Article from:

Risks & Rewards

August 2012 – Issue 60

SOCIETY OF ACTUARIES Investment Section

Risks & Rewards

ISSUE 60 AUGUST 2012

PRICING AND HEDGING FINANCIAL AND INSURANCE PRODUCTS

PART 1: COMPLETE AND INCOMPLETE MARKETS

By Mathieu Boudreault

This paper is the first excerpt of the article: "Pricing and hedging financial and insurance products" which will be available from the Society of Actuaries' website. Comments are welcome.

Suppose your insurance company has issued a new block of equity-indexed products. To manage these policies, you use stochastic scenarios based upon your economist's best estimates regarding equity index returns and yields on investment-grade bonds. On the grounds of these assumptions, you determine that the company can spend 300 bps per year over the next five years for an equity-based guarantee. In order to manage the risk underlying this guarantee, you contact the investment bank but the required derivatives cost 700 bps! Why is this possible and what can we do about it? To make sense of it, we have to better understand the modern financial mathematics that underpins active risk management.

In the latter situation, the bank does not necessarily have a smarter or more risk-averse economist. Banks however price their derivatives to be consistent with the other instruments available, i.e. stocks, bonds and plain vanilla instruments such as swaps, futures and options. They use these instruments to hedge their positions and the price they charge is consistent with the cost of the hedging strategy. Otherwise, arbitrage opportunities could arise. Thus, the key to modern financial mathematics is no-arbitrage pricing.

The primary purpose of this paper is to explain in plain English without any cumbersome formulas (almost!) how financial mathematics applies in modern finance and in today's insurance industry. I describe how arbitrage-pricing and risk-neutral pricing are equivalent and I illustrate with simple examples how to deal with complete and incomplete markets. When possible, I try to link these concepts to traditional and equity-linked insurance.

CONTINUED ON PAGE 4

> "… WHEN TWO PORTFOLIOS HAVE EXACTLY THE SAME PAYOFFS IN EXACTLY THE SAME SCENARIOS, THEN BOTH PORTFOLIOS SHOULD HAVE THE SAME PRICE."

ARBITRAGE-PRICING LAYS DOWN THE LAW (OF ONE PRICE)

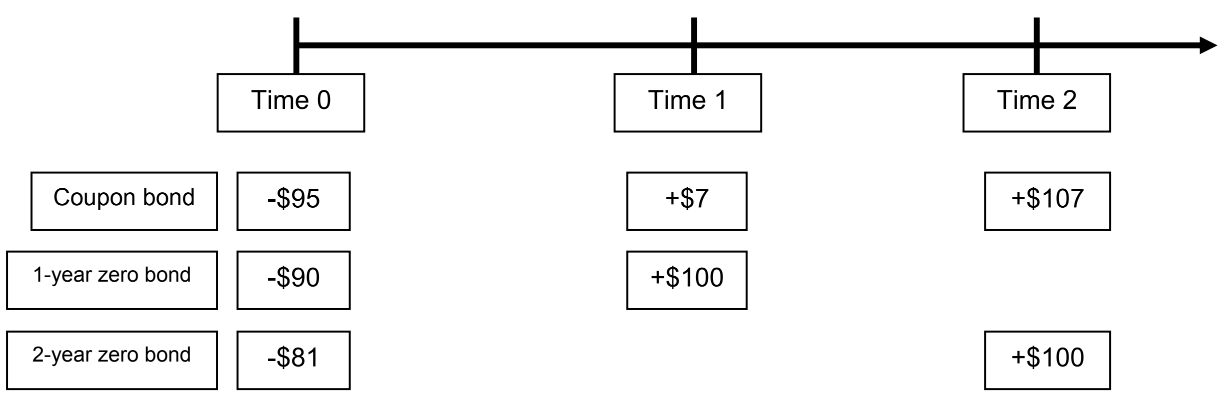
The mathematics of financial engineering mainly deals with the pricing[1] and hedging of financial assets known as derivatives. Contrarily to stocks that are priced according to their future cash flow potential (future dividends and capital gains), derivatives' pricing usually takes the dynamics of the stock as given. One of the objectives of financial engineering is to compute the price of a derivative in this context and find an appropriate risk management strategy. To meet these objectives, we need to define the most basic concepts of financial mathematics which are *arbitrage* and the *law of one price*.

There is an arbitrage opportunity when a zero-investment (net) may yield a profit, without any loss possibility. We often say there is *no free lunch* in a market where arbitrage opportunities do not exist. It is important to make sure no arbitrage opportunities arise because it would mean investors would have a much easier way to make profits without assuming risk. Thus, derivatives and other financial products are priced to avoid these. This leads to the law of one price that stipulates that when two portfolios have exactly the same payoffs in exactly the same scenarios, then both portfolios should have the same price (or cost) to avoid arbitrage opportunities.

Example 1: In a fixed-income market (see Figure 1), a two-year coupon bond with annual coupons of 7 percent trades at \$95 (face value of \$100). Moreover, a one-year zero-coupon bond trades at \$90 and a two-year zero-coupon bond trades at \$81. Given that there is no credit risk, no liquidity risk, no transaction costs, no taxes, etc., is there an arbitrage opportunity?

