Pricing Bounds and Bang-bang Analysis of the Polaris Variable Annuities

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The “Polaris Choice IV” VAs are recently issued by the subsidiary insurance companies of the American International Group.

Three riders are structured into the Polaris:

- **Polaris Income Plus Daily**
- **Polaris Income Plus**
- **Polaris Income Builder**

Pricing the **Polaris Income Plus Daily** is the major focus of our work.
The Polaris Income Plus Daily has several distinguishing features:

1. **Withdrawal-dependent step-up**: the income base can step up to the high water mark of the investment account over certain monitoring period depending on policyholder’s age at first withdrawal.

2. **Withdrawal-dependent protected income**: the guaranteed withdrawal amount depends on the first withdrawal time.

These provisions encourage the policyholder not take excess withdrawal during the early phase of the contract life.
Step-up of Income Base

**Figure 1:** Step-up mechanism of the income phase before first withdrawal (Resource: Page 9 of the client brochure.)
Step-up of Income Phase

Figure 2: Step-up mechanism of the income phase after first withdrawal (Resource: Page 10 of the client brochure.)
### Withdrawal-dependent Income Payment

**Maximum Annual Withdrawal Amount (MAWA)**

<table>
<thead>
<tr>
<th>Age at 1st Withdrawal</th>
<th>Covered Persons</th>
<th>Income Option 1</th>
<th>Income Option 2</th>
<th>Income Option 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MAWA</td>
<td>PIP</td>
<td>MAWA</td>
<td>PIP</td>
</tr>
<tr>
<td><strong>45-59</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Single Life</td>
<td>3.75%</td>
<td>2.75%*</td>
<td>3.75%</td>
<td>2.75%*</td>
</tr>
<tr>
<td>Joint Life</td>
<td>3.25%</td>
<td>2.75%*</td>
<td>3.25%</td>
<td>2.75%*</td>
</tr>
<tr>
<td><strong>60-64</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Single Life</td>
<td>4.75%</td>
<td>2.75%*</td>
<td>4.75%</td>
<td>2.75%*</td>
</tr>
<tr>
<td>Joint Life</td>
<td>4.25%</td>
<td>2.75%*</td>
<td>4.25%</td>
<td>2.75%*</td>
</tr>
<tr>
<td><strong>65-71</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Single Life</td>
<td>6.0%</td>
<td>4.0%</td>
<td>7.0%</td>
<td>3.0%</td>
</tr>
<tr>
<td>Joint Life</td>
<td>5.5%</td>
<td>4.0%</td>
<td>6.5%</td>
<td>3.0%</td>
</tr>
<tr>
<td><strong>72+</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Single Life</td>
<td>6.5%</td>
<td>4.0%</td>
<td>7.5%</td>
<td>3.0%</td>
</tr>
<tr>
<td>Joint Life</td>
<td>6.0%</td>
<td>4.0%</td>
<td>7.0%</td>
<td>3.0%</td>
</tr>
</tbody>
</table>

Figure 3: Calculation scheme of MAWA and PIP (Resource: Page 17 of the client brochure.)
The pricing model should capture the following features of the Polaris:

- **Dynamic withdrawals** ⇒ stochastic optimal control framework
- **Path-dependent payoffs**
  ⇒ auxiliary state and control variables should be introduced:
  - one state variable to record the step-up value
  - one state variable to record the first-withdrawal time
  - one state variable to record the death benefits
  - one control variable to model the decision of starting withdrawal

- Five-dimensional state process and a bivariate control process.
The minimal super-hedging cost of the writer:

\[ V_1(x) = \sup_{\pi_1} \mathbb{E} \left[ \sum_{k=1}^{N-1} \varphi(k-1)H_k(X_k; \pi_k) + \varphi(N-1)G(X_N) \right]. \]

