Introduction

Hans Bühlmann

During the late 1980s and early 1990s we have seen insurance and financial markets interact more closely than ever before. New tools for managing insurance risk emerge frequently, usually with roots in financial markets. A strong motivating factor has been the record number of natural catastrophes and their impact on insured losses. The potential for earthquake and storm losses in the U.S. exceeds insurance market capacity. Reinsurers, banks, security exchanges, and others are developing products, sometimes involving creation of new organizations or joint ventures, to profitably meet this demand. The 1995 Bowles Symposium provided a forum to explore some of these products and to discuss new possibilities.

The Chicago Board of Trade (CBOT) futures on catastrophic loss indices provided an early example of these products, perhaps the first exchange traded insurance derivative securities. Over-the-counter products such as bonds with coupon or principal payments tied to an insurance portfolio loss ratio are examples of privately placed insurance securities. There is more information available on the publicly traded examples, and, naturally, the majority of the symposium papers treat these products. This introduction was written after the symposium to serve as a guide to readers of the papers that were presented in Atlanta. The symposium was organized in three sessions, as follows:

Session 1. *Interplay between Capital Markets and Insurance*
- Leader: Hans Bühlmann
- Presenters: Stephen P. Lowe, Aaron Stern
- Audience discussion led by Frank Pierson, Jon Roberts

Session 2. *Insurance Futures*
- Leader: Paul Embrechts
- Presenters: Prakash Shimpi, Morton Lanc, Patrick L. Brockett
- Audience discussion led by Paul Embrechts

Session 3. *Actuarial Bridge between Insurance and Finance*
- Leader: James C. Hickman
- Presenters: Elias Shiu, Hans Gerber, Graham Lord, Frank Sabatini
- Audience discussion led by Jim Hickman

The symposium began with a welcome from John Hogan, Dean of the College of Business Administration at Georgia State University. The first Thomas P. Bowles, Jr. Chairholder, Professor Hans Bühlmann, was introduced by Professor Samuel H. Cox of GSU. In the opening presentation, Bühlmann showed that, in general terms, all securitization can be viewed as cross-hedging. He illustrated this fundamental notion with catastrophe insurance and reinsurance, but the principle was applied generally. This created the environment for the other contributors.

Continuing in the first session, Stephen P. Lowe discussed U.S. property/casualty risk-based capital and some of the implications of capital structure, product pricing, and insurance portfolio strategy. He showed that actuarial theory and financial theory of asset portfolios converge in practice. In the third presentation in the first session, Aaron B. Stern described the interplay between catastrophe exposures and capital markets. Stern argued that establishing catastrophe insurance as an asset class will provide capital to cover potential earthquake or hurricane losses of $50–100 billion. The presentations in the first session were practice-oriented. They did not refer to the difficult mathematical problems encountered in developing pricing and reserving methods for insurance derivative securities.

The second session, led by Professor Paul Embrechts, provided several advances in the modeling of insurance derivatives, although the first presentation was nonmathematical. The audience participated fully in a vigorous discussion, which ended as we adjourned for lunch. Professor Harry Panjer gave a luncheon speech on the research activities of the Society of Actuaries Foundation.

In the beginning of the second session, Prakash Shimpi discussed the concept of tradable risk in general terms while using the CBOT futures as an illustration. This led to a vigorous and completely nonmathematical discussion of why insurance risk is traded. Dr. Morton Lane followed this with a more technical discussion, showing how insurance market prices reveal information about the underlying loss distribution. This interesting notion was conveyed clearly without heavy mathematical machinery. In the third presentation,

Professor Embrechts laid out the sophisticated mathematical modeling required for a rigorous valuation of insurance derivative securities. As it is applied in the usual financial setting, arbitrage pricing requires (or is roughly equivalent to) complete markets. The financial market may not be complete—financial prices may not contain adequate information about insurance risk. Professor Embrechts showed how this difficulty can be approached through the use of utility theory or general equilibrium techniques. Professor Brockett referred to a related problem as information asymmetry. In his presentation he showed how to determine optimal bounds on insurance derivative prices based on the available information. If the information is complete, the methods coincide with arbitrage-pricing methods. The second session ended after a very active discussion with various members of the audience. A beer and wine social hour on the patio concluded the symposium activities for the day.