Solution: A two-year coupon bond can be constructed with one-year and two-year zero coupon bonds. Indeed, 0.07 unit of the one-year zero and 1.07 unit of the two-year zero-coupon bond yield exactly the same cash flows than the two-year coupon bond. The zero-coupon bond portfolio costs \$92.97 while the exactly equivalent coupon bond trades at \$95. Thus, there is an arbitrage opportunity.

Figure 1: Illustration of the cash flows of the bonds available in the market



In the previous example, we have replicated the cash flows of a coupon bond with a set of zero-coupon bonds. Replication (or hedging) is not a new concept since it can be seen as a type of immunization.

Example 1 (cont'd): a very stable insurance line of business is such that we know with almost certainty that 200 people will die in year one and 350 people will die in year two. The insurance benefit is $1,000. How can we exactly hedge the insurance cash flows with the bonds available (given there is no credit risk, no liquidity risk, no transaction costs, no taxes, etc.)?

Solution: the company has to pay $200,000 at time one and $350,000 at time two. Thus, by buying 2,000 units of the one-year zero-coupon bonds and 3,500 units of the two-year zero-coupon bonds, the cash flows are exactly hedged. The cost of this immunization strategy is 2,000 x $90 + 3,500 x $81 = $463,500.

A portfolio of assets is said to be a *replicating portfolio* if it is specifically designed and dynamically updated such that it exactly replicates the cash flows of an asset or a derivative. By the law of one price, the cost of the replicating portfolio should also represent the true and unique price of the derivative. Otherwise, arbitrage opportunities would exist.

In practice, exploiting an arbitrage involves accounting for *market frictions*, regulations and other restrictions. However, mathematical finance textbooks usually assume a frictionless market. In such a market, the following assumptions hold: no transaction costs, perfectly liquid and divisible assets, lending and borrowing interest rates are the same (thus no default from both sides of the transaction), no taxes, no restrictions on buying and selling (and short-selling), etc. None of these assumptions are observed in practice but they might be approximately true for large investment banks. Indeed, the volume of transactions for investment banks is huge, meaning that transaction costs are minimal and assets are approximately divisible (a block of 100 stocks is small with respect to their volume of transactions). Moreover, before 2008, these banks had the best credit rating possible, meaning lending and borrowing rates were very close to the risk-free rate.

In the insurance industry, public policy prevents individuals and corporations from actively trading insurance contracts. If it were the case, that would introduce an incentive to cause the covered event! Thus, even if there are many identical policies with different prices, a policyholder cannot make arbitrage profits out of these contracts (by selling the costliest, which is not even allowed) and will typically buy the cheapest available.

One might also wonder if there are arbitrage opportunities in actual markets. First, arbitrageurs, investment banks and hedge funds use a looser definition of arbitrage, being able to accept a small amount of risk when exploiting an opportunity. However, how small the risk is depends on many factors and the case of Long-Term Capital Management illustrates how difficult it can be to exploit arbitrage opportunities without any substantial risk. Nowadays, arbitrage opportunities may exist in very tiny time windows over assets that are cross-listed on different markets. These opportunities do not last long: a few thousandths of a second and are exploited by supercomputers with complex algorithms.[2]

In conclusion, the absence of arbitrage (and the law of one price in many cases) should guide how derivatives are priced, no matter what are the assumptions for the evolution of the stock price, or of the underlying market. Market frictions, regulations and other restrictions in the financial

CONTINUED ON PAGE 6

and insurance industry simply make it more difficult (or impossible) to exploit arbitrage opportunities. That does not invalidate the principles underlying no-arbitrage pricing.

SIMPLE CASE: COMPLETE MARKETS

Introduction and assumptions

To illustrate how we should price and hedge a claim under complete markets, we will assume that there is a financial market where only two assets are traded: a risk-free bond (also known as Treasury bond) and a risky asset (say a stock). The bond is risk-free in the sense that default does not exist in such a market so that the value of the bond grows with the risk-free rate. The initial value of the stock is observed and its price at the end of the period can only take two different values: this is the single-step binomial tree. Thus, the stock is risky in the sense that at time 0, there is uncertainty on whether the stock will go down or go up. The two terminal values are fixed and known by every market participant at inception. We further assume there are no market frictions and there are no arbitrage opportunities between the stock and the bond. Consequently, if one invests in the stock (compared to an equivalent investment in the bond), it should be possible to lose or make money out of the stock. Alternatively, the stock cannot *always* earn more (or always earn less) than a risk-free bond.

