each point on the horizontal axis, the sum of the GOM of each MF at that point is one.

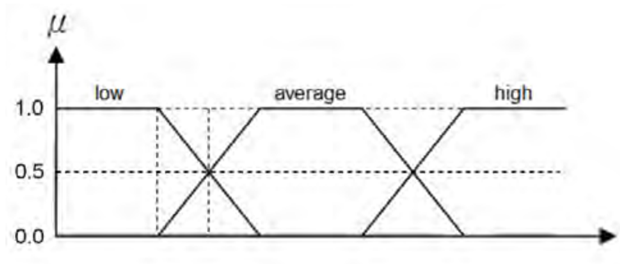

Figure 4: Example of a proper fuzzy partition

2.6.1 A well-defined fuzzy partition [van den Berg et al (2011, pp. 6-7)]

A proper fuzzy partition will be characterized as "well defined" if

$$\sum_{k=1}^{K} P(\tilde{A}_k) = 1\,. \qquad (13)$$

Moreover, under the concept of the conditional probability, a proper fuzzy partition, given $\tilde{B}$, is well defined, since:

$$\sum_{k=1}^{K} P(\tilde{A}_k \mid \tilde{B}) = 1 \qquad (14)$$

3 The Mamdani Fuzzy Inference System (FIS)

In this article we use the Mamdani FIS, a representation of which is shown in Figure 5[4]:

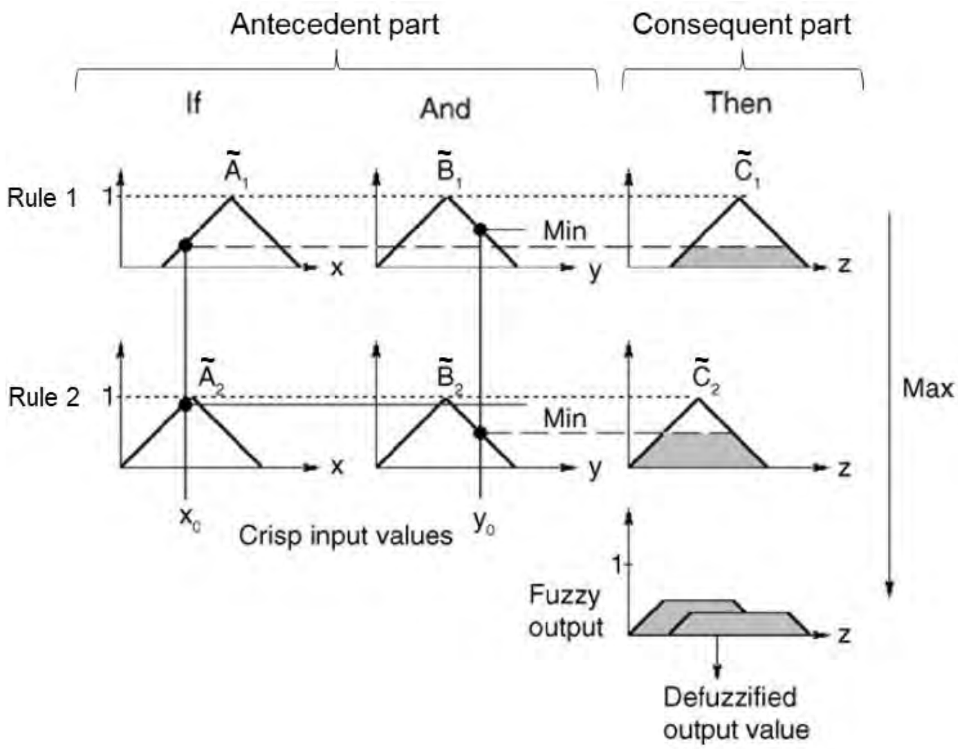

Figure 5: The Mamdani FIS

In this case, there are two crisp inputs, x_0 and y_0, and three sets of membership functions, $\tilde{A}_j$ and $\tilde{B}_j$, which constitute the antecedent part, and $\tilde{C}_j$, which constitutes the consequence part, j=1,2, each set of which is a component of the rule

Rule i: if x is $\tilde{A}_i$ and y is $\tilde{B}_i$ then z is $\tilde{C}_i$,

where the conjunction "and" is interpreted to mean the fuzzy intersection. The minimum of the fuzzy inputs in the first two columns gives the levels of the firing (shown by the dashed lines) and their impact on the inference results (shown by the shaded areas in the third column). Taking the union of the shaded areas of the first two rows of column three results in the fuzzy set show in the third row, which represents the overall conclusion.

