SOCIETY OF ACTUARIES

Article from:

Risks and Rewards Newsletter

August 2006 – Issue No. 48

Implications of Real World Customer Behavior in Risk Neutral Hedging

by Mark Evans

Some have suggested using real world assumptions for determining customer behavior when calculating hedge positions to support minimum guarantees for variable annuities. Real world is also referred to as realistic. This article discusses a simple simulation model that analyzes the implications of this approach.

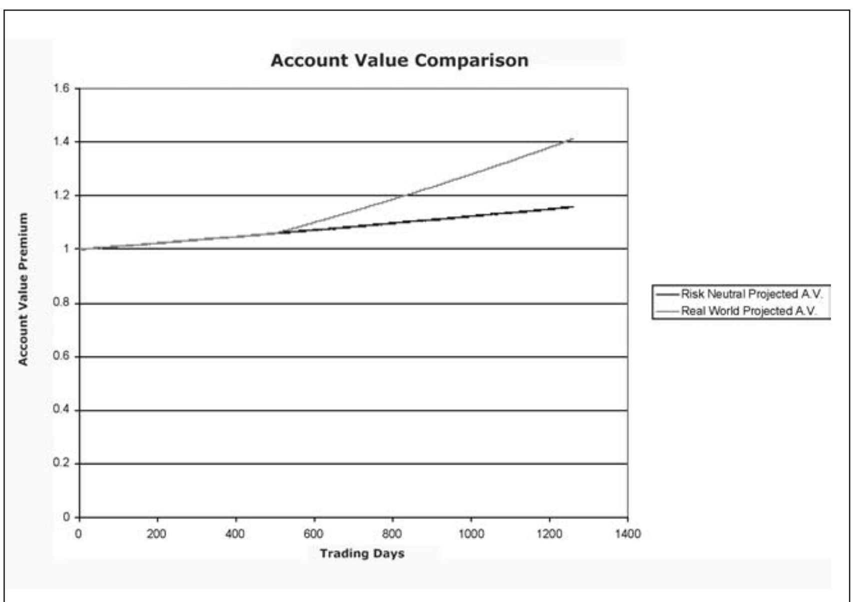
This involves what happens at future nodes in Monte Carlo simulations used to calculate option values and associated greeks. Real world customer behavior might be employed in a hedging program by basing customer behavior at a specific future node in a specific simulation path not on the account value for that node projected on a risk neutral basis from the current time, but on a real world projection from the current time. This will be referred to as the real world shadow account.

Table 1 below gives an example. Here, we are at the end of the second policy duration for a contract being hedged. The lower curve represents a risk neutral path starting at year two. The upper curve represents the real world shadow account starting from the same point at year two. When using the real world shadow account approach to hedging, all calculations are based on risk neutral projection of account value, except that customer behavior will be based on the real world shadow account. In this article we will also look at hedging results where customer behavior is based on the risk neutral projections of account value. This will be referred to as risk neutral customer behavior.

This article will show that the use of a real world shadow account will result in an under-hedge or partial hedge. This will result in hedge income falling short of option payoffs for adverse (falling market) paths.

This article starts by describing the model used to simulate the hedging process. In this model, interest rates and volatilities are assumed to be fixed. Hence, only delta hedging is required. This is achieved by the use of futures contracts. Next, some modeling considerations are discussed that provide additional background to help support the remainder of the article and information helpful to the reader wishing to reproduce these results. Then, the article presents the numerical results of the modeling followed by a discussion of why the results occur. Lastly, other considerations are discussed.

Table 1



Model Description

The model is based on a guaranteed return of 125 percent of premium at the end of 10 years. In other words, this is a 10-year European put struck at 125 percent. The premium in the model is $100,000 so the strike is $125,000. The customer has the option to surrender the contract for the account value at the end of five years. Otherwise, there are no decrements for mortality, surrender or partial withdrawal. There are no fees or other deductions from the account value. The risk free interest rate is 3 percent, the real world equity growth rate is 10 percent and the volatility is 14 percent for both. The entire account value is in equities. Thus the account value growth equals the assumed equity growth. Unless otherwise stated, typical Black-Scholes assumptions are used.