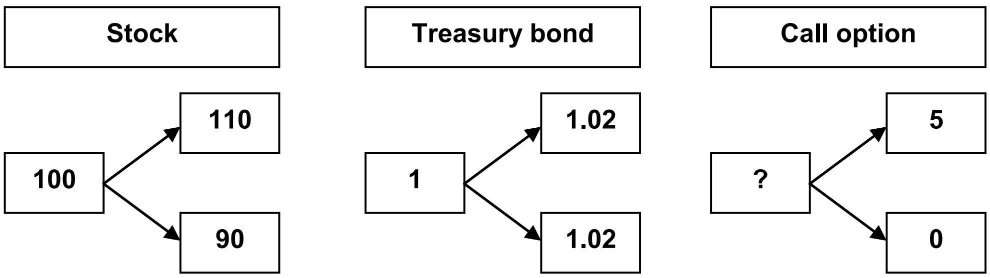
Replicating portfolio

Example 2: A stock currently trades at \$100 (see Figure 2) and can take two different values at the end of the year: \$110 or \$90. A Treasury bond trades at \$1 and will be worth \$1.02 at the end of the period. According to the analysts, the probability that the stock will be worth \$110 at the end of the year is 75 percent. What should be the price of a call option with strike price \$105 in order to avoid arbitrage opportunities?

Solution: We will try to find a replicating portfolio that exactly replicates the cash flows of the option. If the stock trades at \$110 (\$90) at the end of the year, the call option is worth \$5 (\$0). Solving a system of two-equations with two unknowns, one gets that a portfolio that holds 0.25 unit of a stock and a loan of \$22.06 exactly replicates the cash flows of the option. The cost of this portfolio is \$2.94, which should be the price of the option.

To price the option in the latter example, we used the law of one price. That is, we first tried to find how to trade in the assets available at time 0 in a way that exactly replicates the cash flows of the option. Thus, to avoid arbitrage opportunities, the cost of the portfolio should correspond to the price of the option.

Figure 2: Illustration of the possible outcomes of the stock, Treasury bond and call option in the single-step binomial tree



One important conclusion can be drawn from this numerical example. In the market that we defined and its assumptions, the replicating portfolio yields the exact same payoff as the derivative, in every possible scenario. Thus, no matter how likely each scenario really is, the replicating strategy will pay off the same amount as the derivative. Hence, the probability (that will be known as *real probability* or *physical probability* later) of observing a rise in the price of the asset is not a relevant input in the price of the option (that avoids arbitrage opportunities). This is because this probability is already an important factor in determining the current price of the stock, which we take as a given when pricing derivatives. If it is felt that the current stock price is inappropriate, then the derivative will be "mispriced," but consistent with the cost of replication. Thus, the replicating strategy **only** tells you how to hedge the derivative given the current stock price and the underlying model (and its assumptions); nothing else.

Finally, in the exact previous setup, i.e., where a risk-free bond and a stock are traded, and the stock only has two possible values at the end of a period, the exact no-arbitrage price of a derivative can be found for all possible payoff values. Indeed, as long as one can find a unique solution to a system of two equations and two unknowns, there will be a unique replicating portfolio associated to this derivative. A market where each possible derivative can be replicated is known as a *complete market*. We often say that in a complete market, all risks can be replicated.

Risk-neutral pricing

In financial mathematics, there also exists another equivalent way to price a derivative, which is known as risk-neutral pricing. In the one-step binomial tree, it is straightforward to check that these two are exactly equivalent. Indeed, by algebraically writing the cost of the replicating portfolio and reorganizing the terms (see Appendix for the details), one can obtain a very interesting expression. Thus, the price of a derivative can be rewritten as the discounted (at the risk-free rate) expectation of its future cash flows, under an alternative probability measure, known as risk-neutral probability measure. The latter is known as risk-neutral because only risk-neutral agents would expect a return equivalent to the risk-free rate (no risk premium) on any risky asset.

Example 2 (cont'd): how can we find the no-arbitrage price of the call option using risk-neutral pricing? What is that price?

Solution: following the derivations in Appendix, we find that the risk-neutral probability of observing \$110 at time one is 60 percent, which is in no way related to the postulated 75 percent determined earlier by the analyst. The price of this option is thus 60 percent x \$5 discounted at 2 percent, which yields \$2.94 as well.

The fact that this expectation was rewritten from the cost of the replicating portfolio further illustrates that the risk-neutral probabilities are really not related to the true (or physical) probability of observing an increase in the price of the stock. Risk-neutral probabilities are **only** useful when the **no-arbitrage price** of a derivative needs to be found. In all other contexts such as risk management, asset management, investment decisions and stress testing, the true probability (determined by the analyst) is what matters.

In example 2, to answer the questions, "What is the probability that the option is in-the-money?" and "Is this option a good deal or a winning bet?" one should use 75 percent, which is the probability postulated by the analyst. Thus, risk-neutral probabilities can be seen as a mathematical convenience so that we can write down the price of a derivative as a simple expectation. In many contexts, it can be very helpful, but at the cost of making simple calculations unintuitive.

Moreover, we argued earlier that no matter what scenario is ultimately realized at the option maturity, the replicating strategy should be effective. Thus, no matter how risky the stock is, or no matter what our perception of risk is (risk aversion), the replicating strategy is the unique way to exactly replicate the derivative's payoffs in every scenario.

CONTINUED ON **PAGE 8**

Thus, being long (short) the derivative and short (long) the appropriate amount of stock yields a risk-free position.

Finally, risk-neutral pricing **does not** imply that investors are risk-neutral. This would be, of course, untrue as stocks entail a significant risk premium. Risk-neutral valuation is a **consequence** of using arbitrage-free pricing and replicating portfolios. The *fundamental theorem of asset pricing* links the absence of arbitrage to the existence of risk-neutral probabilities.[3]

Investment guarantees and equity-linked insurance

We illustrate in this section how basic investment guarantees can be represented in a complete market environment.

