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Residual Risk When Hedging Delta and Rho of Equity Options

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This article explores the effectiveness of hedging delta and rho of equity options. This provides insight into the frequency and severity of losses due to not hedging volatility risk (vega) or other higher order risks, often known as "greeks." Ten-year equity put option strategies were chosen to represent the risk of hedging guaranteed benefits attached to variable annuities, while one-year put and call strategies were modeled to investigate the risk of hedging equity index interest credited to fixed indexed annuities. In both cases, the value of the option was compared to the value of delta/rho hedges in the tail of both actual historical and simulated scenarios for equity returns and volatilities. The historical path of interest rates and equities was generally used to highlight the hedge impact for different implied volatility assumptions in each example.

FRAMEWORK: DATA AND MODELS FOR INTEREST RATES AND EQUITIES

To simulate the investment environment for hedging, a model was built in Visual Basic for Applications in Excel. Input data included the daily closing value of the S&P 500 index price from Jan. 2, 1962 through Sept. 23, 2014 and daily treasury yields for one, two, three, five, seven, and 10 year bonds. Any missing Treasury yields were estimated using interpolation. A cubic spline was used to interpolate Treasury yields at six-month intervals, and corresponding present value factors for each six month period were boot-strapped. Then for intervening discount factors, the model assumed a constant interest rate during each six-month segment of the curve. Thus an entire yield curve was built for each business day. The model captures the short rate each day for a given put issue date as the one-year Treasury yield. Thus each day's short rate came from a new yield curve as the model moved from one business day to the next. Lastly, the short rates from any given put issue date to the exercise date were accumulated to build a discounting curve for the put. That will be referred to as the Actual Interest Curve. It is used to accumulate and discount actual payoffs for evaluating effectiveness.

Besides the bulk of the simulations that used historical equity returns, two tests were done using a stochastic volatility model to change volatility quarterly and generate equity returns. This provided two paths of stochastically generated equity index scenarios.

In these two runs, equity returns were generated with the following algorithm:

> Let $\sigma(t)$ be the volatility for quarter t. Let ϵ be a random normal variable. Then the stochastic volatility for the quarter is calculated by two steps. First, the intermediate variable v is calculated. Based on a random number, one of three formulas is used to calculate v. The formulas and probability attached to each are as follows:
>
> 99%: $v = .1*\exp\{.07\epsilon - .07^2/2\}$
>
> .5%: $v = .4$
>
> .5%: $v = .65$

Then once this calculation is done, we set $\sigma(t) = \max\{v, .65\ \sigma(t-1)\}$. We use this volatility, historical treasury rates, dividends, and a risk premium of about 2 percent to generate the stochastic scenarios.

This procedure is roughly calibrated to historical S&P 500 returns.

10-YEAR PUT OPTION HEDGING

For the long-dated case, the model sold a 10-year ATM put for each trading day from 1962 to 2004 with a notional amount of 100 and implied volatility of 27 percent. The last put was sold on 9/27/2004 for a total of 10,758 puts. The model hedged delta with S&P futures. For simplicity, it assumed futures expire on each trading day. The model hedged rho with a zero-coupon treasury note that had a maturity date equal to the put expiry date.

Simplifying assumptions were made about futures and treasuries mechanics. Transaction costs such as ticket fees, roll costs and initial margin were not reflected. Futures and treasuries were rebalanced daily at the close.

Tests Performed

The model tested conventional delta/rho hedging of an at-the-money 10-year option with various volatility assumptions and daily rebalancing based on the indicated risk statistics of the option.

The model varied the equity index volatility used to calculate delta and rho from 16 percent to 35 percent, resulting in 20 separate test runs. For any given test run, implied volatility was held constant for all puts at all tenors.

The model also looked at a reduced trading algorithm whereby delta and rho were calculated at two different equity volatilities and no trade was made if the two volatilities suggested trades of opposite sign. If the suggested trades had the same sign, then the smaller of the two trades was made. This was tested as a range around 17 percent volatility and also tested as a range around volatility in the high 20s.

A special run was done with randomly shuffled daily equity returns to investigate how autocorrelation of equity volatility affects the result.

A second test was run using historical returns on a put struck at 50 percent. This was done with volatilities ranging from 20 percent to 33 percent.

Statistics Calculated

All values were discounted using the Actual Interest Curve.

