Article from:

Risks & Rewards

March 2013 – Issue 61

PRICING AND HEDGING FINANCIAL AND INSURANCE PRODUCTS PART 2: BLACK-SCHOLES' MODEL AND BEYOND

By Mathieu Boudreault

This paper is the second excerpt of the article, "Pricing and Hedging Financial and Insurance Products," which will be available from the Society of Actuaries' website upon completion. Comments are welcome.

In 1973-1974, Fischer Black, Myron Scholes and Robert Merton provided the first tools to rationally value a financial derivative.[1] Those scientific contributions also helped launch the first U.S. options exchange in Chicago in 1973, known as the Chicago Board Options Exchange (CBOE). Since then, the market for derivatives has exploded to astronomic proportions. In 2008, 1.2 billion contracts were traded on the CBOE, and nearly $200 trillion of derivatives were traded just in the United States alone.

Over the years that followed their publication, the Black-Scholes' model has quickly become the industry and academic standard to price and hedge financial derivatives. Even though many academics and professionals (and Fischer Black himself!) acknowledged numerous holes in the approach and provided solutions to these, the Black-Scholes' model still was widely used in 2012. It is an extremely useful and simple model that often acts as a starting point to understand the dynamics of simple and complex derivatives. Built upon the first excerpt, this paper discusses in length the Black-Scholes' model, its weaknesses and its alternatives (such as the Heston model (see Boudreault 2012)) using concepts introduced in the first paper.

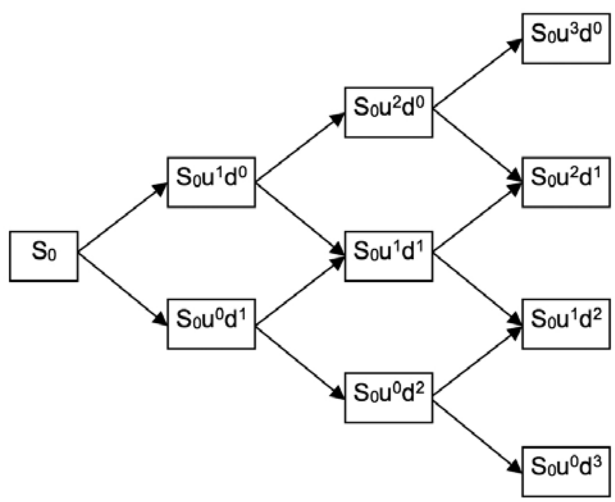
FROM THE SINGLE-STEP BINOMIAL TREE TO BLACK-SCHOLES

Construction of Black-Scholes' Model

Let us recall the single-step binomial tree presented in the first excerpt. The main purpose of the single-step binomial tree was to represent the outcomes of a very simple market in order to replicate the cash flows of a derivative. In this frictionless[2] market where a stock and a Treasury bond are traded, we have assumed that the stock could only take two possible values at the end of the period. The only risk in this market is related to the uncertainty regarding whether or not the stock will increase at the end of the period.

Of course, having only two possible values at the end of a period is unrealistic. Instead, the time horizon is split up into smaller time steps. To do so, we assume that at the end of each period, the stock can increase by a factor of u or decrease by a factor of d. A cell in the binomial tree is known as a *node*. For a given time period, the stock price at each node corresponds to all its possible realizations at that time. Figure 1 shows the possible outcomes of the price of the stock at each time period after three periods.

Figure 1: Evolution of the stock price in a 3-step binomial tree



At time three, there are four nodes and hence four possible stock prices. The price at the uppermost (lowermost) node corresponds to the case where the stock has increased (decreased) on three occasions out of three periods (trials). If a stock price increases (decreases) it is equivalent to a

CONTINUED ON **PAGE 28**

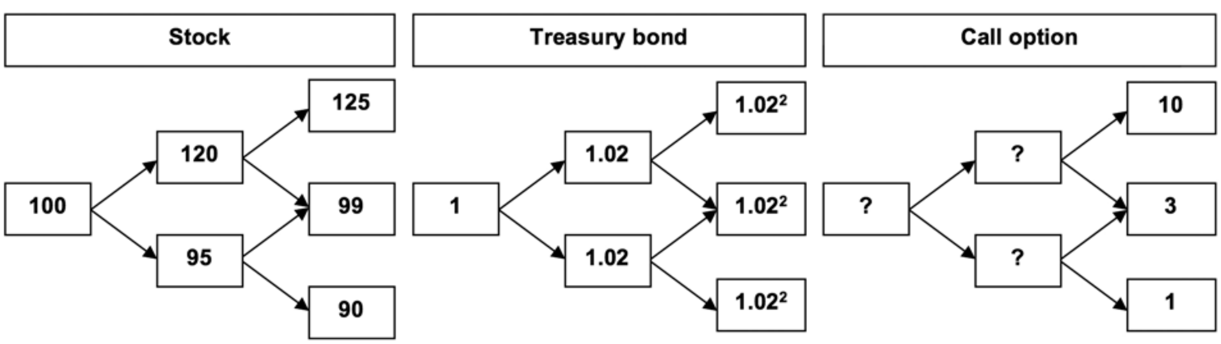
"success" ("failure") in the probability sense, and the price at this node corresponds to three successes (failures) out of three trials. To obtain the given price at node 1 ($S_0u^1d^2$), the stock price has increased only once in three periods, whereas there were two decreases. Hence, there was one success out of three trials. Consequently, the probabilities attributed to stock prices at a given time period are linked to a binomial distribution, which explains the name of the model.