Example 3: a 65-year-old individual invests $100 in an equity-linked insurance policy that provides an investment guarantee. The underlying stock chosen by the policyholder may take two possible values at the end of the period: $90 or $110. A minimum return of 3 percent is guaranteed on the policy, upon death or survival, and the risk-free rate is 2 percent. According to mortality tables, this individual has a 1 percent probability of death by the end of the period. In a frictionless market, what is the no-arbitrage premium that should be paid by the policyholder for the investment guarantee?

Solution: because the payoffs of the contract do not change if the individual survives or not, the policy can be seen as a stock and a plain vanilla put option in a complete market. When the market is frictionless, the premium paid by the policyholder for the investment guarantee is the no-arbitrage price of the put option, which has a strike price of $103. As discussed earlier, one can use replicating portfolios or risk-neutral pricing to find the price of the option and/or the risk management strategy for this liability. Because the binomial tree is the same as in example 2, using risk-neutral pricing, we have that the price is 40 percent x $13 discounted at 2 percent, which is $5.10. The risk management strategy for this policy can be found with replicating portfolios. On one hand, the insurance company sells a share of a stock, which can be hedged by buying a stock as well. It also sells a put option, which can be replicated by selling 0.65 units of the stock (for proceeds of $65) and investing what remains in the Treasury bond (the difference between the stock position and the value of the put).

In that example, the individual invests $100 and pays an additional premium of $5.10 at time 0 so that at maturity, the payment is either $110 or $103. This is because we have assumed that the premium is paid up front instead of a penalty on the return. Even though the policy has considerably reduced the volatility of the returns in the stock, it may be difficult to call this product an investment guarantee because in the down scenario, the investor loses $2.10. In an arbitrage-free market, it is impossible to always earn more (or always less) than the Treasury bond without assuming some level of risk.

A more reasonable payment scheme for this contract could be $104 in the up scenario and $101 in the down scenario, which is similar to a participating policy that penalizes the upside, for a "guarantee" against the downside. It can be found that for an initial investment of $100.78 ($100 plus the initial premium), this price does not create any arbitrage opportunity.

Conclusion

Using no-arbitrage pricing yields two typical methods to price a basic financial derivative: finding the replicating portfolio or risk-neutral pricing. These two approaches are exactly equivalent. Moreover, risk-neutral probabilities are only relevant when one deals with finding the price of a derivative under no-arbitrage. In all other contexts, physical or real probabilities should be used.

It is obvious that representing the evolution of a stock by such a simplistic model cannot be realistic. However, a time step can be a year, month, day, hour, minute, or a second, etc. Repeating the one-step binomial tree at each period over a longer time horizon is one way to make the latter approach more realistic. It turns out that using a single-step binomial tree at every instant for the price of the stock (with appropriately chosen possible outcomes)[4] results in continuous rates of returns that are normally distributed. This is the Black-Scholes' model that will be thoroughly discussed in the next excerpt.

BEING MORE REALISTIC: INCOMPLETE MARKETS

Introduction and assumptions

In the one-step binomial tree model, we have assumed that the stock only took two possible values at the end of the period. It resulted that the market was complete, meaning that all possible payoffs of a derivative could be replicated. We will now relax this assumption through a simple market model in which we will be able to draw very valuable conclusions.

Assume that under normal market conditions, the stock can take two values at the end of the period. Under extreme circumstances (say a crash period, default of the company, etc.), the stock may take a third possible value. We will once again assume that by investing in the stock, the investor may make more or less money than by investing in the risk-free bond, so that there is no arbitrage between the stock and the bond.

Replicating portfolio

Example 2 (cont'd): Under an extreme circumstance (read: bankruptcy), the stock depicted in Example 2 might take the value 0 (or close to). Is it possible to replicate the payoffs of the call option over all possible scenarios (see Figure 3)?

Solution: To create a replicating portfolio that works under each of the three scenarios, we need to find the solution of a system having three equations and two unknowns. A unique solution does not exist in this context.

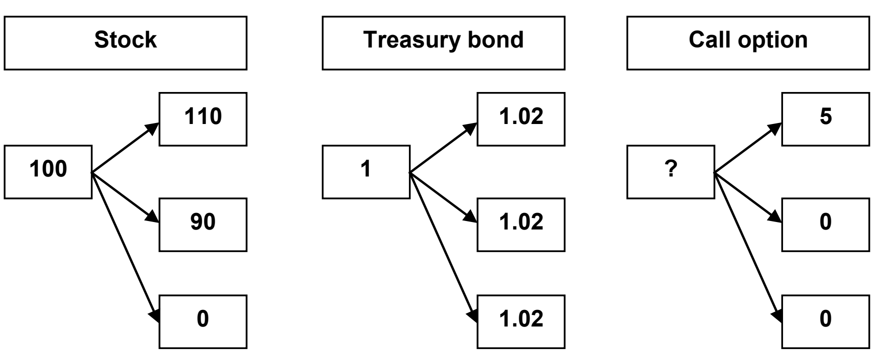
When there are two assets that are traded and three possible outcomes (that we have interpreted as normal and extreme scenarios), it is generally impossible to find a unique replicating portfolio that will work in each scenario. Such markets are known to be *incomplete markets*. In incomplete markets, some derivatives may have a unique replicating portfolio (attainable claims), but the vast majority do not. Thus, incomplete markets are truly what are observed in reality, with some risks that cannot be hedged.