To simulate delta hedging, the model uses daily rebalancing during the first five years. It produces two sets of greeks, one set projecting the account value on a risk neutral basis for the purpose of determining customer behavior and one set using a real world shadow account. The first set will be referred to as risk neutral greeks, and the second set will be referred to as real world greeks. The same random normal variables are used for both sets of greeks. Delta is calculated by shocking up and down by 1 percent, taking the difference of the two shocked paths, and dividing by 2 percent. Delta is applied to each trading day's percentage stock market change, increased by interest imputed from the corresponding short position or futures position. Hedging cash flows are accumulated at 3 percent. The hedging simulation assumes an initial cash position equal to the option value calculated assuming risk neutral customer behavior.

The persistency factor is equal to (-1) times the put delta at the end of five years where the strike is $125,000 and the current asset level is equal to the account value at the end of five years. The put delta is based on a 10 percent interest rate. Note this corresponds to the real world equity growth rate. This will be discussed at greater length later in this article. Volatility is based on 14 percent. Since the customer receives the total return and there are no account fees, the dividend is assumed to be zero. Thus the persistency factor approaches 100 percent as the account value at the end of five years approaches zero and the persistency factor approaches zero percent as the account value at the end of five years approaches infinity. For the graph on the previous page, the persistency factor is .019 for the real world projected account value while it is .075 for the risk neutral projected account values.

This method is arbitrary and other methods could be employed. It does have the desirable characteristics of:

1) Causing the customer to be more likely to retain the contract when the option is more valuable since the delta is related to the probability of payoff,
2) Smoothness,
3) Being continuous,
4) Being well-behaved, and
5) Being intuitive.

The method is not intended to represent an optimal exercise function. In practice, variable annuity minimum guarantees tend to be priced using "semi-optimal" exercise functions where customers are more likely to persist when embedded options are of greater value, but customers do not behave in an entirely optimal fashion.

Since the customer can only surrender at the end of five years, after that point the notional amount of the option is fixed so the option value at the end of five years can be valued analytically using the standard Black-Scholes formula for a European put. The inputs for calculating the option value are the same as for the delta calculated in the paragraph above except the interest rate is 3 percent to reflect the market price based on capital markets pricing.

This is much simpler than any actual variable annuity minimum guarantee, but this is path dependent and contains dynamic customer behavior. Thus, this model permits analysis of real world customer behavior.

Modeling Considerations

The initial option values, one based on risk neutral projection of account values to drive customer behavior and one based on a real world shadow account, are based on 2 million scenarios. This can actually be done very quickly, because here we are not trying to produce daily results. We just need one random variable to determine the account value at the end of five years. As mentioned earlier, at the end of five years, customer behavior is applied and then the value of the option at that point is directly calculated by the Black-Scholes formula for a put. The large number of scenarios increases the accuracy of the option value that in turn is very important for simulating the hedge. Statistical sampling errors in the initial value tend to have a larger impact on the simulation than corresponding errors in subsequent greek calculations.

In practice, variable annuity minimum guarantees tend to be priced using "semi-optimal" exercise functions where customers are more likely to persist when embedded options are of greater value, but customers do not behave in an entirely optimal fashion.

The model uses daily rebalancing assuming 252 trading days annually during the first five years, antithetic scenarios and parallel shock paths for calculating delta to reduce modeling error. For each model day in simulating hedging along a given path, greeks are determined using 2000 random scenarios plus associated shock scenarios. Each random scenario corresponds to a single random normal variable which is multiplied by the square root of the time from the model day to the fifth contract anniversary. This produces an account value that determines the value of the five-year put option, and the delta to determine the persistency factor. The product is then discounted back to the current model day.