Up to now, we have not attributed any unit of time to a period. Hence, this tree could represent the evolution of the stock over three months, for example. One could also assume that a time period can be a week, a day, an hour, a second, etc. When the time period is infinitely small, the values of *u* and *d* are appropriate (a Cox-Ross-Rubinstein tree for example) and if the tree is valid over a finite time horizon, then the distribution of the stock price is lognormally distributed. In other words, the continuously compounded asset return is normally distributed. The set of all possible stock prices on any continuous time horizon is represented by a stochastic process known as *geometric Brownian motion* (GBM). The purpose of Black-Scholes' model is to find the no-arbitrage price of derivatives under the assumption that the stock is traded at every instant. As in the single-step binomial tree model, the market is assumed to be frictionless where only a Treasury bond and a stock are traded. Unfortunately, the fact that the time step is infinitely small (or that the stock is continuously traded) complicates the mathematics and hides the intuition behind the most important results.

Replicating (Hedge) Portfolio

The replicating (hedging) procedure in a general binomial tree works very similarly as in a single-step binomial tree, with the important exception that the replicating portfolio needs to be dynamically updated every time the stock price changes. It all boils down to solving a set of two equations with two unknowns at each node and each time period, valuing the portfolio from the maturity of the derivative to its inception. The next example and Figure 2 illustrate how this works in a two-step binomial tree.

Figure 2: Evolution of the stock price, Treasury bond and a derivative in a two-step binomial tree.



"IN ORDER TO FIND THE NO-ARBITRAGE PRICE OF THIS OPTION, WE HAVE TO FIND THE REPLICATING STRATEGY AT EACH TIME STEP AND NODE."

Example: Suppose that the evolution of the price of the stock is given by the two-step binomial tree of Figure 2. Assume as well that interest rates are flat at 2 percent and that the derivative pays off an amount given in the rightmost tree. Using replicating portfolios, what is the no-arbitrage price of this derivative, along with the strategies that should be followed at each time step and node?

Solution: In order to find the no-arbitrage price of this option, we have to find the replicating strategy at each time step and node. It is important to note that in a two-step binomial tree there are three one-step binomial trees to consider.

- When (or given that) the stock price is \$120 after one year, there are two possible values at the end of the second year, that is, \$125 or \$99. In that particular tree, we want to replicate payoffs of \$10 and \$3 respectively in the upper and lower node. Since the Treasury bond is worth 1.0404 after two years, and solving for two equations and two unknowns, we find that we need 0.2692 units of stock and a loan of 22.74 units of the Treasury bond. At time 1, the option value is the cost of buying the latter portfolio, i.e., 0.2692 units of a stock worth \$120 and a loan of 22.74 units times 1.02. Hence, the option is worth \$9.12 when the stock is \$120 at time 1.
- When (or given that) the stock price is \$95 after one year, there are two possible values at the end of the second year, that is, \$99 or \$90. In that particular tree, we want to replicate payoffs of \$3 and \$1 respectively in the upper and lower node. Using a similar reasoning, we find that we need 0.2222 units of a stock and a loan of 18.26 units of a Treasury bond. The option is worth \$2.48 at time 1 when the stock is \$95.
- Finally, at time 0, we want to replicate an option that is worth \$9.12 (\$2.48) when the stock is worth \$120 (\$95) at time 1. Solving for the two equations and two unknowns, we find that 0.2654 units of a stock and a loan of \$22.28 are necessary at time 0 to replicate the payoffs of the option at time 1. The cost of that portfolio, i.e., the initial price of this option, is \$4.26.

We see that the replicating strategy is dynamic: at time 0 we start by buying 0.2654 units of a stock. At time 1, depending on the value of the stock, we either buy additional stock (from 0.2654 to 0.2692) when the stock goes up to \$125, or sell some units of the stock (from 0.2654 to 0.2222) when the stock goes down to \$95. Thus, applying the appropriate strategy in this context will ensure (within the scope of the example) that we can replicate the payoffs of the derivative, no matter what is ultimately observed at the end.