What happens if one ignores the third outcome?

In fact, the replicating portfolio may work very well but once in a while, it may not work. The following example illustrates the situation.

Figure 3:

Illustration of the possible outcomes of the stock, Treasury bond and call option in the single-step trinomial tree



CONTINUED ON **PAGE 10**

"… NEW ASSETS AND FINANCIAL DERIVATIVES HELP COMPLETE THE MARKET."

Example 2 (cont'd): The risk manager analyzes the credit risk of the firm using reports from rating agencies. He figures that the probability of default (stock is worthless) is 2 percent by the end of the period. He decides to hedge the normal scenarios. Analyze the appropriateness of the strategy.

Solution: By hedging the first two scenarios, one obtains the same replicating portfolio as in the one-step binomial tree section. According to the rating agencies, that would mean that 98 percent of the time, the replicating portfolio would work and exactly replicate the payoffs of $5 or $0 when the stock is respectively worth $110 or $90. However, if the company does default, a loan still has to be repaid ($22.50 of capital and interest) with a stock that is worthless.

How does the risk manager replicate his risks in this context?

A risk manager will never leave such a possibility open without taking any risk attenuation measures. In this simple market, there are no financial assets available to exactly replicate the extreme outcome. The risk manager will have to use judgment in assessing the risk of his positions. He may choose to replicate any pair of outcomes and choose the pair that is the most appropriate. He may also pick a strategy that yields a minimal loss under each scenario.

What is the true price of a derivative?

Unfortunately, for a derivative that does not have a unique replicating portfolio, there is no unique price. Only a range (an interval) of prices makes sure that the derivative does not introduce arbitrage opportunities. The seller and buyer of the derivative will have to agree on a price in the latter range. In this case, it is very likely the buyer and seller will both assume a level of risk, as perfect replication does not exist.

Completing the markets

The introduction of new assets and financial derivatives help complete the market. In other words, those additional assets may help a risk manager attain a greater level of replication of its cash flows. The following example illustrates how financial innovation may help deal with credit risk.

Example 2 (cont'd): A second risky asset is now available in the market (see Figure 4). This product pays off $100 if the stock is worthless, and 0 otherwise. It trades at $3. How does this product affect the risk management and pricing of the call option?

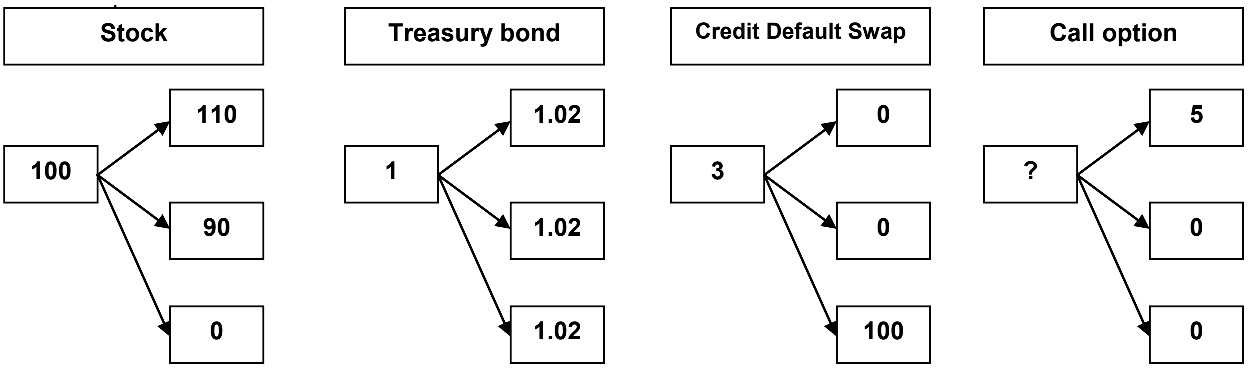
Solution: It can be seen that this asset acts as an insurance against default. This is a simplistic representation of what is known as a credit default swap (CDS). In order to replicate the three possible outcomes of the call option, we now have three assets. This yields three equations and three unknowns. We find that we need 0.25 share of the stock, a loan of $22.06 and 0.225 unit of this insurance. In case of default, we still need to repay the loan with interest, which is $22.50. The insurance will pay off only in case of default, in order to pay back the loan. The cost of the insurance is 0.225 times $3, which is 67.5¢. Because we have found a unique replicating portfolio, the unique no-arbitrage price of this derivative is $2.94+67.5¢, i.e., $3.61.