Defuzzification converts the fuzzy overall conclusion into a numerical value that is a best estimate in some sense. A common tactic in insurance articles is to use the center of gravity

---

[4] Adapted from Shapiro (2004: 404).

(COG) approach, which defines the numerical value of the output to be the abscissa of the center of gravity of the union. In practice, this is computed as $\Sigma_j w_j x_j$, where the weight w_j is the relative value of the membership function at x_j, that is, $w_j = \mu(x_j) / \Sigma_j \mu(x_j)$.

3.1 Probabilistic fuzzy viewpoint

The probabilistic fuzzy viewpoint is illustrated in Figure 6[5].

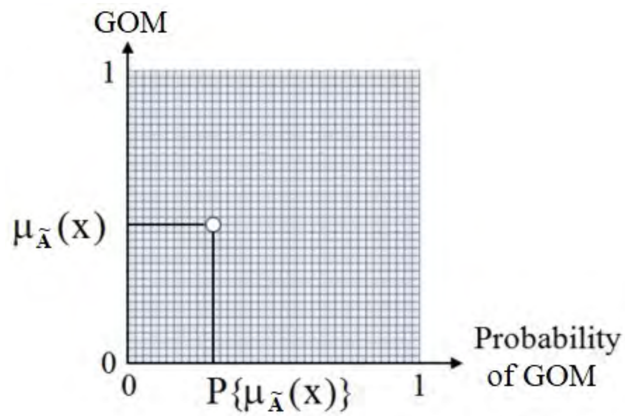

Figure 6: Probabilistic fuzzy view

Here, on the one hand, the vertical axis represents the fuzzy logic perspective, and is concerned with GOM, which extends the classical crisp (binary) notion. On the other hand, the horizontal axis represents the probability perspective, and shows the probability of a given GOM. Thus, GOM and probability of GOM are simultaneously addressed under the probabilistic fuzzy point of view.

3.2 Rule 1 of a Probabilistic Mamdani FIS

The probabilistic Mamdani FIS derives from merging the Mamdani FIS and the probabilistic fuzzy view, an example of which is shown in Figure 7.

[5] Adapted from Meghdadi and Akbarzadeh-T (2001) Figure 1.

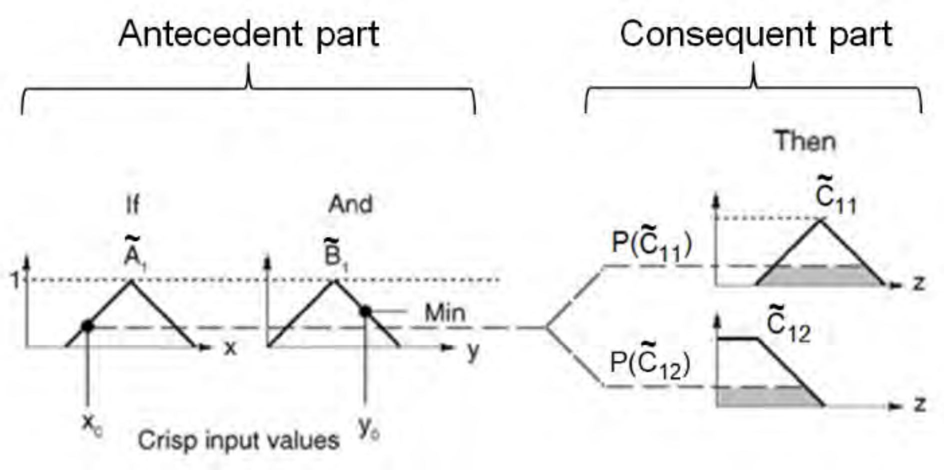

Figure 7: Rule 1 of a probabilistic Mamdani FIS

As indicated, Rule 1 of this probabilistic Mamdani FIS takes the form:

Rule 1: if x is $\tilde{A}_1$ and y is $\tilde{B}_1$ then z is $\tilde{C}_{1j}$ with probability $P(\tilde{C}_{1j})$, j= 1,2.

