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Hedging European Call Option on a Non-dividend Paying Stock in a Random Interest Rate Environment using Futures

by Daniel Hui

Steve Stone explained in the August 2007 issue of Risks and Rewards that a short European call option on a non-dividend paying stock can be hedged by a long stock position and a borrowing in money market account if the interest rate is fixed. However, real markets do not have a single fixed interest rate. On the other hand, some companies implemented hedging using futures instead of stock because futures are more liquid and less costly. This is true on both counts. Does Black-Scholes formula provide any insights on how to hedge an option using futures in the stochastic interest rate environment? Can we just calculate the option delta and purchase futures accordingly?

Metalgesellschaft (MG), a large German trading company, hedged one long forward contract on heating oil with one short futures contract. They sold oil contracts with delivery commitments extending over 10 years. For the delivery commitments of each barrel of oil, they bought one barrel of a short duration futures contracts. This may seem to be a perfect hedge of the delivery commitments on a pure accounting

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perspective. Unfortunately, as the market moved against MG in 1993, they had to engineer a 2.2 billion dollar rescue. The company was in intensive care for a long time. What went wrong there? We will have more on this later.

In this article, I would like to lay out a generic framework on how to hedge a European call option on a non-dividend paying stock using futures in a stochastic interest rate environment. In doing this kind of analysis, stochastic calculus is unavoidable. Formulas (1) - (3) and (6) are used in the derivation. If you accept these formulas, you can carry out and understand the following analysis.

Ito-Doeblin Formula

Stochastic calculus is built on probability theory and Lebesgue integral. Interested readers can refer to the book "Stochastic Calculus for Finance II Continuous-time Model" by Steve Shreve. Here we accept the Ito-Doeblin formula without deriving it. W. Doeblin, a French soldier, derived a similar formula to Ito's in the Second World War and he died shortly after. The document he submitted was not discovered until recently. Some authors now refer to the formula as Ito-Doeblin formula.

We will use lower case letter for real variable. We will use upper case letter for stochastic variable that depends on a drift term and a stochastic or random term $\omega(t)$. For brevity of exposition, I will drop $\omega(t)$ from the stochastic variable, but its presence is understood.

The following are the Ito-Doeblin formulas and the Ito product rule. We will use these formulas several times in this analysis. We assume there is a function f(t, X(t)) which depends on the real variable t and the stochastic variable X(t). f_X denotes the partial derivatives of the function f with respect to X(t).

<u>One-dimensional</u>

$$df(t,X(t)) = f_t(t,X(t))dt + f_x(t,X(t))dX(t) + \tfrac{1}{2} f_{xx}(t,X(t))dX(t)dX(t) \tag{1}$$

<u>Two-dimensional in compact notation</u>

$$df(t,X,Y) = f_t dt + f_x dX + f_y dY + \tfrac{1}{2} f_{xx} dX\, dX + f_{xy} dX\, dY + \tfrac{1}{2} f_{yy} dY\, dY \tag{2}$$

<u>Ito product rule</u>

$$d(X(t),Y(t)) = X(t)dY(t) + Y(t)dX(t) + dX(t)dY(t) \tag{3}$$

Stochastic variable differs from real variable in that the quadratic variation for stochastic variable is not zero. If X(t) is a real variable, the last term in equation (1) will disappear.

Martingale is a concept that the current state, based on the available information is our best guess of a stochastic quantity in the future. There is no up or down trend. Martingale is defined as follows.

<u>Definition of martingale</u>:

$M(s) = \tilde{E}[M(t) \mid \Im(s)]$ for $0 < s < t < T$ and M(t)

is an adapted stochastic process and $\Im(s)$ is the information set available at time s. $\tilde{E}$ is the risk-neutral expectation under $\tilde{P}$ probability.

It can be shown that if a stochastic variable, say X(t), is a martingale, the coefficient of the dt term in dX(t) is zero. We will apply this concept again and again in the following analysis.

