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# Another Perspective on Black-Scholes Option Formulas

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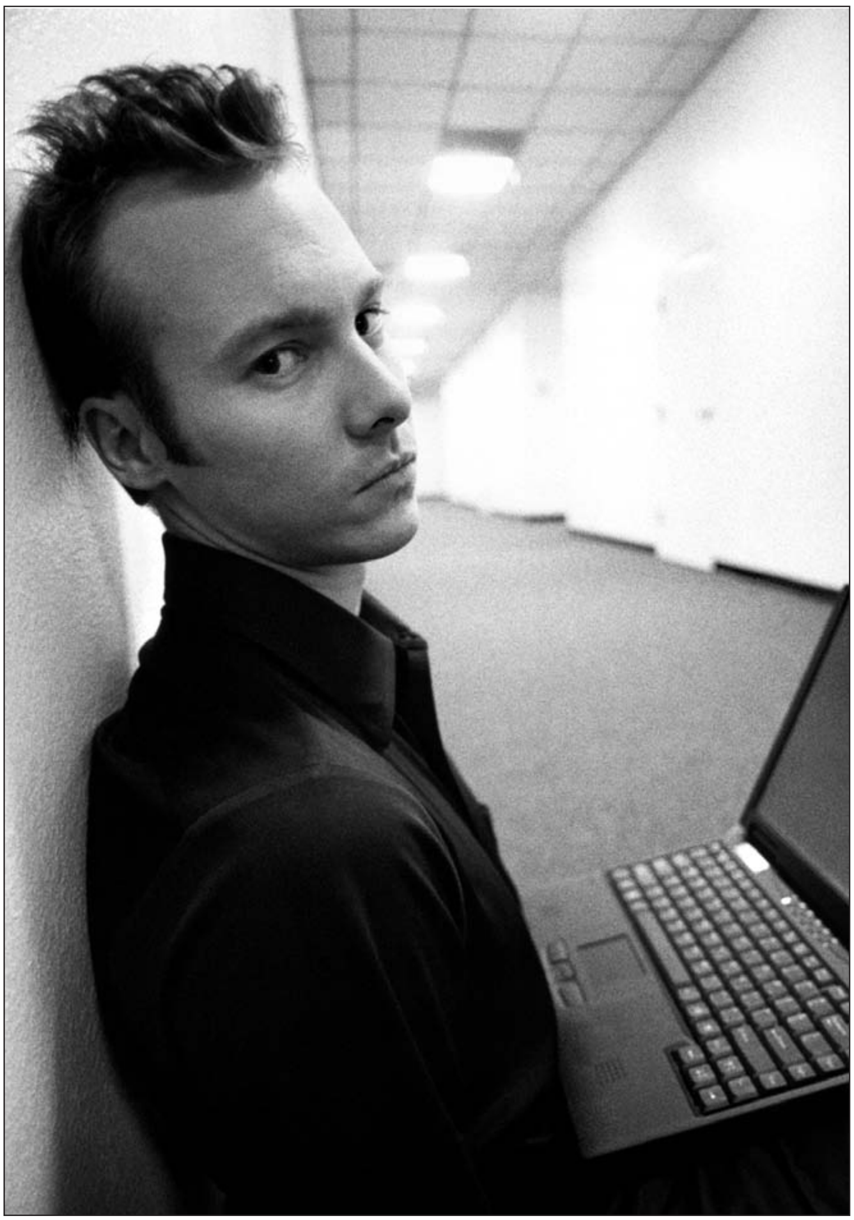
This article shows a different form of the Black-Scholes formula for European calls and puts under risk-neutral assumptions, that permits a comfortable verbal interpretation. The derivation of this alternative form appears at the end of the article.

Consider the Black-Scholes equation for a European call option on a stock with initial price 1, no dividend, strike $X$, time to expiry $t$, and volatility $\sigma$. Let $r$ be the risk-free rate. Let C be the value of a call option and P the value of a put option.



Let $a = [\ln(X) - (r - \sigma^2/2)t]/(\sigma\sqrt{t})$ and let $b = a - \sigma\sqrt{t}$.

First, consider the case $X = \exp(rt)$, i.e. the call is struck at the money forward. Then later we will show that:

$$C = (1/\sqrt{2\pi})\int_b^a \exp(-\varepsilon^2/2)d\varepsilon .$$

Now $a - b = \sigma\sqrt{t}$ and $(1/\sqrt{2\pi})\int_b^a \exp(-\varepsilon^2/2)d\varepsilon$ is the probability under the normal distribution of a value being between $b$ and $a$ standard deviations. Then the area represented by the above integral has width equal to one standard deviation (adjusted for time) and height corresponding to the normal distribution frequency. In other words, we are integrating across the probability function by one standard deviation of the stock price, or the option price is equal to a one standard deviation move times the probability of a move that size. This has a nice intuitive feel.

Returning to the more general case where $X$ can take any value, we will show that the equation can be written as,

$$C = (1/\sqrt{2\pi})\int_b^a \exp(-\varepsilon^2/2)d\varepsilon - (X - \exp(rt))\exp(-rt)(1/\sqrt{2\pi})\int_a^{-\infty} \exp(-\varepsilon^2/2)d\varepsilon .$$

The first term is equivalent to the result we received when $X = \exp(rt)$. The second term can be described as the difference between the strike and the at-the-money forward price times the probability that the option pays off. So the option price is a one standard deviation move times the probability of a move that size with an adjustment term adjusting for the difference between the actual strike and the at-the-money forward price.

A similar analysis exists for a European put option of price P.