4 The Probabilistic Fuzzy System[6]

This section presents an introduction to PFSs, a discussion of their architecture, and a review of the key features of their methodology. The topics covered include: conceptualizing the relationship between the MFs, the conditional pdf of the output distribution, and the mean and variance of the output distribution.

4.1 Conceptualizing the relationship between the MFs

We begin the PFS discussion with Figure 8[7], which provides a representation of the relationship between the antecedent and consequent MFs, given Rule q.

[6] Much of the material of this section is based on van den Berg et al (2013).

[7] Adapted from van den Berg et al (2011) Figure 5.

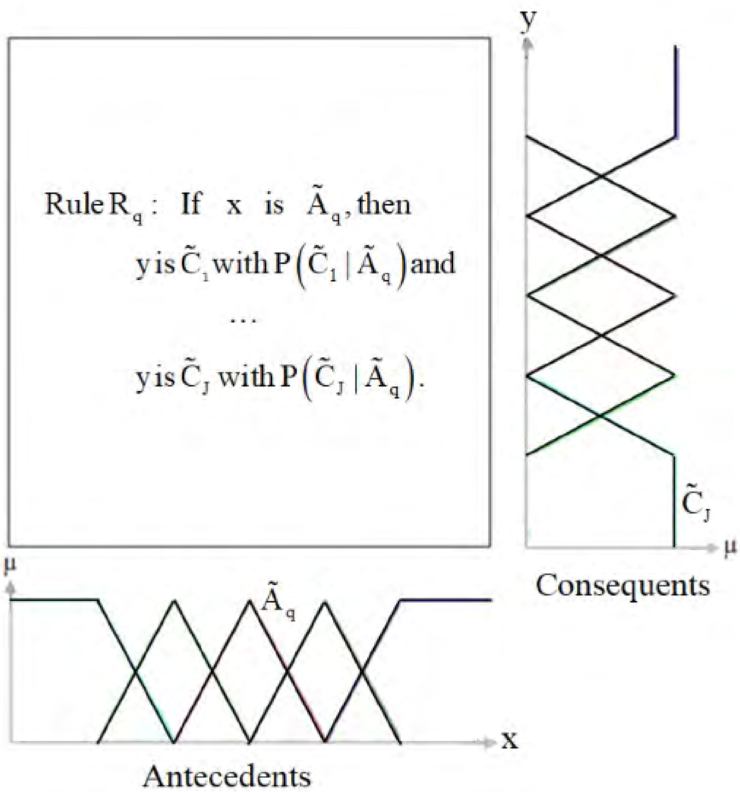

Figure 8: The relationship between the MFs

From Almeida et al (2012), we note that the estimated $\hat{P}(\tilde{C}_j \mid \tilde{A}_q)$ is given by

$$\hat{P}(\tilde{C}_j \mid \tilde{A}_q) = \frac{\sum_{k=1}^{K} \mu_{\tilde{C}_j}(y_k) \otimes \mu_{\tilde{A}_q}(x_k)}{\sum_{k=1}^{K} \mu_{\tilde{A}_q}(x_k)} \tag{15}$$

4.2 The conditional pdf of the output distribution

A probabilistic fuzzy system consists of a system of rules R_q, q = 1, ... , Q, of the type:

$$R_q : \text{If } x \text{ is } \tilde{A}_q, \text{then } f(y) \text{ is } f(y \mid \tilde{A}_q) \tag{16}$$

We follow van den Berg et al (2013, p. 873), and write that a reasonable formulation of the output of the fuzzy system is

$$f(y\,|\,x) = \frac{\sum_{q=1}^{Q} \mu_{\tilde{A}_q}(x)\; f(y\,|\,\tilde{A}_q)}{\sum_{q=1}^{Q} \mu_{\tilde{A}_q}(x)} = \sum_{q=1}^{Q} \overline{\mu}^{(1)}_{\tilde{A}_q}(x)\; f(y\,|\,\tilde{A}_q) \tag{17}$$