Daniel Hui, MBA, FSA, CFA works for AIG ERM Quantitative Analysis. He can be contacted at Daniel.Hui@aig.com

A Model of the Economy

Before we proceed, we build the model of the economy and the available assets in the economy. It will be shown later that our primary hedging assets are futures and zero coupon bond. We need a model for both.

Interest Rate

The interest rate is assumed to be random and follow a generic short rate model in the risk-neutral (RN) world. Depending on the selection of the function β(t,R(t)) and γ(t,R(t)), this will become either the Ho-Lee, the Hull-White model or any other short rate model.

$$dR(t) = \beta(t,R(t))dt + \gamma(t,R(t))d\tilde{\omega}_1(t) \tag{4}$$

Stock

The stock is assumed to follow the geometric Brownian motion in the RN world:

$$dS(t) = R(t)S(t)dt + \sigma(t)S(t)d\tilde{\omega}_2(t) \tag{5}$$

and the stock and interest rate processes are assumed to be correlated:

$$d\tilde{\omega}_1\, d\tilde{\omega}_2 = \rho\, dt; d\tilde{\omega}_1 d\tilde{\omega}_1 = d\tilde{\omega}_2 d\tilde{\omega}_2 = dt;\ dt\, dt = d\tilde{\omega}_1 dt = d\tilde{\omega}_2 dt = 0 \tag{6}$$

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Where $d\widetilde{\omega}_1$ *and* $d\widetilde{\omega}_2$ are Brownian motions under $\widetilde{P}$ probability in the risk-neutral world.

Therefore

$dR\,dR = \gamma^2(t, R(t))dt$ and $dS\,dS = \sigma^2 S^2(t)dt$

Discount Process

Discount process is:

$$D(t) = e^{-\int_0^t R(u)\,du}$$

Let $I(t) = \int_0^t R(u)\,du$ and dI(t)=R(t)dt.

We apply the one-dimensional Ito-Doeblin formula to get

$dD(t) = df(I(t)) = f'(I(t))dI(t) + \frac{1}{2}f''(I(t))dI(t)dI(t)$,

which is equivalent to

$$dD(t) = -R(t)\cdot D(t)\cdot dt \qquad (7)$$

Zero Coupon Bond

The price of a zero coupon bond at time t that pays $1 at time T is B(t,T). This is a function of time t and interest rate R(t). Therefore, B(t,T) is the discount present value at time t of $1 pay at time T.

$$f(t, R(t)) = B(t,T) = \frac{1}{D(t)}\widetilde{\mathrm{E}}[D(T)\cdot 1 \mid \Im(t)]$$

where $\Im(t)$ is the information set available at time t.

D(t) B(t,T) is a martingale in the RN world. This means that the trend is flat. To see this, we apply iterative expectation:

$$D(s)B(s,T) = \widetilde{\mathrm{E}}[\widetilde{\mathrm{E}}[D(T)\mid \Im(t)] \mid \Im(s)] = \widetilde{\mathrm{E}}[D(T)\mid \Im(s)]$$

Since we cannot look ahead, what we know at time s is the information we have. What we may find out at later time t is irrelevant at this point. This implies that the drift term or dt term in d(D(t) B(t)) must be zero. Only the stochastic term remains.

$$\begin{aligned} d(D(t)B(t,T)) &= B(t,T)dD(t) + D(t)dB(t,T) + dD(t)dB(t,T) \\ &= -R(t)D(t)B(t,T)dt + D(t)(f_t dt + f_R dR(t) + \tfrac{1}{2}f_{RR}dR(t)dR(t)) \\ &= D(t)(-Rf + f_t + \beta f_R + \tfrac{1}{2}\gamma^2 f_{RR})dt + D(t)\gamma f_R d\widetilde{\omega}_1 \\ &= D(t)\gamma(t,R(t))f_R(T,R(t))d\widetilde{\omega}_1(t) \end{aligned}$$