It should be noted that the CDS acts as a fundamental asset, just like the stock and the bond. In a market represented by a trinomial tree where only a stock and a bond are traded, one cannot replicate the payoffs from the CDS just like the call option could not be replicated earlier in Figure 3. In trinomial trees, one requires any combination of three assets to replicate a fourth one.[5] It is straightforward to think that to replicate the call option, one needs positions in the stock, the bond and the CDS. But if the current price of the option is known, then one could replicate the CDS payoffs with the stock, bond and call option.

Risk-neutral pricing

One can also find the price of a financial derivative in a one-step trinomial tree using risk-neutral pricing. According to the risk-neutral pricing principles, we need to find the prob-

Figure 4:
Illustration of the possible outcomes of the stock, Treasury bond, CDS and call option in the single-step trinomial tree



abilities such that we expect a return of the risk-free rate on all risky assets traded in this market. As discussed earlier, risk-neutral pricing or replicating portfolios are equivalent and are the consequence of using the absence of arbitrage to price derivatives. With two assets (risky and risk-free) and three outcomes, we have an infinite number of risk-neutral probabilities, which will also yield a range of prices (rather than a unique price) that avoid arbitrage opportunities.

When we add a third asset, as in the credit risk example, we can solve for unique risk-neutral probabilities. Relating to Example 2, we can have a real probability of default (as given by Moody's or Standard and Poor's for example) and a risk-neutral default probability, which is once again, totally unrelated to the true default probability.

Investment guarantees and equity-linked insurance

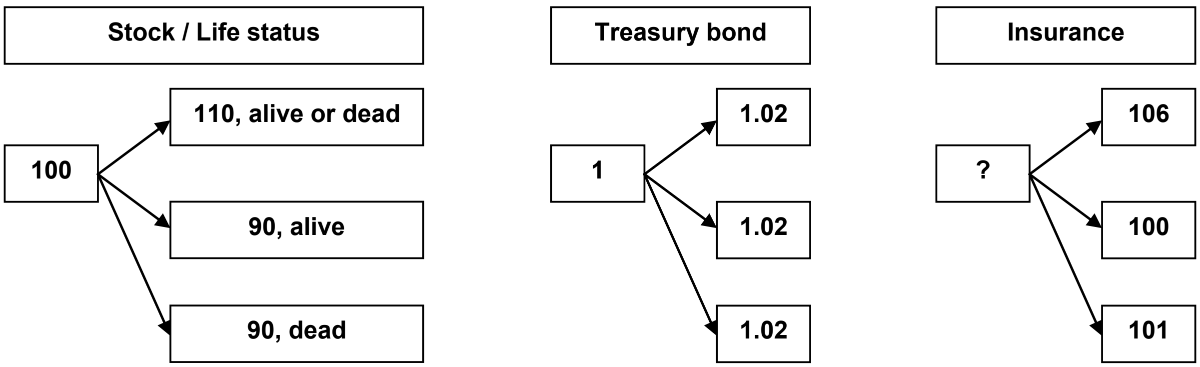
We have used credit risk as a way to interpret market incompleteness and introduced credit default swaps to complete this market. Strictly from a financial engineering viewpoint, mortality risk creates market incompleteness. As it will be seen in the following examples, traditional actuarial techniques can be used to deal with this issue.

In example 3, the payment upon death or survival was exactly the same. Thus, even though the insurance company faces mortality risks and incomplete markets, it was possible to find a unique replicating portfolio and a unique price. This is an example of an attainable claim.

Example 3 (cont'd): we now assume that the payment upon death or survival is different. Suppose that upon death, the minimum return is 1 percent whereas upon survival, the minimum return is 0 percent. In both cases, the upside is capped at 6 percent. What is the no-arbitrage price of this policy assuming frictionless markets?

CONTINUED ON **PAGE 12**

Figure 5:
Illustration of the possible outcomes of the stock, Treasury bond and the insurance in a single-step trinomial tree



Solution: this is an additional example (see Figure 5) where there are more outcomes (three) than the number of assets available in the market (two). One cannot find a unique no-arbitrage price or a unique replicating portfolio.

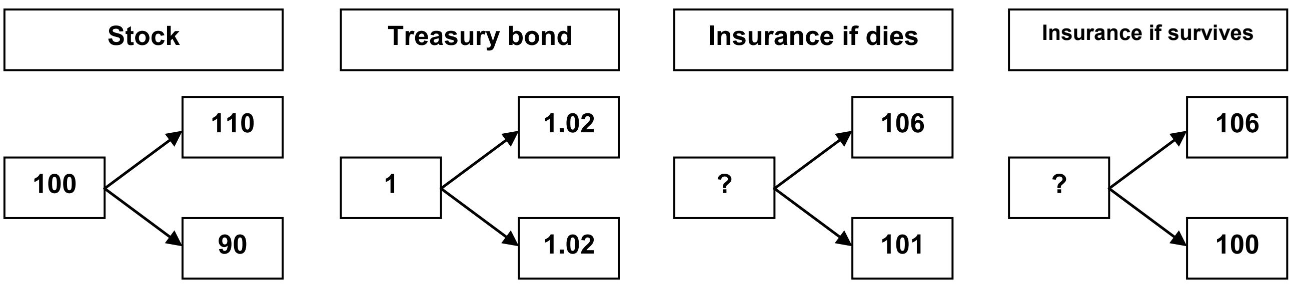
Public policy of course forbids insurance companies to monetize their policies so that we cannot complete markets as with credit risk. Thus, insuring the life of one individual is like a bet: it remains risky. However, the role of insurance companies is to pool these risks to better predict the total loss in a portfolio. Since mortality risk is generally independent[6] from one life to the other, the insurer can predict relatively well the number of deaths at each time period.