For each 10-year hedging simulation, the hedging cost was expressed as a percentage of the initial notional amount. "Hedge slippage" was measured as the incremental cost of the dynamic hedge program vs. the initial cost of the option (which assumed implied volatility of 27 percent). If real-world experience evolved exactly as the Black-Scholes formula indicates, then the average hedging cost in the output tables below would be the same as the price of the option. Since the real-world historical scenario excludes a market risk premium, the average hedging cost should be expected to be less than the price of the option. However, the tail of the hedging cost distribution indicates the amount of unexpected losses the hedger would have experienced by limiting the program to a first-order delta-rho strategy.

For each hedging volatility, the average hedging cash flows were calculated and the percentile results assuming an initial cash position equal to the price of a 27 percent volatility put were tabulated. The model evaluated hedge slippage at the 90 percent, 95 percent, 97.5 percent, 99 percent and 99.9 percent point in the distribution as well as the maximum observed difference. The put issue date for each of the above percentiles was also captured.

For each hedging volatility, the hedge efficiency was calculated as the square root of the quantity of one minus the ratio of the variance of accumulated hedged results to the variance of unhedged put payoffs.[3]

Numerical Results

The average historical realized equity volatility across all the 10-year puts is almost exactly 16 percent.

For the basic historical test, the average cost of hedging is fairly insensitive to the hedging volatility, but the dispersion of results by the various measures above were all minimized around 28 percent to 30 percent volatility. The volatility assumption also impacted which dates corresponded to the highest hedging cost. Results were similar for a 50 percent strike except that lowering the volatility assumption reduced the average hedging cash values at the cost of increasing the dispersion of results. Hedge efficiency could not be calculated for the 50 percent strike as there was never actually a payoff.

The results are summarized in the following tables which show the tail of the distribution of realized hedging cost (assuming an initial cash position corresponding to an option premium calculated at a 27 percent volatility for the percentile calculations) as a percent of the initial notional amount, at various assumed implied volatility assumptions.

TABLE 1:
10-Year Put, 100% Strike

Pct'ile\Volatility	16%	20%	25%	30%	35%
90	0.96	2.42	2.85	2.62	2.33
95	-0.50	1.68	2.47	2.28	1.57
97.5	-2.18	-0.21	2.19	2.15	1.38
99	-4.72	-1.08	1.95	1.85	1.25
99.9	-6.46	-2.39	1.29	1.49	0.91
100	-7.17	-3.39	0.67	1.36	0.74
Avg Hedge CF PV	-5.20	-5.21	-5.13	-4.96	-5.12
Std Dev(Hedge)	3.68	2.96	2.52	2.46	2.62
Hedge Efficiency	56%	75%	82%	83%	81%

TABLE 2:
10-Year Put, 50% Strike

Pct'ile\Volatility	20%	25%	30%
90	0.11	-0.12	0.05
95	-1.26	-0.88	-0.15
97.5	-2.52	-1.30	-0.34
99	-3.20	-1.58	-0.44
99.9	-3.59	-1.82	-0.61
100	-3.70	-1.88	-0.65
Avg Hedge CF PV	-0.72	-0.91	-1.06
Std Dev(Hedge)	1.14	0.82	0.64
Hedge Efficiency	n/a	n/a	n/a

At a 100 percent strike, realized actual interest convexity is worth about 1 percent of equity volatility. That is to say, with historical interest rates, the average cost of hedging corresponds to 17 percent volatility. If interest rates are levelized and frozen, then the average cost of hedging corresponds to 16 percent volatility.

At a 50 percent strike, hedging costs correspond to 21 percent to 23 percent volatility depending on the hedging volatility assumption. This sounds like volatility skew, but when constant

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dividends and interest are used instead of actual, the hedging costs correspond to a 17 percent volatility implying that most of the added cost is due to interest convexity, not volatility skew. This makes sense, since it is hard to imagine that volatility skew matters as much on long-dated options as on short-dated options.

The reduced trading algorithm worked slightly better when the band was around 17 percent as compared to a constant 17 percent hedging volatility, but at the higher volatility test mentioned above, it performed noticeably worse. Given the complexity and unimpressive results of the reduced trading algorithm, this does not seem like something worth further consideration.