In practice, the system of rules takes the form

$$\hat{R}_q : \text{If } x \text{ is } \tilde{A}_q, \text{then } f(y) \text{ is } \hat{f}(y\,|\,\tilde{A}_q) \tag{18}$$

from which the estimate of the conditional fuzzy probability of y, given x, is:

$$\hat{f}(y\,|\,x) = \sum_{q=1}^{Q} \overline{\mu}^{(1)}_{\tilde{A}_q}(x)\; \hat{f}(y\,|\,\tilde{A}_q) \tag{19}$$

where:

$$\hat{f}(y\,|\,\tilde{A}_q) = \sum_{j=1}^{J} \frac{\hat{P}(\tilde{C}_j\,|\,\tilde{A}_q)\; \mu_{\tilde{C}_j}(y)}{\int_{-\infty}^{\infty} \mu_{\tilde{C}_j}(y)\, dx}. = \sum_{j=1}^{J} \hat{P}(\tilde{C}_j\,|\,\tilde{A}_q)\, \overline{\mu}^{(2)}_{\tilde{C}_j}(y) \tag{20}$$

Thus,

$$\hat{f}(y\,|\,x) = \sum_{q=1}^{Q} \sum_{j=1}^{J} \overline{\mu}^{(1)}_{\tilde{A}_q}(x)\; \hat{P}(\tilde{C}_j\,|\,\tilde{A}_q)\;\; \overline{\mu}^{(2)}_{\tilde{C}_j}(y) \tag{21}$$

4.3 The mean and variance of the output distribution

We end this section with a statement of the estimates of the mean and variance of the conditional pdf. [van den Berg et al (2013, p. 874)]

The estimated expected output

$$\hat{E}(y\,|\,x) = \sum_{q=1}^{Q} \mu^{(1)}_{\tilde{A}_q}(x)\; \hat{E}(y\,|\,\tilde{A}_q) = \sum_{q=1}^{Q} \sum_{j=1}^{J} \mu^{(1)}_{\tilde{A}_q}(x)\; \hat{P}(\tilde{C}_j\,|\,\tilde{A}_q)\;\; E\left\{ y^1 \,|\, \overline{\mu}^{(2)}_{\tilde{C}_j} \right\} \tag{22}$$

where $\bar{\mu}^{(1)}_{\tilde{A}_k}(x)$ is the normalized MF associated with the antecedent part of the rule base, and $E\left\{y^k \mid \bar{\mu}^{(2)}_{\tilde{C}_j}\right\}$ is the k-th order MF centroid of the fuzzy set $\tilde{C}_j$.

The estimated conditional variance of the output distribution is

$$\begin{aligned}\hat{V}ar(y \mid x) &= \hat{E}(y^2 \mid x) - \left(\hat{E}(y \mid x)\right)^2 \\ &= \sum_{q=1}^{Q} \mu^{(1)}_{\tilde{A}_q}(x)\ \hat{E}(y^2 \mid \tilde{A}_q) - \left(\hat{E}(y \mid x)\right)^2 \\ &= \sum_{q=1}^{Q}\sum_{j=1}^{J} \mu^{(1)}_{\tilde{A}_q}(x)\ \hat{P}(\tilde{C}_j \mid \tilde{A}_q)\, E\left\{y^2 \mid \bar{\mu}^{(2)}_{\tilde{C}_j}\right\} - \left(\hat{E}(y \mid x)\right)^2\end{aligned} \tag{23}$$

5 Assigning MFs to fuzzy variables

A number of articles have discussed methods for assigning MFs to fuzzy variables. An excellent overview of the different approaches during the first 25 years following Zadeh's 1965 seminal article are given by Dombi (1990). He segregated the approaches into: heuristically based MFs; MFs based on reliability concerns with respect to the particular problem; MFs based on more theoretical demand, such as axiomatically justified or a probability distribution; MFs related to control, where either one defines the functions and identifies the system parameters, or works with a given system and identifies the MF; and MFs as a model for human concepts. More recent reviews on the topic include those of Bilgic and Turksen (1995) and Smithson and Verkuilen (2006).

