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MORE TECHNIQUES FOR BETTER ATTRIBUTIONS

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Joshua Boehme, in the August 2013 issue of Risks and Rewards, describes the Aumann-Shapley technique and applies it to attribution. This article further elaborates on theoretical and practical considerations of Aumann-Shapley and compares it to similar methods.

Boehme presents the Aumann-Shapley formula:

$$a_i = (v_i - u_i)\int_0^1 \frac{\delta f\{(1-z)u + zv\}}{\delta x_i} dz$$

The subscript i represents the ith component attribution, so a_i is the portion of change attributed to component i, and u and v are the vectors of the attribution components at the beginning and end of the attribution period.

Boehme goes on to mention several advantages of using this as an attribution tool. This will be expanded here, but first consider the following formula:

$$a_i = (v_i - u_i)\int_0^P \frac{\delta f(t)}{\delta x_i} dt$$

Here, instead of integrating with respect to z, we integrate with respect to t, which represents time where we are integrating over the attribution period (0 P). This integral follows the path of the security (or portfolio) over time. This produces the theoretically correct attribution. This will be referred to as the time-based method. Aumann-Shapley assumes implicitly that the attribution factors evolve in a linear fashion in unison. Actually, they follow various paths, and depending on the factor may be very non-linear and fall well outside the boundary set by the beginning and ending values during the attribution period. As it turns out, that is usually not that bad of a simplification, but it does introduce a source of error with the Aumann-Shapley approach.

The time-based method also has some practical limitations. In theory, it requires continuous observation of all attribution factors, which is usually impossible. As with Aumann-Shapley, this is normally solved with approximate integration. With Aumann-Shapley, approximate integration involves some sort of discrete summation on the variable z. For the time-based method, discrete summation is by time. Given that the time-based method is the theoretically correct approach, using more time steps in the attribution is more productive than using smaller increments of z in the Aumann-Shapley approximate integration, assuming observations of the attribution components are available.

For many attribution components, there is a legitimate cross term. This can be captured by the step-through approach by shocking variables one at a time and then in combination. The change due to shocking the combination that differs from the sum of the individual shocks is due to cross terms, and it is appropriate to show those as a separate attribution component. Time is unique in that it should not have a true cross term. Assuming various market parameters move according to Brownian motion, there should not be a correlation between those components and time. Time can be addressed by using Aumann-Shapley or other methods to determine its contribution without leaving a cross term. Any cross term mechanically produced where time is involved should theoretically mean revert to zero over the long term. An implication of this is that we can lump all market-driven parameters into a single category, leave theta in its own category, and then calculate a separate theta attribution component. Now, it turns out that Aumann-Shapley, average step-through, and average partial derivatives will produce about the same result in most instances. The reason for this can be shown mathematically. For small changes in financial related instruments, the first and second order terms are the most important. Given that, approximate f with the following formula:

$$f = \sum_{i=1}^{n} B_i x_i + \sum_{i=1}^{n}\sum_{j=1}^{i} B_{ij} x_i x_j$$

Then

$$\frac{\delta f}{\delta x_i} = B_i + B_{ii}x_i + \sum_{j=1}^{n} B_{ij}\, x_j$$

We can think of the various B as partial derivatives of the first and second order (B_{ii} corresponds to 1/2 times the second partial derivative). Now, regardless of whether we use Aumann-Shapley with exact integration, Aumann-Shapley with Simpson's rules, average step-through, or average partial derivative, we get the following result:

$$a_i = (v_i - u_i)\left\{B_i + \frac{1}{2}B_{ii}(u_i + v_i) + \frac{1}{2}\sum_{j=1}^{n} B_{ij}\,(u_j + v_j)\right\}$$

With two variables for component 1 we get:

$$a_1 = (v_1 - u_1)\left\{B_1 + B_{11}(u_1 + v_1) + \frac{1}{2}B_{12}(u_2 + v_2)\right\}$$

We can see an interesting verbal interpretation of the above. If we do a linear transformation so that $u = 0$ and v is then the change over the time period, we see that a_1 is just Taylors's with an added term equal to half the cross term. The proofs for these are pretty straightforward with the possible exception of the average step-through. There we will get a cross term for each step for a_i. The key to the proof is recognizing that for any given cross component j, half the step's contribution to the average will be based on u_j and half will be based on v_j.