Applying the same logic of the previous example when the stock is traded continuously means that the risk manager has to continuously trade in the stock and the bond to make sure the replicating portfolio has the same value as the option. Once again, under no-arbitrage arguments, the initial cost of that replicating strategy will correspond to the price of the option. This price is given by the well-known Black-

Figure 3: Black-Scholes' formula

$$
\begin{aligned}
C_t &= S_t\Phi\left(d_1\right) - Ke^{-r(T-t)}\Phi\left(d_2\right) \\
d_1 &= \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\left(T-t\right)}{\sigma\sqrt{T-t}} \\
d_2 &= d_1 - \sigma\sqrt{T-t}
\end{aligned}
$$

CONTINUED ON PAGE 30

" WHEN PRICING A DERIVATIVE, WE DO NOT ASSUME THAT INVESTORS ARE RISK-NEUTRAL . "

Scholes' formula in the case of a plain vanilla call option. Figure 3 shows the famous Black-Scholes' formula where t represents today, S_t is the current stock price, r is the risk-free rate, T is the maturity of the option, K is the strike price, σ is the volatility of log-returns and Φ is the cumulative distribution function of a standard normal random variable.

We can rewrite the Black-Scholes' formula in order to illustrate how many units of the stock and of the Treasury bond are necessary in every instant to exactly replicate the payoff of a call option, if the portfolio is continuously updated. Let B_t be the accumulated value of \$1 invested for t years at the continuously-compounded risk-free rate, which is also the value of one unit of a risk-free Treasury bond. Figure 4 shows how the formula can be rewritten.[3]

The value at any time t of a call option (C_t) is given by the value of its replicating portfolio, which has a portion invested in stocks ($\Delta_t^{(S)} \times S_t$) and a portion invested in the risk-free Treasury bond ($\Delta_t^{(B)} \times B_t$). The latter, being

Figure 4: Black-Scholes' formula rewritten as the value of the replicating portfolio

$$
\begin{aligned}
C_t &= \Delta_t^{(S)} \times S_t + \Delta_t^{(B)} \times B_t \\
B_t &= 1 \times e^{rt} \\
\Delta_t^{(S)} &= \Phi(d_1) \\
\Delta_t^{(B)} &= -Ke^{-rT}\Phi(d_2)
\end{aligned}
$$

always negative, is in fact a loan at the risk-free rate. The number of units of stocks and bonds has to be dynamically updated because the stock price changes continuously (d_1 and d_2 are both functions of the current stock price).

Hence, as in the single-step binomial tree, if the risk manager can indeed trade at every instant the right number of stocks and Treasury bonds, the value of the replicating strategy will exactly match the payoff of the derivative at its maturity, no matter what path is ultimately observed and no matter how likely each path really is (real-world probability). As in the single-step binomial tree, the price of a derivative only reflects the cost of the positions necessary in the stock and the bond. This is exactly why the mean return on the stock (*mu*) is not a relevant input in the Black-Scholes' formula; this formula only provides the cost of the replicating strategy given the latest price of the stock. The mean return *mu* is already a very relevant parameter when pricing the stock alone. However, no matter how likely the stock is to attain some level, the price of a derivative will always reflect the cost of its replication; there is no need to further account for *mu*.

Risk-Neutral Pricing

In the single-step binomial tree, we were able to express the cost of the replicating portfolio as a discounted expectation of future cash flows. This expectation was taken with a different probability measure q and cash flows were discounted at the risk-free rate. This probability measure is known as a risk-neutral measure because only risk-neutral investors would expect a return equivalent to the risk-free rate on risky assets.

When we solve for the exact dynamics of the cost of the replicating portfolio in Black-Scholes' model, it turns out that we arrive at a similar expression, that is, a discounted expectation of future cash flows, where the discount rate and the mean return on the stock are both the risk-free rate. This should have been expected because the Black-Scholes' model is a limiting case of the binomial tree and in excerpt # 1, we found the cost of the replicating portfolio as a special type of expectation.

Once again, the risk-neutral probability measure has nothing to do with the true probability associated with the stock price. When pricing a derivative, we **do not** assume that investors are risk-neutral. In fact, a short position in the derivative and an appropriate amount of stock ($\Delta_t^{(S)} \times S_t$)

yield a risk-free position. If the position is risk-free, then its cash flows should be discounted at the risk-free rate. Thus, risk-neutral valuation is only the consequence of the fact that we can trade in stocks and bonds to replicate the payoffs of a derivative. Under no-arbitrage pricing of derivatives, the level of risk premium included in the stock is useless.

Conclusion

The Black-Scholes' model is the continuous-time equivalent of applying the one-step binomial tree at every instant. Under the absence of arbitrage, one can equivalently price a derivative using a replicating portfolio or by risk-neutral arguments. Although the mathematical tools are more complex (solving a partial differential equation or using the Girsanov theorem), the results are exactly the same, i.e., it is a type of expectation of discounted cash flows.