A catalogue of methods for the development of MFs appears in Sivanandam et al (2007, chapter 4), where it is noted that the assignment of MFs to fuzzy variables can be done intuitively or by using some algorithms or logical procedures. Among the methods they listed and discussed were: intuition, where the development of the MF is based on the human's own intelligence and understanding, and requires the thorough knowledge of the problem and the linguistic variable; inference, which involves the knowledge to perform deductive reasoning, and forms the MF from the facts known and knowledge; and rank ordering, where the polling concept and pairwise comparisons are used to assign membership values by a rank ordering process. They also mention the role of the other soft computing methodologies, neural networks and genetic algorithms, in the MF assignment process.

Other notable articles that addressed the development of MFs , but were not mentioned in the foregoing, include Chen and Otto (1995), who presented methods for constructing MFs using measurement theory and constrained interpolation, where the former offers a suitable framework for constructing a MF in cases where the membership is based on subjective preferences, and Buckley (2005 §2.8), who showed how to develop triangular-shaped fuzzy MFs based on confidence intervals.

5.1 Using cluster analysis to develop membership functions

Figure 9 shows a representation of how cluster analysis may be used to induce a fuzzy model[8], where the context is the relevance of health to longevity.

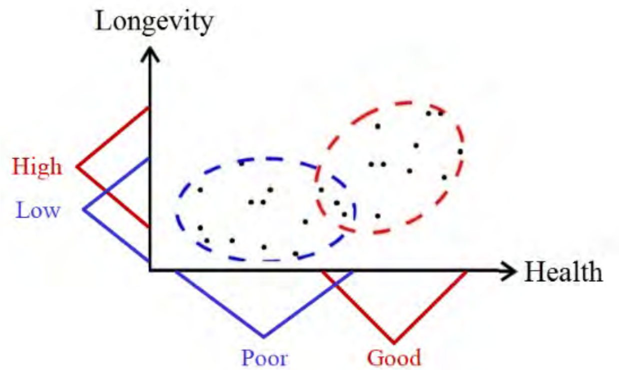

Figure 9: Using cluster analysis to obtain MFs

Starting with the data, clusters are formed, which are used to coordinate the MFs. Specifically, the data is clustered using a clustering algorithm. In this case, there are two clusters. Then, MFs are projected from the clusters: here, good and poor health, on the horizontal axis, and high and low longevity, on the vertical axis. As a final step, the MFs are used to develop fuzzy rules with respect to each cluster. Almeida et al (2008) provide a detailed discussion of this methodology.

6 Comments

The purpose of this article has been to present some preliminary observations with respect to PFSs. To this end, we presented some basic concepts that are pertinent to the PFSs dialogue, discussed the Mamdani Fuzzy Inference System (FIS) and a probabilistic Mamdani FIS, and gave an overview of PFSs, which included its conceptualization, the conditional pdf and the mean and variance of the output distribution, and the assigning MFs to fuzzy variables.

There were a number of topics that were not addressed. Some of the topics include: the best FIS model to use, the most appropriate clustering algorithm to use, the efficiency of the PFS model, and the question of optimal design. Moreover, while some simple applications were mentioned, there are a number of application areas that can be explored, including such topics as underwriting, financial planning, and measurement error, where fuzzy variables and random variables both contribute to the uncertainty.

The foregoing limitations notwithstanding, to the extent that this article provides an impetus for further study in this area, it will have served its purpose.

---

[8] Adapted from van den Berg (2004) Slide 21.

7 References

Almeida, R. J., Baştürk, N., Kaymak, U., Milea, V. (2012) "A multi-covariate semiparametric conditional volatility model using probabilistic fuzzy systems," in: IEEE Conference on Computational Intelligence for Financial Engineering Economics (CIFEE), 1-8.

Almeida, R. J., Kaymak, U. (2009) "Probabilistic fuzzy systems in Value-at-Risk estimation," Intelligent Systems in Accounting, Finance & Management 16, 49-70

Almeida, R. J., Kaymak, U., Sousa, J. M. C. (2008) "Fuzzy Rule Extraction From Typicality and Membership Partitions," IEEE International Conference on Fuzzy Systems, 1964-1970.