Thus, while the model is a stochastic on stochastic model, it can be run on a PC in several hours using VBA.

turn to page 12

From Page 11

The model can use previously determined normal random variables read in from a spreadsheet to produce daily stock changes, or it can produce normal random variables on the fly to produce daily stock changes. For simulated hedging, greeks calculated after time zero are always based on normal random variables produced on the fly.

The following outline summarizes the modeling steps:

1. Calculate initial option values
    a. 2 million scenarios.
    b. First five years covered by one random variable.
    c. Calculate persistency based on delta of a put.
    d. Calculate option value of five-year put based on account value at end of five years.
    e. Discount step d. above five years at 3 percent to get present value for scenario.
    f. Done based on both a risk neutral account value projection to predict customer behavior and using a real world shadow account.

2. Calculate random paths
    a. Use an option value based on risk neutral customer behavior as the initial hedge cash position.
    b. 252 trading day steps per year for five years for a total of 1,260 steps.
    c. Calculate delta for each day based on 2000 scenarios.
    d. Each of the 2000 scenarios uses a random variable to predict index change from trading day to the end of five years.
    e. Use delta to determine hedge position and the hedge cash flows.
    f. Accumulate with interest at 3 percent.
    g. Compare with five year put option value at the end of five years to determine hedge effectiveness.
    h. Done based on both a risk neutral account value projection to predict customer behavior and using a real world shadow account.

Table 2

Hedging Simulation—Very Bad Market

Hedging Simulation		Risk Neutral Behavior			Real World Behavior		
Period	Stock Price	Option Value	Cash Position	Delta	Option Value	Cash Position	Delta
0	100,000	4,860	4,860	-21,297	878	4,860	-5,321
126	87,922	8,355	8,352	-32,087	2,177	5,855	-11,457
252	76,869	13,572	13,624	-45,445	4,728	7,935	-22,241
378	66,916	21,147	21,185	-59,033	9,740	12,202	-38,835
504	60,956	27,778	27,866	-67,211	15,683	16,897	-53,433
630	59,821	30,079	30,031	-70,576	19,135	18,336	-60,850
756	60,309	30,469	30,482	-74,837	21,038	18,281	-67,227
882	54,699	39,230	39,100	-77,586	31,510	25,994	-78,092
1008	53,441	42,624	42,531	-79,444	37,307	29,085	-81,908
1134	54,105	43,381	43,287	-83,324	40,608	29,576	-85,221
1260	52,594	47,651	47,554	0	47,651	33,591	0

Model Results

The initial option value using risk neutral customer behavior is 4,860 while the value is 878 using the real world shadow account. Obviously these are very different option values, but we will see that the assumptions used to justify the 878 option value leads to an inadequate hedge.

The model was used to calculate payoffs for 250 paths based on predetermined sets of normal random variables. The drift rate used for these paths was 10 percent. A detailed analysis was performed on the path producing the lowest account value at the end of five years, and therefore the largest option payoff. The account value at the end of five years was 52,594 producing an option value at the end of five years of 47,651. Using risk neutral greeks, the hedges produced a cash position of 47,554 at the end of five years while using the real world greeks produced a cash position of 33,591. The risk neutral difference is due to daily versus continuous rebalancing and statistical error in calculating delta. The cause for the significant shortfall using real world greeks is illustrated in the following table on the bottom.

Table 2 is taken from the hedging simulation. It shows the hedge cash position and delta at the end of every six months. The actual simulation produces this same information for each trading day. In the earlier periods, the real world behavior delta is much lower because the real world shadow account results in a lower expected persistency. For example, at the end of one year, the stock price is 76,869, which is used as the starting point for calculating both the risk neutral behavior delta and the real world behavior delta at that point. Both the risk neutral behavior analysis and the real world behavior analysis shown above are following the same real world base path, only the hedging approach varies. The risk neutral behavior delta is -45,445 while the real world behavior delta is -22,241. This is because the real world behavior deltas are calculated assuming a projected account value at the end of five years that is (1.1/1.03)^4 = 1.3 times that used to calculate the risk neutral behavior deltas. The higher account values produce a lower estimated persistency.