Once again, consider the case $X = \exp(rt)$, i.e. the put is struck at-the-money forward. Then we will show that

$$P = (1/\sqrt{2\pi})\int_b^a \exp(-\varepsilon^2/2)d\varepsilon.$$

Returning to the more general case where X can take any value, we will show that the equation can be written as,

$$P = (1/\sqrt{2\pi})\int_b^a \exp(-\varepsilon^2/2)d\varepsilon - (X - \exp(rt))\exp(-rt)\,(1/\sqrt{2\pi})\int_{-\infty}^a \exp(-\varepsilon^2/2)d\varepsilon.$$

These results have analogous interpretations to the call results.

The remainder of this article shows how the above formulas are developed.

Under the lognormal assumption for stock price change we can represent the value of the stock at time $t$ as:

$$\exp[(r - \sigma^2/2)t + \varepsilon\sigma\sqrt{t}\,]$$

Then we can write the value of the call option, as:

$$C = (1/\sqrt{2\pi})\int_a^\infty \{\exp[(r - \sigma^2/2)t + \varepsilon\sigma\sqrt{t}\,] - X\}\,\exp(-rt)\exp(-\varepsilon^2/2)d\varepsilon.$$

Manipulating the equation for C,

$$C = (1/\sqrt{2\pi})\int_a^\infty \exp[-(\varepsilon^2 - 2\varepsilon\sigma\sqrt{t} + \sigma^2 t)/2]d\varepsilon - \exp(-rt)\,(X/\sqrt{2\pi})\int_a^\infty \exp(-\varepsilon^2/2)d\varepsilon.$$

If we make the substitution $y = \varepsilon - \sigma\sqrt{t}$ into the first integral above, then we have:

$$C = (1/\sqrt{2\pi})\int_b^\infty \exp[-y^2/2]dy - \exp(-rt)\,(X/2\sqrt{\pi})\int_a^\infty \exp(-\varepsilon^2/2)d\varepsilon.$$

Next we substitute $\varepsilon = y$ to facilitate further development which yields,

$$C = (1/\sqrt{2\pi})\int_b^\infty \exp[-\varepsilon^2/2]d\varepsilon - \exp(-rt)\,(X/\sqrt{2\pi})\int_a^\infty \exp(-\varepsilon^2/2)d\varepsilon.$$

Further manipulation leads to the usual expression for the Black-Scholes equation for a call option, but heading in a different direction allows this formula to be viewed as discussed above.

First, consider the case $X = \exp(rt)$. Then,

$$C = (1/\sqrt{2\pi})\int_b^\infty \exp[-\varepsilon^2/2]d\varepsilon - (1/\sqrt{2\pi})\int_a^\infty \exp(-\varepsilon^2/2)d\varepsilon.$$

$$= (1/\sqrt{2\pi})\int_b^a \exp(-\varepsilon^2/2)d\varepsilon.$$

Returning to the more general case where X can take any value, the equation can be written as,

$$C = (1/\sqrt{2\pi})\int_b^\infty \exp[-\varepsilon^2/2]d\varepsilon - (X - \exp(rt))\exp(-rt)\,(1/\sqrt{2\pi})\int_a^\infty \exp(-\varepsilon^2/2)d\varepsilon - \exp(rt))\exp(-rt)\,(1/\sqrt{2\pi})\int_a^\infty \exp(-\varepsilon^2/2)d\varepsilon.$$

$$= (1/\sqrt{2\pi})\int_b^a \exp(-\varepsilon^2/2)d\varepsilon - (X - \exp(rt))\exp(-rt)\,(1/\sqrt{2\pi})\int_a^\infty \exp(-\varepsilon^2/2)d\varepsilon.$$

A similar derivation exists for a European put option of price P.

$$P = (1/\sqrt{2\pi})\int_{-\infty}^a \{X - \exp[(r - \sigma^2/2)t + \varepsilon\sigma\sqrt{t}\,]\}\exp(-rt)\,\exp(-\varepsilon^2/2)d\varepsilon.$$

$$= \exp(-rt)\,(X/\sqrt{2\pi})\int_{-\infty}^a \exp(-\varepsilon^2/2)d\varepsilon - (1/\sqrt{2\pi})\int_{-\infty}^a \exp[-(\varepsilon^2 - 2\varepsilon\sigma\sqrt{t} + \sigma^2 t)/2]d\varepsilon.$$

$$= \exp(-rt)\,(X/\sqrt{2\pi})\int_{-\infty}^a \exp(-\varepsilon^2/2)d\varepsilon - (1/\sqrt{2\pi})\int_{-\infty}^b \exp(-\varepsilon^2/2)d\varepsilon.$$

Once again, consider the case $X = \exp(rt)$. Then,

$$P = (1/\sqrt{2\pi})\int_{-\infty}^a \exp[-\varepsilon^2/2]d\varepsilon - (1/\sqrt{2\pi})\int_{-\infty}^b \exp(-\varepsilon^2/2)d\varepsilon.$$

$$= (1/\sqrt{2\pi})\int_b^a \exp(-\varepsilon^2/2)d\varepsilon.$$

This is the same result we obtained for the call option.

Returning to the more general case,

$$P = (X - \exp(rt))\exp(-rt)(1/\sqrt{2\pi})\int_{-\infty}^a \exp[-\varepsilon^2/2]d\varepsilon + \exp(rt))\exp(-rt)(1/\sqrt{2\pi})\int_{-\infty}^a \exp(-\varepsilon^2/2)d\varepsilon - (1/\sqrt{2\pi})\int_{-\infty}^b \exp(-\varepsilon^2/2)d\varepsilon.$$

$$= (1/\sqrt{2\pi})\int_b^a \exp(-\varepsilon^2/2)d\varepsilon + (X - \exp(rt))\exp(-rt)\,(1/\sqrt{2\pi})\int_{-\infty}^a \exp(-\varepsilon^2/2)d\varepsilon.$$

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