Pricing Bounds and Bang-bang Analysis of the Polaris Variable Annuities

Zhiyi (Joey) Shen

Department of Statistics and Actuarial Science
University of Waterloo

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Polaris Choice IV

- The "Polaris Choice IV" VAs are recently issued by the subsidiary insurance companies of the American International Group.
- Three riders are structured into the Polaris:
  - *Polaris Income Plus Daily*
  - *Polaris Income Plus*
  - *Polaris Income Builder*
- Pricing the *Polaris Income Plus Daily* is the major focus of our work.

Polaris Income Plus Daily

- The *Polaris Income Plus Daily* has several distinguishing features:
  1. **Withdrawal-dependent step-up**: the income base can step up to the high water mark of the investment account over certain monitoring period depending on policyholder's age at first withdrawal.
  2. **Withdrawal-dependent protected income**: the guaranteed withdrawal amount depends on the first withdrawal time.
- These provisions encourage the policyholder not take excess withdrawal during the early phase of the contract life.

Step-up of Income Base

Figure 1: Step-up mechanism of the income phase before first withdrawal (Resource: Page 9 of the client brochure.)

Step-up of Income Phase

Figure 2: Step-up mechanism of the income phase after first withdrawal (Resource: Page 10 of the client brochure.)

Withdrawal-dependent Income Payment

Maximum Annual Withdrawal Amount (MAWA)
(as a percentage of your Income Base)

		Income Option 1		Income Option 2		Income Option 3
Age at 1st Withdrawal	Covered Persons	MAWA	PIP	MAWA	PIP	MAWA/PIP
45-59	Single Life	3.75%	2.75%*	3.75%	2.75%*	3.00% for life
	Joint Life	3.25%	2.75%*	3.25%	2.75%*	2.75% for life
60-64	Single Life	4.75%	2.75%*	4.75%	2.75%*	3.50% for life
	Joint Life	4.25%	2.75%*	4.25%	2.75%*	3.25% for life
65-71	Single Life	6.0%	4.0%	7.0%	3.0%	5.00% for life
	Joint Life	5.5%	4.0%	6.5%	3.0%	4.50% for life
72+	Single Life	6.5%	4.0%	7.5%	3.0%	5.25% for life
	Joint Life	6.0%	4.0%	7.0%	3.0%	4.75% for life

Figure 3: Calculation scheme of MAWA and PIP (Resource: Page 17 of the client brochure.)

Model formulation

- The pricing model should capture the following features of the Polaris:
  - **Dynamic withdrawals** $\Rightarrow$ stochastic optimal control framework
  - **Path-dependent payoffs**
    $\Rightarrow$ auxiliary state and control variables should be introduced:
    - one state variable to record the step-up value
    - one state variable to record the first-withdrawal time
    - one state variable to record the death benefits
    - one control variable to model the decision of starting withdrawal
- Five-dimensional state process and a bivariate control process.

Stochastic Control Framework

- The minimal super-hedging cost of the writer:

$$V_1(\mathbf{x}) = \sup_{\boldsymbol{\pi}_1} \mathbb{E}\left[\sum_{k=1}^{N-1} \varphi(k-1) H_k(\mathbf{X}_k; \pi_k) + \varphi(N-1) G(\mathbf{X}_N)\right].$$

  - $\boldsymbol{\pi}_1 = (\pi_k)_{1 \le k \le N-1}$: policyholder's decisions
  - $\varphi(k)$: discount factor
  - $H_k(\mathbf{X}_k; \pi_k)$: intermediate liability = death benefits + withdrawal
  - $G(\mathbf{X}_N)$: terminal liability

- The standard DPP argument implies the Bellman equation:

$$\begin{cases} V_N(\mathbf{x}) & = G_N(\mathbf{x}), \\ V_n(\mathbf{x}) & = \sup_{\pi_n \in D_n} \left\{ \underbrace{H_n(\mathbf{X}_n; \pi_n)}_{\text{withdrawal value}} + e^{-r\Delta t} \underbrace{\mathbb{E}^{\pi_n}_{n,x}\left[V_{n+1}(\mathbf{X}_{n+1})\right]}_{\text{continuation value}} \right\} \\ & \qquad n = N-1, N-2, \ldots, 1. \end{cases}$$

Major Challenges and Results

- The pricing problem poses challenges in two aspects.
  - Complex optimization problem $\Rightarrow$ no guarantee for global optimizer.
  - Large dimensionality of state process: $\Rightarrow$ computationally prohibitive.
- Our major results are summarized as follows:
  1. Show the existence of the Bang-bang solution for a synthetic contract.
  2. Solve for the Bang-bang solution: Monte Carlo + regression.
  3. Use the Bang-bang solution as an upper bound for the hedging cost of the real contract.