Bilgiç, T., Türkšen, I. B. (1995) "Measurement of membership functions: theoretical and empirical work," in: Dubois, D., Prade, H. (Eds.), Fundamentals of Fuzzy Sets 1, Kluwer Academic, Boston, 195-200.

Buckley, J. J. (2005) Fuzzy Probabilities. Physica-Verlag, Berlin Heidelberg.

Chen, J. E., Otto, K. N. (1995) "Constructing membership functions using interpolation and measurement theory," Fuzzy Sets and Systems 73, 313-327.

Dombi, J. (1990) "Membership function as an evaluation," Fuzzy Sets and Systems 35, 1-21.

Dubois, D. and Prade, H. (1980) Fuzzy sets and systems - Theory and Application, Chestnut Hill, MA: Academic Press, Inc.

Kaymak, U., van den Bergh, W.-M., van den Berg, J. (2003) "A fuzzy additive reasoning scheme for probabilistic Mamdani fuzzy systems," in: Proceedings of the 2003 IEEE International Conference on Fuzzy Systems 1, 331-336.

Meghdadi, A. H., Akbarzadeh-T, M.-R. (2001) "Probabilistic fuzzy logic and probabilistic fuzzy systems," IEEE International Conference on Fuzzy Systems 3, 1127-1130.

Moura, G., Roisenberg, M. (2015) "Probabilistic Fuzzy Naive Bayes," Brazilian Conference on Intelligent Systems, 246-251

Nguyen, H. T., Wu, B. (2006) Fundamentals of Statistics with Fuzzy Data, Springer-Verlag Berlin Heidelberg

Ruspini, E. H. (1969) "A new approach to clustering," Inf. Control 15, 22-32.

Shapiro, A. F. (2004) "Fuzzy logic in insurance," Insurance: Math. Econom. 35, 399-424.

Shapiro, A. F. (2013) "Modeling future lifetime as a fuzzy random variable," Mathematics and Economics 53 (2013) 864–870

Singpurwalla, N. D., Booker, J. M. (2004) "Membership functions and probability measures of fuzzy sets," Journal of the American Statistical Association 99 (467), 867-877.

Sivanandam, S. N., Sumathi, S., Deepa, S. N. (2007) Introduction to Fuzzy Logic Using MATLAB. Springer-Verlag, Berlin Heidelberg.

Smithson, M., Verkuilen, J. (2006) Fuzzy Set Theory: Applications in the Social Sciences. Sage Publications, Thousand Oaks, CA.

van den Berg, J., Kaymak, U., Almeida, R. J. (2011) "Function Approximation Using Probabilistic Fuzzy Systems," ERIM Research Program, ERS-2011-026-LIS, http://hdl.handle.net/1765/30923

Van Den Berg, J., Kaymak, U., Almeida, R. J. (2013) "Conditional density estimation using probabilistic fuzzy systems," IEEE Transactions on Fuzzy Systems 21(5), 869-882.

Van Den Berg, J., Kaymak, U., Van den Bergh, W.-M. (2004) "Financial markets analysis by using a probabilistic fuzzy modelling approach," International Journal of Approximate Reasoning 35(3), 291-305.

Viertl, R. (2011) Statistical Methods for Fuzzy Data, John Wiley & Sons, Ltd.

Zadeh, L. A. (1965) "Fuzzy sets," Information and Control 8 (3), 338-353.

Zadeh, L. A. (1968) "Probability measures of fuzzy events," Journal of Mathematical Analysis and Applications 23, 421-427.

Zadeh, L. A. (1975) "The concept of linguistic variable and its application to approximate reasoning (parts 1–3)," Information Sciences 8, 199-249. 301-357, 1976, 9, 43-80.

Zadeh, L. A. (1981) "Fuzzy systems theory: a framework for the analysis of humanistic systems," in: Cavallo, R.E. (Ed.), Recent Developments in Systems Methodology in Social Science Research. Kluwer, Boston, 25-41.