It should also be emphasized that the replicating strategy only tells us how to perfectly hedge the derivative given that the model's assumptions are observed. In the more realistic case where the model's assumptions do not hold, the hedge will not be perfect and may result in a random profit or loss. The next sections discuss the weaknesses of the Black-Scholes' model and how practitioners and academics have dealt with these issues.

EVIDENCE AGAINST BLACK-SCHOLES' MODEL

The Black-Scholes' model is extremely useful in various settings. However, one has to be careful because most of its underlying assumptions do not hold in practice. This has been largely documented, as for example in papers by Fischer Black titled, "The Holes In Black-Scholes" and "How To Use The Holes In Black-Scholes."

Deviations between real market dynamics and the one given by a model can have very small or very large consequences on risk management. Indeed, serious mispricing of derivatives can lead to arbitrage opportunities and losses to its

issuer. For risk management purposes, *hedging errors*, which is the difference between the value of the replicating portfolio and the derivative's payoff at maturity (or any time period), are difficult to predict and can result in profits or losses ultimately. In both cases, it is extremely important to validate each assumption and the potential risk associated with a deviation to the true market dynamics.

Geometric Brownian Motion

When applied to modeling stock (or asset) prices, the GBM requires that the continuously-compounded returns are independent and identically distributed as normal random variables. This is unfortunately not the case in practice for various reasons.

Behavior Of Observed Returns

Financial econometricians constantly study the behavior of asset returns (individual stocks and portfolios) at various time horizons (daily, monthly, etc.). One basic exercise they often do is compute various descriptive statistics of these returns as in Table 1. They have generally found that asset returns show negative asymmetry, which means that "the left tail is longer and the mass of the distribution is

CONTINUED ON **PAGE 32**

Table 1: Descriptive statistics of the log-returns observed on the S&P500 from Jan. 1, 1957 to Nov. 9, 2012. Source: Federal Reserve of St. Louis' Economic Data (FRED)

Frequency	# Data	Mean	Std. dev.	Variance*100	Skewness	(Excess) Kurtosis
Daily	14063	0.02%	1.00%	0.0101	-1.0271	27.87
Weekly	2914	0.12%	1.80%	0.0325	-0.7027	6.28
Monthly	670	0.51%	3.60%	0.1293	-0.7320	4.15
Quarterly	223	1.56%	6.29%	0.3956	-0.9974	3.85
Yearly	55	6.24%	12.94%	1.6754	-1.0943	3.91

concentrated on the right" (Wikipedia). This is contrary to a normal distribution as the latter is perfectly symmetric around the mean. Moreover, returns show fat tails, which is shown by a significant excess kurtosis. In a normal distribution, the kurtosis is three, so that the excess kurtosis is zero. Although, the asymmetry is close to -1 at every observation frequency, the kurtosis is clearly much more significant at the daily level. Thus, asset returns clearly are not normally distributed and deviations from normality are greater at higher frequencies. We still have to investigate whether or not historical asset returns are independent and identically distributed as in a random walk.

A lot of academic and professional research has been devoted in the last several decades toward the predictability of asset returns and the efficient markets hypothesis (EMH). However, the EMH taken alone is very difficult to assess because an equilibrium pricing model has to be assumed, so that rejection of the EMH may as well be related to the failure of the model to fit prices. Being central to the EMH, the random walk hypothesis is often investigated to determine the ability to "predict" asset returns.

Campbell, Lo and MacKinlay (1996) (Chapter 2) document three types of random walks. The simplest is one where increments are independent and identically distributed (i.i.d.). The second form of random walk is one where increments are independent, but not necessarily identically distributed. The third type of random walk involves uncorrelated increments. There are various statistical tests to check for the validity of each type of random walk. However, it is not plausible that returns are identically distributed (first type of random walk) because throughout the financial history, there have been changes in the economic and regulatory environments, in addition to technological advances. Moreover, it has been shown numerous times in the financial econometrics literature that squared asset returns are autocorrelated. Thus asset returns are not independent and this would lead to a rejection of the second type of random walk. Therefore, most empirical research testing the predictability of asset returns is focused toward the third type of random walk.