Professor James C. Hickman opened the third session Friday morning. The "actuarial bridge" is not one bridge, Dr. Hickman explained. This session illustrated three such "bridges," which allow for interaction of actuarial and financial theory and practice. Arbitrage pricing was conveniently stated in terms of a Martingale measure. In the final session, Hans Gerber and Elias Shiu revealed a surprising use of an old actuarial technique—the Esscher transformation of a loss distribution can be applied to the distribution of returns of a security to easily obtain an explicit representation of an equivalent Martingale measure. Their presentation was based on work that won the Annual Prize of the Society of Actuaries and the Halmstead Prize of the Actuarial Education and Research Fund.

In applying arbitrage pricing to insurance products, or more generally, in simulating blocks of insurance business, getting a reasonable model of the term structure of interest rates is often the most important step. In the second presentation, Graham gave an overview of several of the principal approaches to term structure modeling. Several of the methods are currently used in financial and actuarial pricing and reserving. Dr. Lord demonstrated some of the difficulties encountered in applying these methods to practical problems. Asset liability management is a key application of term structure modeling. Frank Sabatini surveyed the key elements of asset liability management, emphasizing life insurance business. The typical life company is faced with what he described as a short straddle challenge.

The symposium ended with a final audience discussion on the interaction of finance and insurance that is fostered by actuarial science.

The symposium was above all intended to promote an understanding of the basic ideas of securitization and to discuss the fundamentals of new emerging products. The interaction between innovations in the financial markets and the needs of the insurance sector has been an excellent topic for bringing financial specialists and actuaries closer together so that they can learn from each other.

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I

Bounds on the Price of Catastrophe Insurance Options on Futures Contracts

by Patrick L. Brockett, Samuel H. Cox, and James Smith

Abstract

Insurers can hedge the risk of assets and interest sensitive liabilities with readily available financial instruments. Similarly, certain types of loss exposures which are strongly correlated with the valuation of special commodities (for example, default risk in mortgage insurance in Houston, Texas is negatively related to the price of oil, etc.) can also be hedged by utilizing the commodity futures market. Until recently, however, the management of other underwriting risks has primarily been limited to reinsurance contracts. In 1992 the Chicago Board of Trade (CBOT) introduced insurance futures contracts based on the loss ratio of a portfolio of actual insurance policies over a given period of time. This is conceptually similar to more commonly traded stock index futures, with the exception that there is no intrinsic underlying tradable asset upon which to base no arbitrage pricing, and there is only partial information available about the dynamics of the underlying loss ratio series. This paper describes methods for determining bounds on the price of derivative instruments, such as options on insurance futures, given only market information.

Introduction

The first recorded use of derivative contracts to insurable risk was an option contract covering grain bound for Venice in the seventh century (Ferrick, Faber, and Dumas Limited [1994]). The CBOT began trading in grain futures in 1840, and more recently (since 1972), the Chicago Mercantile Exchange and many other exchanges throughout the world trade in currency futures and other financial derivatives. The rapid growth of the use of financial futures and options (and other financial derivative products such as interest swaps, look back options, compound options, etc.) has shown a development from the arena of *product* risk management (for example, bonds, swaps, currency futures and options, etc.) to an arena of *exposure* risk management (for example, interest rate, stock market, foreign exchange rate options and futures, etc.). The latest step in this process was the CBOT's introduction in December 1992 of insurance futures to manage exposure in the general area of the insurance line upon which the contract is based. Cox and Schwebach (1992) discuss the mechanics of these insurance futures, their purpose and use, and how they fit into the financial market in general. Insurance futures are based on the loss ratio for a portfolio of actual insurance policies over a given period. The loss ratio is analogous to the stock market index underlying a stock index futures. An important difference is that the insurance portfolio loss ratio is not readily available, and there is no purchasable set of insurance contracts by which to price the futures contract (in contrast to the stock market). The insurance futures contract is also the basis for put and call options, and option spreads.