When the daily returns were randomly shuffled, assuming a level 16 percent volatility resulted in 99 percent hedge efficiency, implying that the shuffling obscured legitimate volatility trends. Hedging costs are primarily a function of volatility, not market direction.

Results for the two stochastically generated paths were very much different from both each other and the historical path based results. For the first path, the percentiles and equivalent volatility looked best for a 23 percent volatility, the average cost dropped as volatility went up, but the standard deviation of the hedge cost was lower for lower volatilities. For the second path, while the average hedge cost was similar to historical, there were a lot of puts with a very high hedge cost, in other words, results were much more dispersed, particularly at the higher hedging volatilities. The divergent results from the stochastically generated paths strongly suggests that the results suggesting using a high volatility to get less divergent results are an overfit to the historical data. Note that there is a lot of overlapping in our 50 years of data, since if we prohibited overlapping data we would have modeled only five puts.

ONE-YEAR PUTS AND CALLS

The study was repeated for one-year puts and one-year calls. The last option was sold on 9/23/2013 for a total of 13,021 options. The average realized volatility across all the one-year options was almost exactly 14.8 percent.

As the hedging volatility increases, the average hedge cash flow worsens, but the dispersion of results in the tail improves while hedge efficiency is nearly constant. The results are similar for both puts and calls which is to be expected due to put/call parity or equivalent gamma (one implies the other). Interest rate volatility impact is different between the two, however.

The results are summarized in the following tables which show the tail of the distribution of realized hedging cost (assuming an initial cash position corresponding to an option premium calculated at a 27 percent volatility for the percentile calculations) as a percent of the initial notional amount, at various assumed implied volatility assumptions.

TABLE 3:
1-Year Put, 100% Strike

Pct'ile\Volatility	16%	20%	25%	30%	35%
90	-6.42	-6.56	-6.81	-6.80	-6.60
95	-8.63	-8.51	-8.35	-8.17	-8.15
97.5	-10.12	-10.30	-10.75	-10.38	-10.78
99	-12.27	-11.92	-12.10	-13.18	-14.29
99.9	-16.89	-15.76	-15.06	-15.45	-16.54
100	-23.10	-20.66	-18.48	-16.90	-16.73
Average Hedge CF PV	-4.70	-4.83	-4.95	-5.04	-5.09
Std Dev(Hedge)	4.65	4.64	4.67	4.72	4.78
Hedge Efficiency	78%	78%	77%	77%	76%

TABLE 4:
1-Year Call, 100% Strike

Pct'ile\Volatility	16%	20%	25%	30%	35%
90	-7.21	-7.22	-7.31	-7.30	-7.14
95	-8.72	-8.85	-8.99	-8.91	-8.97
97.5	-9.39	-9.74	-10.15	-10.44	-10.92
99	-11.70	-11.15	-11.81	-12.33	-13.21
99.9	-15.92	-15.02	-14.32	-14.98	-16.11
100	-22.01	-19.58	-17.39	-15.81	-16.26
Average Hedge CF PV	-7.02	-7.16	-7.28	-7.36	-7.42
Std Dev(Hedge)	5.74	5.80	5.90	5.99	6.07
Hedge Efficiency	84%	84%	83%	83%	82%

CONCLUSIONS

As we have seen, the best volatility to use for delta hedging is revealed only in retrospect which is of little practical use. Using something drastically different than a reasonable expectation of future volatility, however, is not practical. On the other hand, a precise prediction of volatility has a smaller impact on hedge efficiency than one might intuitively expect. Second, we see that hedging only first order risk results in hedge efficiency significantly lower than 100 percent. Hedging only the first order risks may be a problem where the unhedged risk is large compared to the overall size of the insurer and/or earnings volatility is a concern.

For questions and comments on this study, as well as much more detailed statistics and associated graphs, please send an email to mark@appliedstochastic.com. ■

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ENDNOTES

1 http://data.treasury.gov:8001/Feed.svc

2 In certain cases the result was also expressed as an equivalent number of percentage points of equity volatility, by comparing hedging cost to the put prices calculated at various volatilities and the forward curve on the put issue date.

3 The standard deviation of the unhedged results was 4.45, 7.36, and 10.62 for the 10-year put, one-year put, and one-year call respectively.

4 Average put costs are .73, .88, 1.05 at 21 percent, 22 percent, and 23 percent volatility respectively.