Empirical evidence (see for example Section 2.8 of Campbell, Lo and MacKinlay (1996)) shows that daily, weekly and monthly returns have positive and statistically significant autocorrelation (at the first lag), thus rejecting the random walk hypothesis. Another way to assess whether returns follow a random walk is to compare the variance of the process at different time horizons: this is known as the variance ratio test. In a random walk (even the third type), the variance of the process grows linearly over time as more increments are added to the total. Hence, the variance of annual returns should be about 12 times the variance

> WHETHER LONG-HORIZON RETURNS (RETURNS OVER PERIODS LONGER THAT AN YEAR) ALSO EXHIBIT MEAN REVERSION IS STILL AN OPEN DEBATE AMONG PRACTITIONERS AND ACADEMICS.

of monthly returns. It turns out that the variance ratio test rejects the random walk hypothesis (third type) at numerous horizons for equal-weighted portfolios. It was found that portfolios formed with the smallest firms have the most significant deviations from the random walk hypothesis. Finally, for individual securities the random walk hypothesis cannot be rejected and this should have been expected because individual stocks show company-specific noise that is mostly attenuated when aggregated into a portfolio.

Computing variance ratios in Table 1 also suggests that the random walk hypothesis is rejected with more recent data. We can see that daily returns are more variable (proportionally) than weekly, monthly or annual returns. This may be a sign of short-term mean reversion. In other words, an extreme daily return is often corrected in the next days, so that over a week, the weekly return has a more reasonable behavior. Short-term mean reversion is also backed by the decreasing kurtosis in Table 1.

Whether long-horizon returns (returns over periods longer than a year) also exhibit mean reversion is still an open debate among practitioners and academics. Proponents of long-term mean reversion often use widely accepted economic theories and models and pretend that mean reversion is supported in these theories. Others use statistical tests to measure their presence. In both cases, opponents of long-term mean reversion argue that the financial history is too short to obtain statistically significant results and that bubbles (NASDAQ or Japan for example) make it difficult to observe such reversion.

Overall, the GBM assumption clearly does **not** hold. (Log-) Returns are not normally distributed: they generally show negative skewness and very fat tails. Moreover, returns do not follow a random walk: they have some form of autocorrelation,[4] as shown by the Box-Pierce and variance ratio tests.

Realistic Alternatives To The GBM

When we dig deeper into both the empirical finance and econometrics literature, we can find that asset returns show the following characteristics:

- Non-constant and possibly stochastic volatility;
- Jumps in asset prices (downward jumps more often than not);
- Jumps in the volatility as well.

While it is the role of behavioral finance and economics to explain why asset returns show these features, actuaries and other risk managers have to account for these characteristics of stock prices in pricing and hedging claims that are linked to the financial markets.

The insurance industry has shown particular interest in extending the Black-Scholes model to reserve their equity-linked insurance policies. For example, the regime-switching log-normal model (RSLN) (Hardy (2001)) has been proposed for equity prices by the Task Force on Segregated Funds of the Canadian Institute of Actuaries whereas a discrete-time stochastic volatility model has been proposed by the American Academy of Actuaries (AAA) for variable annuities. It is important to note that the RSLN model features stochastic volatility (one possible value per regime) and by construction, the volatility jumps at each regime switch. The AAA's model does feature a much greater spectrum of values for the volatility, but it lacks jumps in both the volatility and the price.

In the financial mathematics literature, more sophisticated continuous-time alternatives to the GBM also exist. The stochastic volatility model of Heston (1993) (discussed below) remains very popular in the finance industry. For a thorough review of financial econometrics models applied to the context of reserving and hedging variable annuities or segregated funds, the reader is invited to look at Augustyniak & Boudreault (2012).

Continuous Trading And Rebalancing

One of the underlying assumptions of the Black-Scholes'

CONTINUED ON PAGE 34

model is that the stock is traded continuously, so that the replicating portfolio should also be updated continuously. Physically this is virtually impossible that the stock be traded in continuous-time! Humans and computers have to take some time to analyze their positions and send orders to execute them. However, these delays between trades are melting down now with supercomputers that execute computations and trades in milliseconds, and massive investments by banks in communications infrastructures. Transaction costs also prevent an investor to trade continuously in a stock to update its replicating portfolio. Consequently, the fact that the replicating portfolio cannot be updated continuously entails a potential for hedging errors. In fact, it is exposed to large moves from the stock between two portfolio updates. This hedging error can be reduced by systematically rebalancing more frequently, say daily or hourly, but this can be very costly.

To reduce transaction costs and potential hedging errors, there are two common solutions. First, practitioners usually update their portfolio only when it deviates significantly from delta-neutrality (or some other criteria). They can also combine this approach with other hedging schemes that involve making sure the portfolio is insensitive to other factors (hedging based upon Greeks are discussed later in the text).

Frictionless Market

In many finance textbooks, either in the Black-Scholes' model or others, it is assumed that markets are frictionless, i.e., there are no transaction costs, assets are perfectly liquid and divisible, lending and borrowing interest rates are the same, there are no taxes and no restrictions on long and short positions, etc. We will briefly discuss the impact of these elements on pricing and hedging derivatives.