The above-listed insurance futures contracts and options have payoffs which are similar to traditional reinsurance in the sense that if one is long in the futures contract, and the loss index goes up indicating increased losses in the insurance market, then these

losses are partially offset by financial gains in the financial market. Thus, the CBOT complex offers an alternative to reinsurance that is easily reversible and low cost. The call spread is the most popular contract. It was designed to compete with reinsurance stop-loss contracts. Insurance futures contracts also provide an investor or speculator easy access to the insurance market since there is no licensing, only margin requirements. The idea behind the introduction of insurance futures is to allow more efficient use of capital. The beneficiaries ultimately are insurance consumers and insurance company owners. In fact, it has been speculated that more capital in the reinsurance market would possibly have helped to alleviate the insurance crisis of the late 1980s. [The market for reinsurance is relatively difficult to enter when compared to financial markets, a fact that some such as Berger, Cummins, and Tennyson (1992) assert may have contributed to the general insurance crisis of 1984–86.] Insurance futures provide an alternative way for capital to be used like reinsurance. The result should be more stable consumer prices.

Although the CBOT has, from time to time, proposed futures on other insurance lines such as health, as of May 1995 there was but a single insurance futures market in operation, and it involved what is known as catastrophe insurance, loss ratios being based upon specified lines of insurance susceptible to catastrophic losses from natural events such as hurricanes, earthquakes, floods, wind damage, etc.* The Insurance Services Office uses actual loss and premium data from participating insurance companies to aggregate these lines of insurance (homeowners property loss, automobile physical damage, etc.), and the CBOT creates a market based upon this loss ratio index. Cox and Schwebach (1992) suggest that prices of insurance futures and options can be priced using the Black-Scholes framework as a crude approximation. D'Arcy and France (1992) give a convincing argument for the viability of catastrophe futures. Boose and Graham (1993) assess the viability of an insurance futures market using an empirical model. A successful catastrophic insurance futures and option market could reduce the threat of insolvency due to lack of capacity to withstand catastrophic loss as occurred in Florida after Hurricane Andrew. Boose and Graham (1993), however, conclude that the positive factors effecting success of the insurance futures market (large size of the insurance market, low transactions cost, high liquidity) are offset by the infrequent release of loss information. The Standard & Poor's 500 stock index is distributed to the market every 15 seconds. In contrast, the loss ratio and related aggregate loss data underlying the insurance futures contract is released to the market only twice: once at the end of the loss period and once at the settlement date, although monthly reports of industry catastrophe losses are available to the public. This lack of information should be taken into account in models used to price insurance and futures options.

Helyette Geman has presented informally joint work with David Cummins (1994) on valuation of insurance futures. Their approach is based on Asian options. This Asian option is based on an average of prices before the exercise date. Since the CBOT contract is based on a loss ratio, of which the numerator is the sum of prior losses, the Asian option feature fits the actual contract very well. However, their approach is based on the assumption that all traders can observe the loss ratio continuously. There is no published model which allows for this important feature; in the actual insurance futures market traders may have different sets of information regarding the underlying index.

The purpose of the current research is to establish a relation between information (or lack of information) about the loss ratio and insurance futures prices based solely on observable market prices. The potential for insurance futures to play an important role in hedging underwriting risk is enormous. However, the absence of models which take into account the lack of information presents a barrier to their success. The results should allow traders to better understand this unusual aspect of the catastrophe insurance futures market. With better understanding, speculators and hedgers may be more willing to enter the market.