Bang-bang Analysis

Theorem 1 (Bang-bang Analysis)

Assume the periodical rider charge is proportional to the investment account. The optimal withdrawal strategies are limited to three choices:

1. *non-withdrawal,*
2. *withdrawal at Maximal Annual Withdrawal Amount or*
3. *complete surrender.*

- In real contract specifications, the rider charge is proportional to the income base and deducted from the investment account. This would break the argument for proving Theorem 1.
- We first make a compromise by assuming the insurance fee is proportional to the investment account and call this modified contract as synthetic contract.

Pricing Bounds

Theorem 2 (Pricing Bounds)

Let $\bar{V}_0$ be the minimal super-hedging cost of the real contract that charges the insurance fees proportional to the income base. Let V_0 be the minimal super-hedging cost of the synthetic contract that charges the insurance fees proportional to the investment account. Then we have $\bar{V}_0 \leq V_0$.

Remark (Economic Insight)

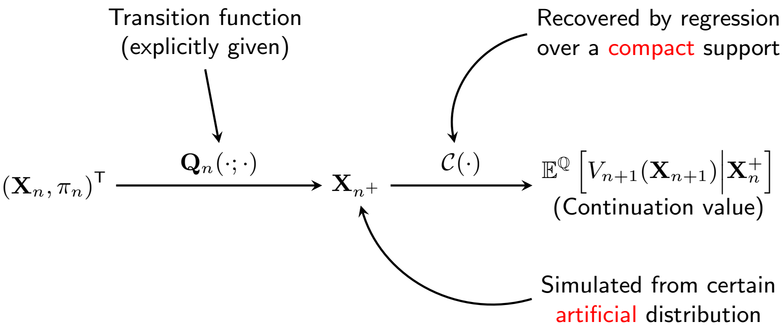
Charging the fees against the income base reduces the insurer's risk exposure.

- V_0 is relatively easier to solve due to the existence of Bang-bang solution.
- A lower-bound for $\bar{V}_0$ can be easily obtained.

Least-Squares Monte Carlo (LSMC) Method

- The LSMC method was first proposed to price American options.
  - The price process is not influenced by the excise rule
    $\Rightarrow$ forward simulation of sample paths [Longstaff & Schwartz 2001].
  - Approximating the conditional expectation by regression.
- Extending the LSMC to general stochastic control problem is nontrivial.
  - The state process depends on the optimal controls unknown in prior
    $\Rightarrow$ sample paths cannot be simulated.
- One possible strategy: guess a initial control sequence, simulate the paths and update the control policies backwards [Huang & Kwok 2016].
  - Convergence to the global optimal solution is not clear.
  - This strategy cannot generate variations in certain state variable.

Our Approach: Pseudo Simulation & Backward Updating



- Conditioning on $\mathbf{X}_{n^+}$, $\mathbf{X}_{n+1}$ can be simulated directly.
- The regression is conducted once to recover $\mathcal{C}(\cdot)$ per time-step.
- $\mathcal{C}\left[\mathbf{Q}_n(\mathbf{X}_n;\pi_n)\right]$ can be computed for different pairs of $(\mathbf{X}_n, \pi_n)^\mathsf{T}$.