- \( \pi_1 = (\pi_k)_{1 \leq k \leq N-1} \): policyholder’s decisions
- \( \varphi(k) \): discount factor
- \( H_k(X_k; \pi_k) \): intermediate liability = death benefits + withdrawal
- \( G(X_N) \): terminal liability

The standard DPP argument implies the Bellman equation:

\[
\begin{cases}
V_N(x) = G_N(x), \\
V_n(x) = \sup_{\pi_n \in D_n} \left\{ H_n(X_n; \pi_n) + e^{-r\Delta t} \mathbb{E}_{n,x}^{\pi_n} [V_{n+1}(X_{n+1})] \right\}
\end{cases}
\]

\( n = N - 1, N - 2, \ldots, 1 \).
Major Challenges and Results

- The pricing problem poses challenges in two aspects.
  - Complex optimization problem $\Rightarrow$ no guarantee for global optimizer.
  - Large dimensionality of state process: $\Rightarrow$ computationally prohibitive.

- Our major results are summarized as follows:
  1. Show the existence of the Bang-bang solution for a synthetic contract.
  2. Solve for the Bang-bang solution: Monte Carlo + regression.
  3. Use the Bang-bang solution as an upper bound for the hedging cost of the real contract.
Bang-bang Analysis

Theorem 1 (Bang-bang Analysis)

Assume the periodical rider charge is proportional to the investment account. The optimal withdrawal strategies are limited to three choices:

1. non-withdrawal,
2. withdrawal at Maximal Annual Withdrawal Amount or
3. complete surrender.

In real contract specifications, the rider charge is proportional to the income base and deducted from the investment account. This would break the argument for proving Theorem 1.

We first make a compromise by assuming the insurance fee is proportional to the investment account and call this modified contract as synthetic contract.
Pricing Bounds

Theorem 2 (Pricing Bounds)

Let $\bar{V}_0$ be the minimal super-hedging cost of the real contract that charges the insurance fees proportional to the income base. Let $V_0$ be the minimal super-hedging cost of the synthetic contract that charges the insurance fees proportional to the investment account. Then we have $\bar{V}_0 \leq V_0$.

Remark (Economic Insight)

Charging the fees against the income base reduces the insurer’s risk exposure.

- $V_0$ is relatively easier to solve due to the existence of Bang-bang solution.
- A lower-bound for $\bar{V}_0$ can be easily obtained.
The LSMC method was first proposed to price American options.
- The price process is not influenced by the excise rule ⇒ forward simulation of sample paths [Longstaff & Schwartz 2001].
- Approximating the conditional expectation by regression.

Extending the LSMC to general stochastic control problem is nontrivial.
- The state process depends on the optimal controls unknown in prior ⇒ sample paths cannot be simulated.

One possible strategy: guess a initial control sequence, simulate the paths and update the control policies backwards [Huang & Kwok 2016].
- Convergence to the global optimal solution is not clear.
- This strategy cannot generate variations in certain state variable.
Our Approach: Pseudo Simulation & Backward Updating

- Transition function (explicitly given)

\[(X_n, \pi_n)^T \xrightarrow{Q_n(\cdot; \cdot)} X_{n+1}\]

- Recovered by regression over a compact support

\[E^Q \left[ V_{n+1}(X_{n+1}) \middle| X_n^+ \right]\]

(Continuation value)

- Simulated from certain artificial distribution

- Conditioning on \(X_{n+1}, X_{n+1}\) can be simulated directly.

- The regression is conducted once to recover \(C(\cdot)\) per time-step.

- \(C[Q_n(X_n; \pi_n)]\) can be computed for different pairs of \((X_n, \pi_n)^T\).
Regression Technique: Shape-Restricted Sieve Estimation

- Primary criteria for the choice of nonparametric regression technique:
  1. avoid computationally costly tuning parameters selection
     ⇒ local methods are not good candidates;
  2. avoid or mitigate the undesirable overfitting
     ⇒ the space of basis functions shouldn’t be too complex;
  3. ensure the regression estimate inherit the convexity and monotonicity
     ⇒ shape-restricted regression problem.