Methodology

In this section we present one way to allow for lack of information. Let $S(t)$ denote the aggregate losses paid during the interval $[0,t]$. The loss ratio on the settlement date T is $S(T)/Q$ where Q is an estimate of the premiums written during the interval $[0,t]$. The value of $S(t)$ is known at only a few points in time (four points for a three-month contract, for example), but trading in the futures contract with a price denoted by $F(t)$ takes place continuously. In a financial market which is "frictionless" (in that traders can buy or sell as many securities as desired at posted market prices

*A new crop of insurance futures and options markets has been opened subsequently.

without incurring transactions costs) and allows no arbitrage, the fundamental theorem of asset pricing implies the existence of a "risk-neutral probability distribution" (or "equivalent martingale measure") such that the current price of any asset is equal to its expected future value discounted at a risk-free rate r. [See Harrison and Kreps (1979) for a detailed discussion of this theory.] For example, since a futures contract requires no cash outlay at time 0, the futures price must be equal to the expected value of the settlement value with the expectation calculated using the risk-neutral distribution:

$$F(t) = E\left[\frac{S(T)}{Q}\right].$$

As another example, a call option written on a futures contract gives the owner the right, but not the obligation, to buy the futures contract for some predetermined "strike price" K at expiration T. Assuming the owner of the option follows the optimal exercise policy (exercise at expiration if and only if the futures price exceeds the strike price), the value of the option at expiration is then given by $(F(T)-K)^+=\max[0, F(T)-K]$, and its value at any earlier time $t(t<T)$ is given by

$$E\left[\frac{(X-K)^+}{(1+r)^{T-t}}\right],$$

where E denotes expectations taken with respect to the risk-neutral distribution.

The values of more exotic options can be similarly represented as the expected value of some payoff function under the risk-neutral distribution. For example, a call option spread is a CBOT contract which provides the owner an exercise value of 100,000* $\{\mathrm{Max}(F(T)-K_1, 0)-\mathrm{Max}(F(T)-K_2, 0)\}$ where the strike prices K_1 and K_2 and exercise date T are specified in the contract. This is equivalent to buying a call option with an exercise price based on a loss ratio of K_1 and selling based on K_2. Let $u(s)$ denote the payoff function:

$$u(s) = \max(s-K_1, 0) - \max(s-K_2, 0)$$
$$= \begin{cases} 0 & \text{if } s \le K_1 \\ s-K_1 & \text{if } K_1 \le s \le K_2 \\ K_2-K_1 & \text{if } s \ge K_2. \end{cases}$$

Suppressing the contract face amount of 100,000, we can write the value at expiration as $u(F(T))$ and at any earlier time t, the price would be given by:

$$E\left[\frac{u(F(T))}{(1+r)^{T-t}}\right],$$

where, again, E denotes expectations taken with respect to the risk-neutral distribution.

In general these risk-neutral distributions will be unique if and only if the market is "complete" in that the underlying risk can be perfectly hedged by trading existing securities [see Harrison and Kreps (1979)]. This completeness assumption underlies the standard Black-Scholes valuation methods and may be appropriate when valuing put and call options on a stock or stock market index where the underlying asset (the stock or stocks making up the index) are continuously traded and prices are continually updated. But, in the case of insurance futures contracts and options on these futures contracts, the underlying asset (the loss ratio S) is not traded and its value is infrequently updated so that, consequently, the assumption of completeness seems inappropriate. In the incomplete markets case, we cannot identify a unique risk-neutral distribution, but we can use available information to restrict the set of possible equivalent risk-neutral distributions and compute upper and lower bounds on the price of any security.

The bounds on securities prices thus depend on what information we assume to be available about the underlying uncertainty. For example, rather than assuming that traders have precise information about the distribution for $S(T)$, we might assume, as in Brockett and Cox (1985) and Cox (1991), that only a range of values is known:

$$\mu_1 \le E[X(T)] \le \mu_2 \text{ and } \sigma_1^2 \le \mathrm{Var}[X(T)] \le \sigma_2^2.$$

Alternatively, as in Smith (1995), we might assume that we know prices for certain securities and want to determine bounds on the prices for others that are consistent with the known prices. For example, we might be given prices for a futures contract and several call option contracts and seek bounds on some other call option contract. We will examine this second example in detail shortly.