In later periods, the real world shadow account and the risk neutral projection converge as time approaches five years. Thus the deltas align more closely, and in fact the real world deltas become slightly larger as the put delta is very close to -1 and the 1 percent shock applied to the account value has a bigger impact on the shadow account than the risk neutral projection. But it is too little, too late as the real world cash position is already hopelessly behind.

Table 3 to the right is based on the path producing the largest account value at the end of five years. Here the understated real world deltas work to the advantage of the real world hedging simulation as the lower deltas result in smaller cash outflows as a result of the rising market.

The hedging simulation was performed for an additional 98 real world paths using both risk neutral greeks and real world greeks yielding the following statistics for the 100 real world paths:

		Risk Neutral	Real World
Error:	Average:	0.19	2,685.31
	Std. Dev.	186.98	2,515.16
	Maximum	621.73	4,481.90
	Minimum	(426.63)	(14,060.66)

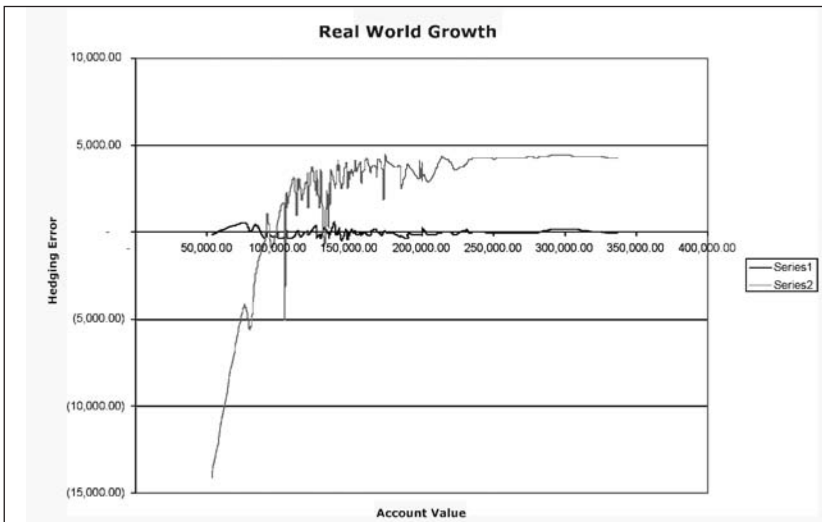
It is also interesting to look at the hedging error as a function of ending five-year account value graphically in Table 4 to the right.

Series 1 in Table 4 shows the hedging error associated with risk neutral that is essentially zero. Series 2 shows the hedging error associated with real world that is very negative for in the money paths, but positive for out of the money paths. For the higher account values, the real world hedging error appears to approach an upper bound. In fact, this upper bound can be calculated easily. When the value of the option at the end of five years is near zero, the real world shadow account results in a lower delta based on an expected option cost of 878. As long as there is not material value to the option at the end of five years, then the hedging costs will approach 878 on a present value basis. Compared to the initial hedging cash position of 4,860, this produces a difference of

Table 3
Hedging Simulation—Very Good Market

Hedging Simulation		Risk Neutral Behavior			Real World Behavior		
Period	Stock Price	Option Value	Cash Position	Delta	Option Value	Cash Position	Delta
0	100,000	4,860	4,860	-21,297	878	4,860	-5,321
126	109,934	2,979	3,059	-15,083	535	4,425	-3,502
252	140,778	710	672	-4,644	101	3,947	-848
378	163,322	192	165	-1,559	25	3,911	-242
504	197,764	28	-2	-289	3	3,942	-39
630	215,939	5	-21	-69	1	3,998	-9
756	226,005	3	-26	-35	1	4,057	-7
882	225,710	1	-28	-12	0	4,117	-3
1008	237,735	0	-29	-4	0	4,178	-1
1134	288,086	0	-29	0	0	4,240	0
1260	335,960	0	-30	0	0	4,303	0

Table 4



3,982 which accumulates with 3 percent interest to 4,616 which is close to the 4,482 maximum hedging error shown on page 13.