Regression Technique: Shape-Restricted Sieve Estimation

- Primary criteria for the choice of nonparametric regression technique:
  1. avoid computationally costly tuning parameters selection
     $\Rightarrow$ local methods are not good candidates;
  2. avoid or mitigate the undesirable overfitting
     $\Rightarrow$ the space of basis functions shouldn't be too complex;
  3. ensure the regression estimate inherit the convexity and monotonicity
     $\Rightarrow$ shape-restricted regression problem.
- Shape-restricted sieve regression is a suitable choice [Wang & Ghosh 2012].
  - Multivariate Berstein polynomials are chosen as basis functions.
  - Linear constraints are imposed on the regression coefficients
    $\Rightarrow$ constrained Least-Squares (CLS) estimation

Model Parameters

Table 1: Parameters used for numerical examples.

Parameter	Value
Volatility σ	0.19
Interest rate r	0.04
Attained age t_0	65
Mortality	DAV 2004R (65 year old male)
Withdrawal times	Yearly
Initial purchase payment	1 unit
Time periods N	30
Rider charge rate η_n	200 bps
Withdrawal penalty k_n	$n=1:8\%,\ n=2:7\%,\ n=3:6\%,$ $n=4:5\%,\ n>4:0\%$
MAWA percentage $G(\xi)$	$1\leq\xi\leq6:5\%,\ \xi>6:5.5\%$
PIP percentage $P(\xi)$	$1\leq\xi\leq6:5\%,\ \xi>6:5.5\%$

Ordinary Least-Squares (OLS) Estimate

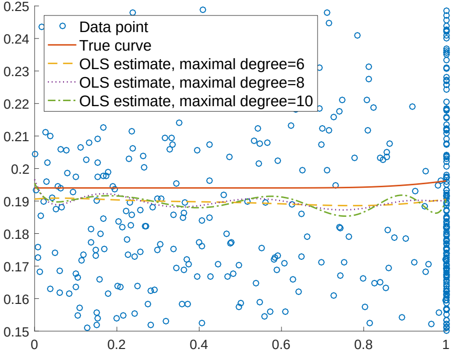


Figure 4: Fitted curves of marginal continuation function using OLS method.

- Overfitting.
- Sensitive to maximal degree.

Constrained Least-Squares (CLS) Estimate

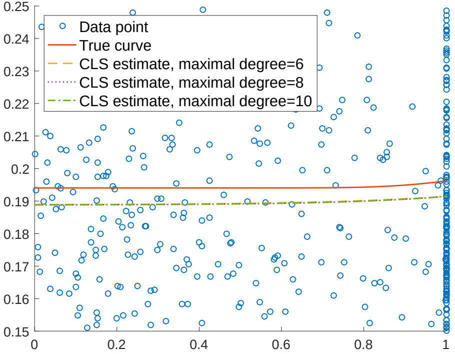


Figure 5: Fitted curves of marginal continuation function using CLS method.

- Mitigate overfitting.
- Robust to maximal degree.
- Economically sensible.

Performance of Pricing Bounds

Table 2: Numerical results of the validation test. The initial purchase payment is 1 unit. The mean and standard deviation are obtained by running the algorithm 40 times.

# of Simulation	Lower Bound		Upper Bound	
	Mean	S.d.	Mean	S.d.
1×10^4	1.0199	0.0140	1.0380	0.0041
3×10^4	1.0207	0.0087	1.0380	0.0029
1×10^5	1.0195	0.0033	1.0379	0.0016

- "Upper Bound" is the minimal super-hedging cost of the synthetic contract.
- "Lower Bound" is obtained by discretizing the feasible set of control and then solving a similar stochastic control problem associated with the real contract.

Summary of Numerical Results

- The numerical result produced by our Monte-Carlo-based algorithm tends to be stable as the number of simulation increases.
- The shape-restricted regression technique has four primary merits:
  1. Mitigating undesirable overfitting problem.
  2. Avoiding computational intensive tunning parameter selection.
  3. Producing economically sensible results.
  4. Good finite-sample performance: less volatile result.
- The pricing bounds are rather sharp: the gap between sub and super hedging costs is less than 3%.

Conclusions

- A risk-neutral pricing framework for the "Polaris Income Plus Daily" rider is established.
- Bang-bang solution is proved to exist for a synthetic contract.
- A new Monte-Carlo-based numerical approach is developed.
- The minimal super-hedging cost of the synthetic contract is shown to be a sharp upper bound for the hedging cost of the real contract.

Thank you!