Example 3 (cont'd): assume that the insurance company insures the life of 10,000 independent individuals aged 65, each with a death probability of 1 percent (according to an appropriate mortality table). These individuals have the same risk characteristics and hold identical portfolios. How can we price and manage the previous equity-linked insurance in this context?

Solution: using the law of large numbers, it is possible to say that approximately 100 deaths will happen and 9,900 people will survive.[7] Thus, the positions of the insurance company are as follows (see Figure 6): (1) short 100 derivatives that pay \$106 in the up scenario and \$101 in the down scenario and (2) short 9,900 derivatives that pay \$106 in the up scenario and \$100 in the down scenario. Using risk-neutral pricing or replicating portfolios, we find that the (no-arbitrage, frictionless market) price of the first contract is \$101.96 while the second is \$101.57. The replicating strategy required is 0.25 (0.3) unit of stock for each of the first (second) contract. Thus, for the 10,000 lives, 100 x 0.25 + 9,900 x 0.3 = 2995 shares of stock are required. The rest of the proceeds are invested in the Treasury bonds.

When one uses risk-neutral pricing in the context of example three, one sees that two types of expectations are used. Conditional upon survival (or death), a risk-neutral expectation is applied to find the no-arbitrage price of the derivative when the individual survives (dies). However, the value of the portfolio is weighted by the true number of deaths and survivors. The weights are determined using a mortality table, which is an observed or real death probability. Overall, those are nested expectations; with the outside expectation taken with real death probabilities and the inside expectation computed with risk-neutral probabilities of observing an increase in the price of the stock.

Figure 6:
Illustration of the possible outcomes of the stock, Treasury bond and the insurance represented in binomial trees



Example 3 showed a practical example where we can manage risks in an incomplete market. However, it is important to understand that the example featured a very large set of independent and identically distributed policyholders, so that assuming 100 deaths is reasonable. In reality, policyholders have different risk characteristics (and hold different portfolios) so that the realized mortality is very likely to deviate (positively or negatively) from expectations. In that case, traditional actuarial techniques are necessary to deal with these deviations that will make the hedge portfolio imperfect.

CONCLUSION

In this paper, we have illustrated fundamental concepts of modern financial mathematics such as arbitrage pricing under complete and incomplete markets. It was shown that under absence of arbitrage, the price of a derivative should correspond to the cost of the replicating portfolio. In a complete market, the price is unique, whereas in an incomplete market, perfect replication is rarely possible, and a range of price prevents arbitrage opportunities. In incomplete markets, buyers and sellers have to assume some level of risk. In all cases, to find the no-arbitrage price of a financial derivative, the replicating portfolio or risk-neutral pricing are equivalent approaches to find such price. The risk-neutral probability is only relevant in the context of finding the price of a derivative under absence of arbitrage; in all other cases, the true probability measure matters.

We have used basic financial engineering and actuarial mathematics to deal with equity-linked insurance. In reality, insurance is very much different from investment banking. First, public policy prevents people and insurers from trading individual life insurance policies just like other basic financial assets. In this case, prices can deviate from their no-arbitrage equivalents, meaning the best an individual can do is opting for the cheapest contract. Second, insurance contracts involve asymmetry of information between the policyholder and the company; the former always knows more about its risks than the latter, requiring the company to underwrite the policy. Finally, people buy insurance and equity-linked products for family estate management and tax considerations.

One should be cautious regarding the latter three arguments. First, rational investors would already account for tax differentials between insurance and financial assets. Indeed, two assets having the same payoffs but taxed differently should have different prices. The difference would only be due to taxes to make sure there is no arbitrage between the

CONTINUED ON **PAGE 14**

"FINANCIAL AND ACTUARIAL TOOLS USED IN THIS PAPER CAN AND SHOULD BE USED TO MANAGE THE RISK OF EQUITY-LINKED INSURANCE."

two assets. Moreover, the fact that markets cannot monetize insurance policies is a major impediment indeed. The danger however is if the contract is underpriced, even when accounting for mortality and underwriting. The rational individual could long (buy) the insurance contract and short (sell) the replicating portfolio from the financial markets, making a "sure" profit.

More importantly, financial and actuarial tools used in this paper can and should be used to manage the risk of equity-linked insurance. The actuary should keep in mind that perfect hedging in incomplete markets (which is the reality) is impossible, but neither dynamic hedging, nor traditional actuarial techniques are perfect methods. Stress-testing is the key.

In the upcoming article, we will discuss the Black-Scholes' model, its imperfections and how we can improve Black-Scholes' for financial and insurance products.

APPENDIX

In this section, we show how we can link replicating portfolios and risk-neutral pricing in the context of the single-period binomial tree. In general, the *fundamental theorem of asset pricing* is used to link the two approaches.