Transaction costs such as bid-ask spreads, commissions and other fees are indubitably the most important market friction in the derivatives market. First they help prevent arbitrage that may come up with very small price differentials between identical products and portfolios. Second, they can significantly increase the true cost of the replicating portfolio, forcing companies to hedge more intelligently or less frequently, which in the latter case is riskier. Different lending and borrowing rates, in addition to taxes, can be seen as an asymmetrical transaction cost structure, further reducing arbitrage opportunities.[5] Finally, the fact that we can only buy an integer number of stocks (or a block of 100 stocks for example) is not a huge issue for investment banks since they trade millions of positions.

Overall, some of these simplifying assumptions on market frictions do have minor impacts, but transaction costs (including tax and interest differentials) can have significant effects for pricing and risk management purposes. The interested reader is referred to Black (1989) for a discussion of these frictions.

Conclusion

We have only listed a subset of the many problems encountered with the Black-Scholes' model, i.e., the fact that observed asset returns do not follow a GBM, that continuous rebalancing of the replicating portfolio is virtually impossible and that market frictions such as transaction costs and restrictions on short selling can reduce the quality of the hedge portfolio and lead to important losses. Two other holes in Black-Scholes also deserve more attention, especially for insurance companies that sell long maturity put options through their equity-linked policies.

Suppose for example that your company has issued an implicit five-year put option in equity-linked policies and you need to replicate its future cash flows with stocks and bonds. Stock markets crash worldwide and the option goes deeper into the money. By following the replicating strategy, you would need to sell more units of the stock, and in times of crises, finding a buyer at the other end can be difficult. To find a buyer, one would need to further reduce the price offered, which increases the hedging loss. That creates a type of liquidity risk. According to Hull (2008), this had further aggravated the October 1987 crash, and it is reasonable to believe a similar issue may have contributed to the current recession.

It is postulated in the Black-Scholes' model that the term structure of interest rates is flat and deterministic over time. This is, of course, unrealistic since the term structure is not constant and it moves randomly over time. In fact, changes in interest rates can be largely explained by random variations in the level, slope and convexity (curvature) of the term structure. Because the volatility of interest rates is much smaller than the volatility in stocks, pricing very short-term options can reasonably be done using deterministic interest rates. However, for long-term contracts, especially equity-linked insurance that matures after five to 20 years, interest rate risk can be important when pricing these contracts.

IMPACTS ON INDUSTRY PRACTICES

The fact that equity prices do not follow a GBM and the inability to continuously rebalance the hedge portfolio are perhaps the two most important holes in Black-Scholes. They have had important and profound impacts on how the industry uses the Black-Scholes' model. This section covers the implied volatility metric and hedging with Greeks.

Implied Volatility

We have seen in Table 1 that equity returns clearly do not follow a normal distribution through the computation of very simple descriptive statistics. We found that asset returns are negatively skewed and they show fat tails. The lack of normality of asset returns also show in the historical price structure of plain vanilla call and put options. We can see that low-strike put options are underpriced with Black-Scholes, meaning the left tail is underestimated with a normal distribution. Similarly, high-strike call options are overpriced with Black-Scholes, meaning the right tail is overestimated with a normal distribution. So instead of moving away from Black-Scholes, practitioners use a different volatility parameter to price options with different strikes. They increase (decrease) the volatility used in Black-Scholes' model for low-strike (high-strike) options: this is the so-called volatility skew (smile) for equity prices. Hence, the implied volatility is the *sigma* parameter in the Black-Scholes' formula that replicates a given market price. The relationship between the implied volatility and the time to maturity is the volatility term structure. Finally, the relationship between the implied volatility, the strike and the time to maturity is the volatility surface.

Practitioners have been using the implied volatility to deal with the asymmetry and fat tails of the return distribution for pricing and hedging purposes. The concept is widely known and used: the CBOE monitors an index known as the VIX, which is, loosely speaking, a compound measure of the implied volatility computed from a portfolio of traded options. Derivatives known as variance swaps are even issued on the VIX so that investors can protect against changes in volatility. However, as discussed by LeRoux (2006), the implied volatility is "the wrong number to plug into the wrong formula to get the right price," describing how it is not necessarily the most adequate solution to deal with Black-Scholes' imperfections.

Hedging With Greeks

To cope with the inability to trade continuously and the lack of normality of equity returns, practitioners have been using Greeks to improve the robustness of their hedge portfolio. For those familiar with the concepts of duration and convexity matching (immunization), hedging with Greeks is very similar. It comes from the Taylor expansion of the option price at the next time period (which is random) around the current stock price (which is known). Truncating the Taylor series to the first or second term, allows us to immunize first-order and second-order changes of the option price with respect to the stock. With fixed-income securities, matching the first-order derivative is known as duration matching but with options, this is known as delta-hedging. When both first- and second-order derivatives are matched, duration-convexity matching is equivalent to delta-gamma hedging. As with duration-convexity match-

> "THERE IS NO CURE TO MODEL RISK: ONE CAN IMPLEMENT MORE THAN ONE MODEL TO COME UP WITH MORE ROBUST RISK MANAGEMENT STRATEGIES, AND OF COURSE, STRESS- AND BACK-TESTING ARE VERY IMPORTANT."

ing, delta-gamma hedging is exposed to the same caveats, i.e., it is only valid for a small time period.[6] Otherwise, the terms that are ignored in the Taylor expansion can become more significant. Thus, the impacts of being unable to continuously update the replicating portfolio (or the lack of normality of asset returns) can be diminished by also hedging gamma.