The general framework of our analysis is concomitant with the mathematics of the general moment problem from statistics [cf., Kemperman (1987)]. Specifically, we assume that we are given $n+1$ real valued functions $f_i(x)$, $i=0, 1, \ldots, n$, whose expectations are assumed known and finite, that is, we know

$$\mu_i = E_p[f_i] = \int f_i(x)dP(x), \text{ for } i = 0, 1, \ldots, n.$$

Though we know these moment values, the specific underlying distribution P is unknown. For convenience we take $f_0(x)=1$ and $\mu_0=1$ so that the first moment constraints specify that we are dealing with a probability measure. We let $\mu=(\mu_0, \mu_1, \ldots, \mu_n)$ and $\mathbf{f}=(\mathbf{f}_0, \mathbf{f}_1, \ldots, \mathbf{f}_n)$ and assume the moment functions are linearly independent. Now, given a real valued *objective function* ϕ, our goal is to compute

$$\inf E_P[\phi] \text{ and } \sup E_P[\phi] \qquad (1)$$

where the infimum and supremum are taken over the set of all distributions P such that $E_P[\mathbf{f}]=\boldsymbol{\mu}$. In our context, we allow P to vary over the set of all risk-neutral distributions consistent with the given information.

There are two different methods for solving these optimization problems, both of which lead to the same answer. In the "primal" approach, one seeks a distribution P that attains or approaches the bounds in (1): the theory of the moment problem [cf., Brockett and Cox (1985), Kemperman (1987), Smith (1995)] shows that the infimum and supremum are obtained by distributions which are discrete at most $n+1$ mass points; therefore, we can restrict our search to this subset of distributions. Alternatively, in the "dual" approach, we seek linear combinations of the given moment functions which lie above (for the upper bound) and below (for the lower bound) the function whose expectation is to be bounded and seek the polynomials with the least and greatest expectations. Under certain specific conditions regarding the moment functions f_i and objective function ϕ (for example, if they form a Chebycheff system of functions for example), we may find explicit formulas for the solutions to these optimization problems [cf., Brockett and Cox (1985)]. In the more general setting, while explicit formulae are not known, the problem can readily be solved using numerical methods. For example, Smith (1995) describes a procedure where one first solves a discretized version of (1) using standard linear programming methods and then obtains an exact solution by "polishing" this approximate solution by solving a low-dimensional, non-linear programming problem.