Because the dt term must be zero, we have:

$$R(t)f(t,R(t)) = f_t + \beta f_R + \tfrac{1}{2}\gamma^2 f_{RR}$$

We substitute this in the derivation for dB(t,R(t)) to get:

$$\begin{aligned} dB(t,R(t)) &= f_t dt + f_R dR + \tfrac{1}{2}f_{RR}dRdR \\ &= (f_t + \beta f_R + \tfrac{1}{2}\gamma^2 f_{RR})dt + \gamma f_R d\widetilde{\omega}_1 \\ &= R(t)f(t,R(t))dt + \gamma(t,R(t))f_R(t,R(t))d\widetilde{\omega}_1(t) \end{aligned} \qquad (8)$$

Futures

The futures price on a stock at t with maturity at T is the expected price of the stock at time T given the available information at time t. This is a function of time t, interest rate and stock price. The futures price is a function of time t, interest rate R(t) and current stock price S(t).

$$g(t, S(t), R(t)) = Fut_S(t,T) = \widetilde{\mathrm{E}}[S(T)\mid \Im(t)]$$

It is easy to see that this is a martingale in the RN world when we apply iterative expectation with 0<u<t<T. We have:

$$Fut_S(u,T) = \widetilde{\mathrm{E}}[Fut_S(t,T)\mid \Im(u)] = \widetilde{\mathrm{E}}[\widetilde{\mathrm{E}}[S(T)\mid \Im(t)]\mid \Im(u)] = \widetilde{\mathrm{E}}[S(T)\mid \Im(u)]$$

The futures price itself is a martingale. The dt term in $dFut_S$ is zero and only stochastic terms remain. We apply the two-dimensional Ito-Doeblin formula in calculating $dFut_S$.

$$\begin{aligned} &dFut_S(t,T) = \\ &g_t dt + g_R dR + g_S dS + \tfrac{1}{2}g_{RR}dRdR + \tfrac{1}{2}g_{SS}dSdS + g_{RS}dRdS \\ &= (........)dt + \gamma g_R d\widetilde{\omega}_1 + g_S \sigma S d\widetilde{\omega}_2 \\ &= \gamma(t,R(t))g_R(t,R(t),S(t))d\widetilde{\omega}_1(t) + g_S(t,R(t),S(t))\sigma(t)S(t)d\widetilde{\omega}_2(t) \end{aligned} \qquad (9)$$

European Call Option

Let c(t,S(t),R(t)) be the price of the European call option on a non-dividend paying stock which pays max(0,S(T)-K) or $(S(T)-K)^+$ at maturity – time T. The call price at time 0 is c(0,S(0),R(0)) and $c(T,S(T),R(T))=(S(T)-K)^+$. Applying the two dimensional Ito-Doeblin formula and omitting the argument (t,S(t),R(t)) in several places, we got:

$$\begin{aligned} &d(D(t)\cdot c(t,S(t),R(t)) \\ &= D(t)(c_t + c_s RS + \beta c_R + \tfrac{1}{2}\sigma^2 S^2 c_{SS} + \rho\sigma\gamma S c_{SR} + \tfrac{1}{2}\gamma^2 c_{RR} - Rc)dt \\ &+ \gamma D c_R d\widetilde{\omega}_1 + D\sigma S c_S d\widetilde{\omega}_2 \\ &= D(t)\gamma c_R d\widetilde{\omega}_1(t) + D(t)\sigma(t)S(t)c_S d\widetilde{\omega}_2(t) \end{aligned}$$

The dt term is zero since the discounted option price is a martingale under risk-neutral measure.

Hedging Portfolio

In this model, we require that the option, the zero coupon bond and the futures mature at the same time T. Let X(t) be the capital of the hedging portfolio at time t and X(0) is the capital at time 0. Assuming that c(0,S(0),R(0)) is known, we can set up X(0)=c(0,S(0),R(0)). If we can match the change in discount portfolio value, d(D(t) X(t)), over time with the change in the discounted option price d(D(t) C(t,S(t),R(t)) by holding the appropriate amount of futures and zero coupon bonds, we will have
X(T)=c(T,S(T),R(T))= $(S(T)-K)^+$ almost surely. This is because we started with the same position X(0)=C(0,S(0),R(0)) and we match the change in value over time. We will end up with the same position at time T.