The issuer of an option is also exposed to other risks than variations of the stock price. Other variables such as changes in interest rates and volatility can have a significant effect on the option price. It is possible to diminish the impacts of changes in these variables by using an approach similar to delta-gamma hedging. Indeed, using a multivariate Taylor series expansion of the option price with respect to many variables, one can develop a hedging strategy that consists in matching the sensitivity of liabilities and assets to many risks. The sensitivity of the options' price with respect to each of many variables is given a name after a Greek letter (most of the time). For example, the first- and second-order derivatives of the option's price with respect to changes in stock price are known as delta and gamma. The first order derivative of the option's price with respect to changes in the risk-free rate is known as rho.

In order to hedge an option with Greeks, one has to compute these sensitivities. First-order, second-order and partial derivatives have to be based upon a model and the industry standard is Black-Scholes. Although a Taylor series expansion shows that using more Greeks improve the quality of hedging, this is only guaranteed to be true when the market model is the same as the one used in computing sensitivities. However, when the market dynamics (say Heston) are different than the one used to build the hedging strategy (Black-Scholes), it is not perfectly clear as to how using more Greeks is going to improve the quality of the hedge.

HESTON MODEL AND BEYOND

Financial econometric models are used in the insurance industry mainly to help represent the risk on securities such as stocks and bonds on both the asset and liability side of the company. For example, it can be used to generate multiple scenarios to price equity-linked insurance, to assess the quality of a hedging strategy, etc. We now discuss how the wider use of complex financial models have an impact on the most basic risk management operations of an insurance company. To lighten the discussion, we will mainly discuss the Heston (1993) model, but most of the remarks also hold for models with stochastic volatility, jumps in prices and jumps in the volatility.

In Heston's model, the variance of the continuously-compounded return is given by a Cox-Ingersoll-Ross process, i.e., the variance is mean-reverting and guaranteed to be positive. There are two sources of risk: the volatility, and given the volatility, the stock price also has a stochastic element. Since only the stock is traded and one cannot buy one unit of volatility (yet), the underlying financial market is incomplete. Incomplete markets have been discussed in a trinomial tree in the first excerpt. Thus, Heston's model has all the caveats of an incomplete market model, that is the price of most derivatives is not unique and the hedge cannot be perfect.

To partly circumvent the non-uniqueness problem, Steven Heston has assumed some particular form of market price of risk, which only depends on one parameter. Contrarily to the Black-Scholes model where the passage from the real-world to the pricing measure is unique and straightforward, the Heston model implies estimating the parameter related to the market price of risk to make the passage from the physical to the risk-neutral probability measures. This parameter should really be taken seriously, especially if one is interested in both risk management and pricing (under no-arbitrage) applications.

No matter how a financial model is going to be used ultimately, one has to determine its parameters based on some set of data. Although estimating a GBM is straightforward, it is generally not the case for the Heston model. Given a specific structure for the market price of risk, Heston obtains (quasi-) closed-form solutions for the price of plain vanilla options (calls and puts), which is very interesting since it can be used in the estimation/calibration process. Indeed, one can minimize the squared deviations between theoretical and observed options prices. The method is straightforward, but will only provide for parameters under the risk-neutral probability measure. For risk management purposes, those parameters will not be adequate, unless one appropriately estimates the parameter related to the market price of risk, or at least performs a sensitivity analysis of this parameter. One can also use the time series of volatility indices such as the VIX to infer the dynamics of the variance part of Heston model. The approach has proven to have some value, and the interested reader is referred to Aït-Sahalia & Kimmel (2007). Finally, estimating the Heston model (or any stochastic volatility model with or without jumps) often relies on the use of particle filters. This is one of the only methods that is statistically efficient with these models.

CONCLUDING REMARKS

It is undeniable that the Black-Scholes' model has been the cornerstone of modern finance and financial mathematics. However, any model is by definition a simplification of reality. To quote George Box, a famous statistician, "All models are wrong, but some are useful" and Black-Scholes' model is widely used because it is indeed very useful. Actuaries and the insurance industry need to master the tools of mathematical finance, including the more complex models of stochastic volatility and jumps, to be more effective at developing new products and hedge them appropriately in the financial markets.