So we see that Aumann-Shapley with exact integration, Aumann-Shapley with Simpson's rule, average step-through, and average partial derivative only differ to the extent higher order moments matter. They all have the common advantages that they recognize the state of the attribution at the beginning and end of the period and capture first and second order moments. They all share a disadvantage that if the underlying components follow a significantly non-linear path and/or fall far outside the bounds of the beginning and ending value, they can perform poorly. Usually, however, the advantages will outweigh the disadvantages.

Extending this to more variables, consider three categories of attribution components: time(theta), market variable, and contractual. Market variable would include equity levels (and other assets) and interest rates. Within equity, for example, we have delta, gamma, vega, and perhaps some higher order greeks. Also, equity might be subdivided by index or, depending on the application, even individual stocks. There would also be correlations between them, so we care about their cross terms. Contractual would include entries and exits via new business, new deposits, surrenders, death claims, partial withdrawals, transfers, etc. With a daily attribution, the contractual items normally occur at the end of the day and therefore can be dealt with last in a step-through process. They should not be included in any averaging. This leads to a general approach. Assuming a daily attribution, use one of the four methods above to separate theta and market variables impact. Each market variable could then be further subdivided. Equity, for example, could be separated into delta, gamma, cross terms, and other. Other would theoretically include higher order greeks and some noise created by the separation of theta and the market variable attribution components.

Consider a six-month European call option struck at the money. Assume constant volatility of 20 percent, level interest rate of 3 percent and a constant dividend of 2 percent. This produces an option cost of 5.8 percent of the strike. We study the period Dec. 31, 2007 through Dec. 31, 2012, issuing 10 S&P 500 options, one each on the last business day of every June and December. We calculate a daily partial derivative attribution and monthly attributions based on the step-through, partial derivatives, Aumann-Shapley (based on many points), Aumann-Shapley based on three points, and average partial derivative methods. Approximate integration for Aumann-Shapley is based on

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Simpson's rule as described by Boehme. Boehme describes all but the average partial derivative method, which is the same as the partial derivative method except the average of the beginning and ending partial derivatives is used instead of the beginning partial derivatives.

With our simplifying assumptions, the two attribution components are equity index level and time. This gives us delta and theta. For the daily partial derivative and monthly partial derivative methods, we also use gamma. The step-through approach uses the averaging technique described by Boehme. With 10 consecutive options of six months each, this gives us 60 months of data to compare these methods. While this is not a huge sample, we will see the results are quite conclusive and furthermore provide justification and explanation for the results.

We sum the results for the daily attribution to get monthly totals. This allows us to compare each of the monthly attributions to the daily and see which performs best.

The following table shows the daily attribution for January 2008:

Date	Index	Time to Expiry	Chg from Delta	Chg from Gamma	Chg from Theta	Chg from Other	Option Value	Delta	Gamma	Theta
12/31/2007	1468.36	0.498289					85.1969	0.536501	0.001895	-86.886089
1/2/2008	1447.16	0.492813	-11.3738	0.4259	-0.4758	0.0080	73.7811	0.495485	0.001944	-86.256485
1/3/2008	1447.16	0.490075	0.0000	0.0000	-0.2362	-0.0003	73.5446	0.495283	0.001950	-86.492251
1/4/2008	1411.63	0.487337	-17.5974	1.2308	-0.2368	0.0174	56.9586	0.425105	0.001973	-82.806428
1/7/2008	1416.18	0.479124	1.9342	0.0204	-0.6801	-0.0080	58.2251	0.432950	0.001991	-84.103801
1/8/2008	1390.19	0.476386	-11.2524	0.6723	-0.2303	0.0099	47.4247	0.380918	0.001972	-80.005480
1/9/2008	1409.13	0.473648	7.2146	0.3538	-0.2190	-0.0077	54.7662	0.418077	0.001999	-83.498522
1/10/2008	1420.33	0.470910	4.6825	0.1254	-0.2286	-0.0046	59.3408	0.440120	0.002008	-85.349529
1/11/2008	1401.02	0.468172	-8.4987	0.3743	-0.2337	0.0075	50.9902	0.400945	0.002002	-82.584717
1/14/2008	1416.25	0.459959	6.1064	0.2322	-0.6783	-0.0222	56.6283	0.430342	0.002030	-85.702886
1/15/2008	1380.95	0.457221	-15.1911	1.2647	-0.2346	0.0115	42.4788	0.358696	0.001989	-79.434624
1/16/2008	1373.2	0.454483	-2.7799	0.0597	-0.2175	0.0043	39.5455	0.342720	0.001974	-77.863995
1/17/2008	1333.25	0.451745	-13.6917	1.5751	-0.2132	-0.0068	27.2091	0.265873	0.001823	-67.448148
1/18/2008	1325.19	0.449008	-2.1429	0.0592	-0.1847	0.0060	24.9466	0.250528	0.001784	-65.153953
1/22/2008	1310.5	0.438056	-3.6803	0.1925	-0.7135	0.0453	20.7906	0.221472	0.001707	-60.849916
1/23/2008	1338.6	0.435318	6.2234	0.6739	-0.1666	-0.0021	27.5192	0.270913	0.001867	-69.629897
1/24/2008	1352.07	0.432580	3.6492	0.1694	-0.1906	-0.0089	31.1382	0.295722	0.001933	-73.660736
1/25/2008	1330.61	0.429843	-6.3462	0.4451	-0.2017	0.0097	25.0452	0.254473	0.001831	-67.398706
1/28/2008	1353.96	0.421629	5.9419	0.4991	-0.5536	-0.0477	30.8850	0.296273	0.001957	-74.762319
1/29/2008	1362.3	0.418891	2.4709	0.0681	-0.2047	-0.0062	33.2130	0.311988	0.001997	-77.277879
1/30/2008	1355.81	0.416153	-2.0248	0.0421	-0.2116	0.0046	31.0233	0.298325	0.001974	-75.590664
1/31/2008	1378.55	0.413415	6.7839	0.5103	-0.2070	-0.0091	38.1014	0.343548	0.002064	-81.960049

The following table shows the daily attribution summed by month for January 2008 through June 2008:

SUM OF DAILY PARTIAL DERIVATIVE

Date	Index	Time to Expiry	Chg from Delta	Chg from Gamma	Chg from Theta	Chg from Other	Option Value	Delta	Gamma	Theta
12/31/2007	1468.36	0.498289	0.0000	0.0000	0.0000	0.0000	85.1969	0.536501	0.001895	-86.886089
1/31/2008	1378.55	0.413415	-49.5722	8.9943	-6.5180	0.0003	38.1014	0.343548	0.002064	-81.960049
2/29/2008	1330.63	0.334018	-19.6830	6.3108	-6.1853	-0.0985	18.4454	0.220341	0.001921	-70.354232
3/31/2008	1322.7	0.249144	-10.8846	8.5686	-5.4607	0.3648	11.0335	0.164651	0.001874	-67.374585
4/30/2008	1385.59	0.167009	8.0150	5.5672	-7.9109	0.1605	16.8654	0.257295	0.002844	-112.188237
5/30/2008	1400.38	0.084873	-1.0771	5.3010	-10.7737	-0.2839	10.0316	0.220218	0.003628	-145.018470
6/30/2008	1280	0.000000	-11.5709	5.9933	-4.4095	-0.0446	0.0000	0.000000	0.000000	0.000000
			-84.7727	40.7352	-41.2581	0.0988				

The "Chg from Other" column in the above corresponds to the nonzero unexplained amount discussed in Boehme. Here it is small relative to the other attribution components because of the small size (daily) of the attribution period. It is only meaningful on days with large market moves. For example, on Sept. 29, 2008 it was -1.93 due to a 107 point drop in the index. Normally, gamma accurately predicts the change in delta, but for large market moves the higher order terms of Taylor's series matter. In this case, using gamma one would have expected delta to be 0.02 at the end of the day, but it was 0.08. Similarly, the large index movement creates a huge drop in theta causing the change from theta to be inaccurate as well. The only other "Chg from Other" with absolute value greater than 1.00 was June 29, 2009 when it was 1.01. This was the next-to-last day of the option where the option finished slightly in-the-money. When an index finishes close to the strike, gamma is extremely high, causing attributions, hedging, etc. to break down. For the 10 six-month options, the total "Chg from Other" for the six months ranged from -2.12 to 1.83. The average magnitude was 0.71.