The real world hedging curve in the graph above is very choppy for the money paths. This has to do with the actual development of the path. If the market drop occurs early when the delta is significantly understated, then significant under-hedging results and there is a large negative hedging error. If

turn to page 14

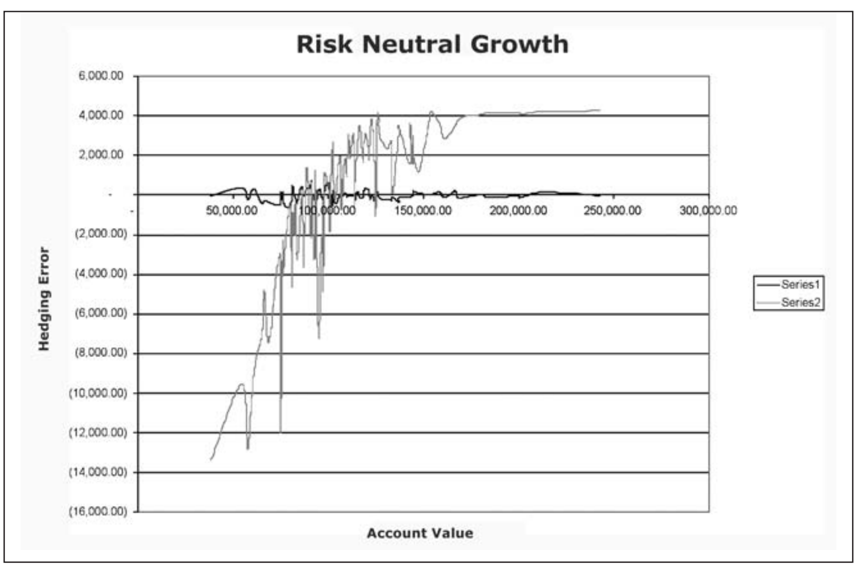
From Page 13

the market does not drop until later, then hedging error is small even if the ending account value is small. A practical implication is that the hedging error is not well behaved in that there is not an easy way to predict or describe it. While one may view use of a real world shadow account as a partial hedge, it lacks proportionality, tail protection, etc. that one might desire from a partial hedge.

The above analysis was performed based on 100 random scenarios that assumed real world drift rates of 10 percent. Under this assumption, the real world hedging produces better results on average, but significantly underperforms in down markets. The point here is that even if one is correct in assuming the expected growth rate is 10 percent, that does not mean that use of a real world shadow account produces an unbiased hedge. This is another example of diversification or averages failing to address capital market risks in the manner they address mortality, morbidity and many other risks.

At this point, the problems with using a real world shadow account in hedging calculations have been demonstrated. Looking at what happens with risk neutral paths provides some additional insight. It is often helpful to look at issues assuming both risk neutral and real world scenarios.

Table 5



A similar hedging simulation was performed on the corresponding 100 risk neutral paths (based on a drift rate of 3 percent) yielding the following statistics:

		Risk Neutral	Real World
Error:	Average:	(0.68)	149.80
	Std. Dev.	248.68	3,839.34
	Maximum	739.91	4,269.28
	Minimum	(590.96)	(13,345.96)

The results appear graphically in Table 5 below. Once again, Series 1 in the above shows the hedging error associated with risk neutral that is essentially zero. Series 2 shows the hedging error associated with real world. We get similar results, except that now the average hedging error is not significantly different from zero for either the risk neutral greeks or real world greeks. The average hedging error for Series 2 would be zero with continuous rebalancing, and infinite scenarios because any hedging strategy will have an expected error of zero across a distribution of risk neutral paths. This is easy to see in the simple case of a hedging strategy consisting of holding cash. The initial cash position would be equal to the risk neutral option value that would then grow at the risk neutral rate. Along any given path, the cash position could dramatically over- or under-perform, but on average would give the correct result. In this example, the bias would still remain with continuous rebalancing, and infinite scenarios causing a large standard deviation of the hedging error per sample path.