- Shape-restricted sieve regression is a suitable choice [Wang & Ghosh 2012].
  - Multivariate Berstein polynomials are chosen as basis functions.
  - Linear constraints are imposed on the regression coefficients
    ⇒ constrained Least-Squares (CLS) estimation
Model Parameters

Table 1: Parameters used for numerical examples.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volatility $\sigma$</td>
<td>0.19</td>
</tr>
<tr>
<td>Interest rate $r$</td>
<td>0.04</td>
</tr>
<tr>
<td>Attained age $t_0$</td>
<td>65</td>
</tr>
<tr>
<td>Mortality</td>
<td>DAV 2004R (65 year old male)</td>
</tr>
<tr>
<td>Withdrawal times</td>
<td>Yearly</td>
</tr>
<tr>
<td>Initial purchase payment</td>
<td>1 unit</td>
</tr>
<tr>
<td>Time periods $N$</td>
<td>30</td>
</tr>
<tr>
<td>Rider charge rate $\eta_n$</td>
<td>200 bps</td>
</tr>
<tr>
<td>Withdrawal penalty $k_n$</td>
<td>$n = 1:8%, \ n = 2:7%, \ n = 3:6%,$</td>
</tr>
<tr>
<td></td>
<td>$n = 4:5%, \ n &gt; 4:0%$</td>
</tr>
<tr>
<td>MAWA percentage $G(\xi)$</td>
<td>$1 \leq \xi \leq 6:5%, \ \xi &gt; 6:5.5%$</td>
</tr>
<tr>
<td>PIP percentage $P(\xi)$</td>
<td>$1 \leq \xi \leq 6:5%, \ \xi &gt; 6:5.5%$</td>
</tr>
</tbody>
</table>
Ordinary Least-Squares (OLS) Estimate

Figure 4: Fitted curves of marginal continuation function using OLS method.

- Overfitting.
- Sensitive to maximal degree.
Constrained Least-Squares (CLS) Estimate

- Mitigate overfitting.
- Robust to maximal degree.
- Economically sensible.

**Figure 5:** Fitted curves of marginal continuation function using CLS method.
Table 2: Numerical results of the validation test. The initial purchase payment is 1 unit. The mean and standard deviation are obtained by running the algorithm 40 times.

<table>
<thead>
<tr>
<th># of Simulation</th>
<th>Lower Bound</th>
<th>Upper Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>S.d.</td>
</tr>
<tr>
<td>$1 \times 10^4$</td>
<td>1.0199</td>
<td>0.0140</td>
</tr>
<tr>
<td>$3 \times 10^4$</td>
<td>1.0207</td>
<td>0.0087</td>
</tr>
<tr>
<td>$1 \times 10^5$</td>
<td>1.0195</td>
<td>0.0033</td>
</tr>
</tbody>
</table>

- “Upper Bound” is the minimal super-hedging cost of the synthetic contract.
- “Lower Bound” is obtained by discretizing the feasible set of control and then solving a similar stochastic control problem associated with the real contract.
Summary of Numerical Results

- The numerical result produced by our Monte-Carlo-based algorithm tends to be stable as the number of simulation increases.

- The shape-restricted regression technique has four primary merits:
  1. Mitigating undesirable overfitting problem.
  2. Avoiding computational intensive tuning parameter selection.
  3. Producing economically sensible results.

- The pricing bounds are rather sharp: the gap between sub and super hedging costs is less than 3%.
Conclusions

- A risk-neutral pricing framework for the “Polaris Income Plus Daily” rider is established.
- Bang-bang solution is proved to exist for a synthetic contract.
- A new Monte-Carlo-based numerical approach is developed.
- The minimal super-hedging cost of the synthetic contract is shown to be a sharp upper bound for the hedging cost of the real contract.
Thank you!