Suppose the assumptions regarding the one-period binomial tree hold. The current stock price is s_0 and its future possible prices are s_1^u and s_1^d in the up and down scenarios respectively. The payoffs of the derivative in the up and down scenarios are c_1^u and c_1^d. We would like to find the current price of the derivative c_0 such that there is no arbitrage opportunity. Let x be the number of stocks that we should hold in the period to exactly replicate the payoffs of the derivative, while y is the number of Treasury bonds. The value of such a bond is one at inception, and $1 + r$ at maturity.

To find the appropriate replicating portfolio, we build the system of equations that allow us to exactly replicate the payoff of the derivative in each scenario. We thus solve for a set of two equations with two unknowns:

$$\begin{aligned} x s_1^u + y(1+r) &= c_1^u \\ x s_1^d + y(1+r) &= c_1^d. \end{aligned}$$

Subtracting the two equations allow us to easily find x which is

$$x = \frac{c_1^u - c_1^d}{s_1^u - s_1^d}.$$

One easily recognizes delta. Furthermore,

$$y = \frac{1}{1+r}\frac{c_1^d s_1^u - c_1^u s_1^d}{s_1^u - s_1^d}.$$

The cost of this portfolio at inception is simply

$$c_0 = s_0 x + 1y = s_0\frac{c_1^u - c_1^d}{s_1^u - s_1^d} + \frac{1}{1+r}\frac{c_1^d s_1^u - c_1^u s_1^d}{s_1^u - s_1^d}$$

by substituting the values of x and y. Now, we isolate $\frac{1}{1+r}$ and put everything on the same denominator. We get

$$\begin{aligned} c_0 &= \frac{1}{1+r}\left((1+r)s_0\frac{c_1^u - c_1^d}{s_1^u - s_1^d} + \frac{c_1^d s_1^u - c_1^u s_1^d}{s_1^u - s_1^d}\right) \\ &= \frac{1}{1+r}\left(\frac{c_1^u(1+r)s_0 - c_1^d(1+r)s_0 + c_1^d s_1^u - c_1^u s_1^d}{s_1^u - s_1^d}\right). \end{aligned}$$

We now group the terms in c_1^u and c_1^d to get

$$c_0 = \frac{1}{1+r}\left(c_1^u\frac{s_0(1+r) - s_1^d}{s_1^u - s_1^d} + c_1^d\frac{s_1^u - s_0(1+r)}{s_1^u - s_1^d}\right).$$

Now, let $q = \frac{s_0(1+r)-s_1^d}{s_1^u-s_1^d}$. We find that $1-q = \frac{s_1^u-s_0(1+r)}{s_1^u-s_1^d}$. Furthermore, because there are no arbitrage opportunities between the stock and the bond, q is necessarily between zero and one (bounds excluded) since investing in the stock implies that we can make more or less money than investing in the risk-free asset. Then, the cost of the replicating portfolio can be written as

$$c_0 = \frac{1}{1+r}(q \times c_1^u + (1-q) \times c_1^d)$$

where q has the characteristics of a probability. Thus, the cost of the replicating portfolio can be rewritten as an expectation (under an alternative probability measure q) of future cash flows, discounted at the risk-free rate. To expect a rate of return equivalent to the risk-free rate is equivalent to having a universe where agents are risk-neutral. Even if we had supposed that the true probability of an increase were p (physical or real probability measure), this quantity would have been irrelevant when pricing derivatives under no-arbitrage.

ACKNOWLEDGMENTS

The author would like to thank Maciej Augustyniak, Dick Joss, Joseph D. Koltisko, Pierre-Laurence Marchand and Jean-François Renaud for their valuable comments.

END NOTES

1 By pricing or price we mean finding the value of a tradeable security as of a given date.

2 Some experts blame these supercomputers and their algorithms for the Flash Crash of May 2010.

3 In a single-step binomial tree (see Appendix), if there is no arbitrage between the stock and the Treasury bond, then the risk-neutral probability is unique.

4 Those are the Cox-Ross-Rubinstein and Jarrow & Rudd binomial trees for example.

5 This is the case when the payoff of any of the three fundamental assets cannot be written as a linear combination of the other two. Otherwise, one of these fundamental assets would be redundant and could not be used to replicate a fourth one. Mathematically, this is the necessary condition to solve a system of three equations with three unknowns, i.e., the matrix built with the payoffs of the fundamental assets should be of full rank.

6 In a population, wars and epidemics are factors that create dependence between lives. However, these risks are often excluded in life insurance policies. Moreover, within a couple, it is generally recognized that spouses' lives are somewhat dependent. Finally, it is often assumed that financial markets do not affect mortality experience and vice-versa.

7 The usual formulation of the law of large numbers in that context is that the mean proportion of deaths goes to 0.01 with certainty. However, in large portfolios, it can be easily seen that the standard deviation of the number of deaths relative to the mean, will to 0. Hence, the error committed by assuming 100 deaths should be small in relative terms.

Mathieu Boudreault, FSA, Ph.D., is associate Professor of actuarial science in the Department of mathematics, at the Université du Québec à Montréal. He can be reached at boudreault.mathieu@uqam.ca.