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From our sample options, using all the methods discussed, we get the following results:

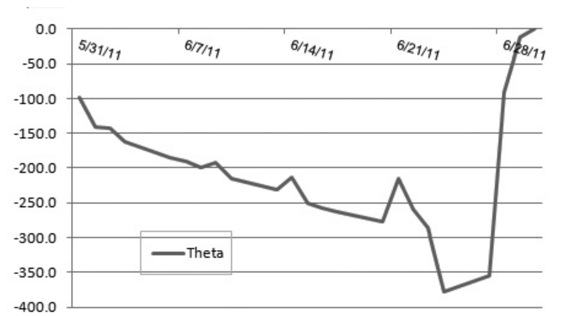
Method	Item	Std Dev Error	June 2011 Theta Error
Sum of Daily Partial Derivative	Other	0.33	
	Theta Adj	0.02	
Partial Derivative	Theta	2.26	10.05
	Other	2.61	
Average Step-Through	Theta	1.74	11.27
	Other	0.00	
Aumann-Shapley	Theta	1.86	11.52
	Other	0.02	
Aumann-Shapley-3 points	Theta	1.78	11.28
	Other	0.22	
Average Partial Derivative	Theta	2.63	14.14
	Other	2.41	

For each method we take the "Chg from Other" for each of the 60 months and determine its standard deviation assuming a mean of zero. For each method we compare the difference between its theta for the month and theta from the Sum of Daily Partial Derivative's theta and determine its standard deviation assuming a mean of zero. For the Sum of Daily Partial Derivative we used Aumann-Shapley-3 points on a daily basis to determine theta. Once we did that, we saw other drop to just 0.02.

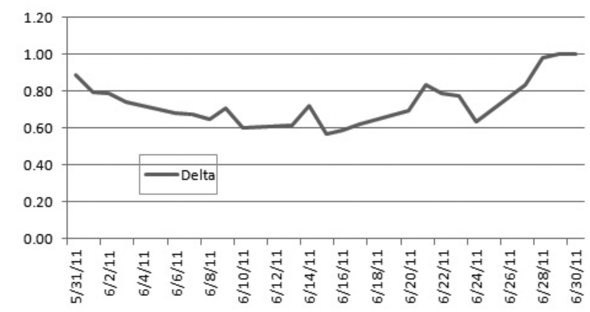
Aumann-Shapley is based on 40 points per month. This results in about the same number of calculations as used for the Sum of Daily Partial Derivative to get an apples-to-apples comparison. Note Average Step-Through, Aumann-Shapley and Aumann-Shapley-3 points all have about the same error in estimating theta. The other is also small in each case, and frankly is irrelevant compared to the theta error. I expected the average partial derivative theta to be more in line with the three other methods as far at theta error, but the "other" error was expected to be more noticeable since that method does not address other like the other three methods.

The theta error is quite large for June 2011 for all five monthly methods. This is the last month of the option period when results are very sensitive. As the graphs below show, both delta and theta took a very non-linear path during June 2011. Furthermore, both greeks frequently were far outside the bounds set at the beginning and ending of the month. The daily attribution handled this well with the other being only 0.41 for the month. That value reduced to 0.004 when using Aumann-Shapley-3 points on a daily basis to calculate theta.

THETA



DELTA



The partial derivative approach happened to do a little better in June 2011, which pulled its standard deviations of error down, but that is misleading since there were 13 months where the theta error was greater than 2.00 while the other four monthly methods had three to five such months.

In summary, Average Step-Through, Aumann-Shapley and Aumann-Shapley-3 points were expected to perform the best given the same time period, and that is the result. In conclusion, attribution should be done with fairly small time steps as opposed to using a lot of points in the Aumann-Shapley calculation, but using Aumann-Shapley-3 points as a last step has some advantages.

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