Using a more sophisticated model does not necessarily guarantee success. No one can pretend to have a fool-proof understanding of a phenomenon. Hence, no one knows how the sophisticated model truly deviates from the true market dynamics. This relates to model risk, i.e., the unexpected consequences related to choosing a model over another, compared to the true dynamics of a phenomenon, which is unknown. There is no cure to model risk: one can implement more than one model to come up with more robust risk management strategies, and of course, stress- and back-testing are very important.

A model can be useful, but it is crucial to understand its limitations. Many authors like Steven Shreve, Pablo Triana, Felix Salmon (in Salmon (2009)), Sam Jones (in Jones (2009)) among others blame users of financial models because they did not understand thoroughly enough the models they were using and their limits. The authors contend that the latter may have instigated (or worsened) the current financial crisis. It also highlights the difficulties a whole company can face when implementing a complex risk management strategy, especially if the modeling department (that can be highly technical) cannot efficiently communicate with the upper management and board (who have very limited technical knowledge).

In the third (and final?) excerpt, we will discuss specific issues tied with equity-linked insurance that have not been discussed yet, such as pricing and hedging lapses, resets, withdrawals and other guarantees.

CONTINUED ON PAGE 38

REFERENCES

Aït-Sahalia, Y. and R. Kimmel (2007), "Maximum Likelihood Estimation of Stochastic Volatility Models," *Journal of Financial Economics* **83**, 413-452.

Augustyniak, M. and M. Boudreault (2012), "An Out-Of-Sample Analysis of Investment Guarantees for Equity-Linked Products: Lessons from the Financial Crisis of the late-2000s," *North American Actuarial Journal* **16**, 183-206.

Black, F. (1988), "The Holes in Black-Scholes," *Risk*.

Black, F. (1989), "How To Use the Holes in Black-Scholes," *Journal of Applied Corporate Finance* **4**, 67-73.

Boudreault, M. (2012), "Pricing and Hedging Financial and Insurance Products, Part 1: Complete and Incomplete Markets," *Risks & Rewards*, August 2012, Society of Actuaries.

Campbell, J.Y, A. W. Lo and A.C. MacKinlay (1996), *The Econometrics of Financial Markets*, Princeton University Press.

Hardy, M. R. (2001), "A Regime-Switching Model of Long-Term Stock Returns," *North American Actuarial Journal* **5** 41-53.

Heston, S. (1993), "A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bonds and Currency Options," *Review of Financial Studies* 6, 327–343.

Hull, J.C. (2008), *Options, Futures and Other Derivatives,* 7th edition, Prentice-Hall.

Jones, S. (2009) "The Formula That Felled Wall Street," *Financial Times*, April 24th, 2009

LeRoux, M. (2006), "A Long-Term Model of the Dynamics of the S&P500 Implied Volatility Surface," Presentation to the Canadian Institute of Actuaries' Stochastic Modeling Symposium, Toronto, April 3, 2006

Salmon, F. (2009), "Recipe for Disaster: The Formula That Killed Wall Street", *Wired* Magazine 17.03

Stone, S. (2007), "Understanding the Black-Scholes equation", *Risk & Rewards*, August 2007, Society of Actuaries.

ACKNOWLEDGMENTS

The author would like to thank Nino Boezio, Dick Joss and Pierre-Laurence Marchand for their valuable comments.

END NOTES

[1] The approach is often attributed to Black & Scholes' seminal paper in 1973, but Merton applied similar principles to value the equity of the firm and its credit risk in 1974. Although some authors use the Black-Merton-Scholes terminology, we will use the more common Black-Scholes term to refer to their 1973 paper.

[2] A frictionless market is one in which there are no transaction costs, no differences in lending and borrowing interest rates, no taxes, no constraints on buying and (short-) selling the assets, etc.

[3] A call option pays the excess of the stock price over the strike price, only when the former is greater than the latter. No matter what is the underlying stock price model, that preceding payoff can always be decomposed as a long position in a derivative that delivers one unit of the stock if its price is greater than the strike, and a short position in a derivative that pays an amount equivalent to the strike, only when the stock price is greater than the strike. Stone (2007) provides further insights into the Black-Scholes formula.

[4] One may be tempted to exploit this finding to make money! However, even if autocorrelations are statistically different from zero, the implied R^2s suggest that a very small percentage of the variance is explained by past returns.

[5] Tax differentials between financial products (such as corporate and municipal bonds) are generally accounted for in prices because authorities work to prevent tax arbitrage.

[6] Applying a delta-hedging strategy continuously is equivalent to applying the theoretical replicating portfolio of Black-Sholes.

Mathieu Boudreault, Ph.D., F.S.A. A.C.I.A. is Associate Professor of actuarial science in the Department of mathematics, at the Université du Québec à Montréal. He can be reached at boudreault.mathieu@uqam.ca.