The hedging portfolio X(t) consists of Δ(t) futures, Γ(t) zero coupon bond B(t,T) which mature at time T and the balance of the portfolio is either lending or borrowing in the money market. The change in value of the hedging portfolio and hence the change in discounted portfolio value are:

$$dX(t) = \Delta(t)dFut_S(t,S(t)) + \Gamma(t)dB(t,T) + R(t)(X(t) - \Gamma(t)B(t,T))dt \tag{11}$$

$$\begin{aligned} &d(D(t)X(t)) \\ &= -R(t)D(t)X(t)dt + D(t)dX(t) \\ &= -R(t)D(t)X(t)dt + D(t)(\Delta(t)(\gamma g_R d\widetilde{\omega}_1 + \sigma S g_S d\widetilde{\omega}_2) + \\ &\quad \Gamma(t)(Rfdt + \gamma f_R d\widetilde{\omega}_1) + R(X - \Gamma B)dt) \\ &= D(t)(\Delta(t)\gamma g_R + \Gamma(t)\gamma f_R)d\widetilde{\omega}_1 + D(t)\Delta(t)\sigma(t)S(t)g_S d\widetilde{\omega}_2 \end{aligned}$$

In order to have d(D(t)X(t)=d(D(t)C(t,S(t)), we need to make the following selections.
Futures :

$$\Delta(t)g_S = c_S(t,S(t),R(t)) \quad \Rightarrow \quad \boxed{\Delta(t) = \frac{c_S}{g_S}}$$

and

Zero coupon bond:

$$\Delta(t)g_R + \Gamma(t)f_R = c_R \Rightarrow \quad \boxed{\Gamma(t) = \frac{c_R - \Delta(t)g_R}{f_R}}$$

Black-Scholes equation tells us that when we hedge a European call option with stock and the interest rate is constant, we need to hold CS(t,S(t)) of stock and a money market account.

When the interest rate is random and we are hedging with futures instead of stock, we will need to hold Δ(t) futures and Γ(t) zero coupon bonds as indicated above. In addition to option delta, we need to know the sensitivity of futures price to changes in stock price as well as interest rate in order to calculate the proper hedge.

Tailing the Hedge

Let us assume that the interest rate is constant. Now we have r instead of R(t). The futures is not random any more. The futures is dependent on the interest rate r and the current stock price S(t).

$$\begin{aligned} &g(t,S(t)) = Fut_S(t,T) = \widetilde{\mathrm{E}}[e^{r(T-t)}S(t) \mid \Im(t)] = e^{r(T-t)}S(t) \\ &g_s(t,S(t)) = Fut_s(t,T) = e^{r(T-t)} \\ &\Delta(t) = \frac{c_s(t,S(t))}{g_s(t,S(t))} = \frac{c_s(t,S(t))}{e^{r(T-t)}} = e^{-r(T-t)}c_s(t,S(t)) \end{aligned}$$

The number of futures contract needed for hedging the call option is the present value of the option delta when the interest rate is fixed. Some authors call this tailing the hedge.

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There were a lot of things that went wrong in MG's case. For one, hedging one forward contract with one futures contract is more than what is needed especially when it is far from maturity. Some estimated that MG needed only about half of the futures contracts to hedge the forward contracts. Stephen Ross did the forensic study for MG. He reckoned that the so-called hedged position was actually riskier than the naked position.

Conclusion

Black-Scholes model shows the hedging of European option using the underlying in a fixed rate environment. This analysis accounts for random interest rate and hedging with futures instead of the underlying. Section 4 shows the hedge ratio using futures in a random interest rate situation and section 5 shows the hedge ratio when the interest rate is assumed to be constant. In each case, the hedge ratio does not equal to the option delta.

This illustrates the pitfall in simply plugging in the arbitrage-free interest rate model, discounting cash flows and calculating the Greeks. There is nothing wrong in doing all these things. The problem is the hedge ratio can be different from the option delta. Without writing down a proper model, it is easy to lose sight of the risk factor that one is hedging against. It would be misguided to rely only on the option Greeks in setting up the hedges. What happened to MG in 1993 was a perfect example. Whoever wants to be the next, please stand up.