The model assumes customer behavior based on a put delta that used the real world equity growth rate of 10 percent for both risk neutral greeks and real world greeks. Unlike the real world shadow account, this customer behavior assumption does not introduce a bias into the hedge result. The calculation is only a function of a judgement of customer behavior, and does not violate any risk neutral principles (subject to earlier disclaimers about optimal exercise). This is an important distinction. A customer behavior function based on risk neutral projected account values during the first five years does not violate any risk neutral principles, regardless of the input parameters with regards to the customers view of the future relative to the end of the fifth contract year. In other words, if one assumes that customer behavior will be based on a put delta of 10 percent and customers really behave that way, the hedge will be accurate. If one assumes that customer behavior will be based on a put delta based on 3 percent and customers really behave that way, the hedge will be accurate. If one assumes that customer expectation will be based on a put delta based on 7 percent and customers really behave that way, the hedge will be accurate. If one assumes that customer behavior will be based on ignoring the time value of money and customers really behave that way, the hedge will be accurate. On the other hand, if one assumes customers will behave one way, and the assumption is wrong, then the hedge will be inaccurate.

Why?

Customer behavior describes reactions to specific conditions at a future point in time. The more accurate the assumptions surrounding customer behavior, the more accurate the hedge. The real world shadow account, however, modifies the specific conditions at a future point in time that are used to predict customer behavior. This modification results in an inaccurate hedge, even if the customer behavior function is correctly predicted.

Conceptually, hedging exchanges an uncertain account value return with a return based on the risk neutral interest rate. If we hedge by shorting an asset, we receive cash when we short that we can earn interest on at the risk neutral rate. If we use futures, the price decay imputes the same interest earning. Hedging cash flows are then invested at the risk neutral rate, not the real world rate associated with equities. Thus our hedging simulation is inconsistent if we assume a real world shadow account at the same time we have exchanged an uncertain equity growth for the risk neutral rate.

One may argue that risk neutral weights adverse paths too heavily. This has been the subject for some debate, but when hedging a path associated with a down market, whether that down market is caused by a negative deviation associated with the volatility portion of Brownian motion or a lower drift rate, is both undeterminable and irrelevant. The fact that a path is a low probability path does not change the fact that the hedge applies to the path you experience and to be fully hedged, one cannot use a real world shadow account. In this example, the cost to be fully hedged is the 4,544 associated with the initial option value using risk neutral customer behavior.

Other Considerations

This article is based on a simple example. The conclusions still apply with stochastic interest rates and/or stochastic volatilities. While this article discusses real world shadow accounts as employed directly in a hedging program, other techniques using real world account projections in some manner to determine customer behavior will lead to an under-hedge as well.

Mark Evans, FSA, MAAA, is vice president and actuary with AEGON USA, Inc. in Louisville, Ken. He can be reached at mevans@aegonusa.com.

Past Highlights of *Risk and Rewards*

10 Years Ago –

The December 1995 issue had two articles on genetic algorithms and another using chaos theory to explain movements in Treasury bond yields. That issue turned out to be pretty much the peak of actuarial interest in artificial intelligence and chaos theory. According to an electronic search of the actuarial library, there was an article in *ARCH* in 2001 on neural networks and genetic algorithms. The only other references to chaos theory appeared in 1996.

15 Years Ago –

In November 1990, former SOA President Jack Bragg explained his theory of economic series. The theory identified four categories of economic periods, each with different implications for inflation, interest rates and the stock market. Coincidentally, the December 1995 issue of *Risks and Rewards* reported that Bragg's theory had been adopted by an actuarial committee on mortgage defaults.