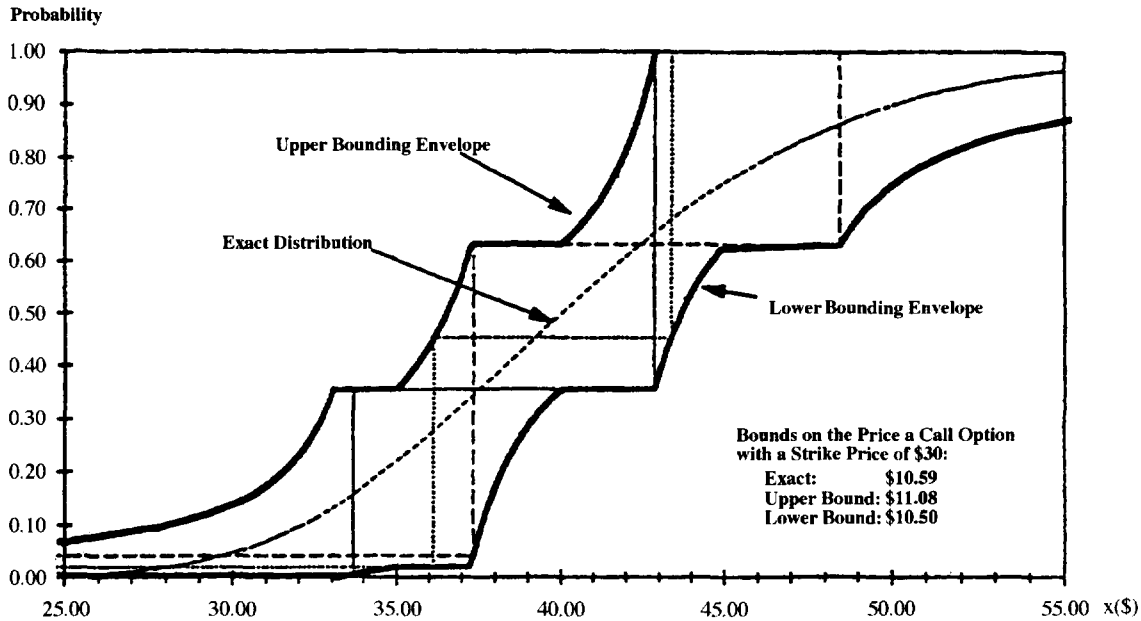
An Example

To illustrate this methodology, we consider a specific example where we are given the current (time 0) value of an asset (X_0) and the prices for a series of call options on this asset that expire at time T. To make the example concrete, suppose the current asset price is \$40 and call options that expire in four months with strike prices \$35, \$40, and \$45 have current prices \$6.26, \$3.08, and \$1.26, respectively. We will then compute bounds on the underlying (cumulative) risk-neutral distribution as well as bounds on a call option with a \$30 strike price. These prices are consistent with the Black-Scholes model with a risk-free discount rate of 5% per year and an annual volatility (σ) of 30%: in the Black-Scholes model, the risk-neutral distribution is log normal ($\ln(X/X_0)$ and is normally distributed with mean $(r-\sigma^2/2)t$ and variance $\sigma^2\, t$) so that the call option with a \$30 strike price would have a price of \$10.59.

To place this example in our general framework, we let $X=[0, \infty]$ represent the possible asset prices at time T. The moments correspond to the current prices for the traded securities, and the moment functions represent the discounted time T payoffs as a function of the underlying asset price at time T. For the asset itself, we take $f_1(x)=x/(1+r)^T$ and have a corresponding moment value $\mu_1=\$40$. For the call options, the moment function is given by $f_i(x)=(x-K_i)^+/(1+r)^T$ where K_i denotes the strike price and the moments μ_i correspond to the given prices for the call options. We can compute bounds on the underlying (cumulative) risk-neutral distribution at some point d by taking the objective function $\phi(x)$ to be a step function with a step at d (since $P(X \le d)=E[\phi]$); by varying the point d we can trace out the entire distribution function. We can compute bounds on the call option with \$30 strike price by taking the objective function $\phi(x)$ to be $(x-30)^+/(1+r)^T$.

The results for this example are summarized in Figure 1. In all cases, the bounds are achieved by distributions with no more than three points of support; a few of these distributions are shown in the figure. Here we find that the bounds on the underlying risk-neutral distribution are quite loose. From the "dual" perspective described above, this is a reflection of the fact that step functions used in computing these bounds are poorly approximated by the given moment functions: a "step" security that pays \$1 if and only if the stock price is less than, say, \$35 cannot be approximated very well by a portfolio consisting of the stock and given call options. The bounds on the value of a call option with a strike price of \$30, however, are somewhat tighter as we can better approximate its payoffs using the given securities.

FIGURE 1
BOUNDS ON THE RISK-NEUTRAL DISTRIBUTION IN THE OPTION PRICING EXAMPLE



Some Extensions

This general framework can be extended to incorporate additional information about the underlying distribution by including this information into the constraint set in the optimization outlined previously. For example, another useful constraint is to allow the external measure to be "not too unusual" in the sense that its deviation from some prespecified potential risk-neutral distribution is not too great. For example, the Black-Scholes model assumes the risk-neutral distribution to be the lognormal distribution, and we may want to allow limited deviations from this distribution. A measure of deviation which can be used in this context is the entropic or information theoretic distance measure

$$I(P_1,P_2) = \int p(x)\ln p(x)dP_2(x)$$

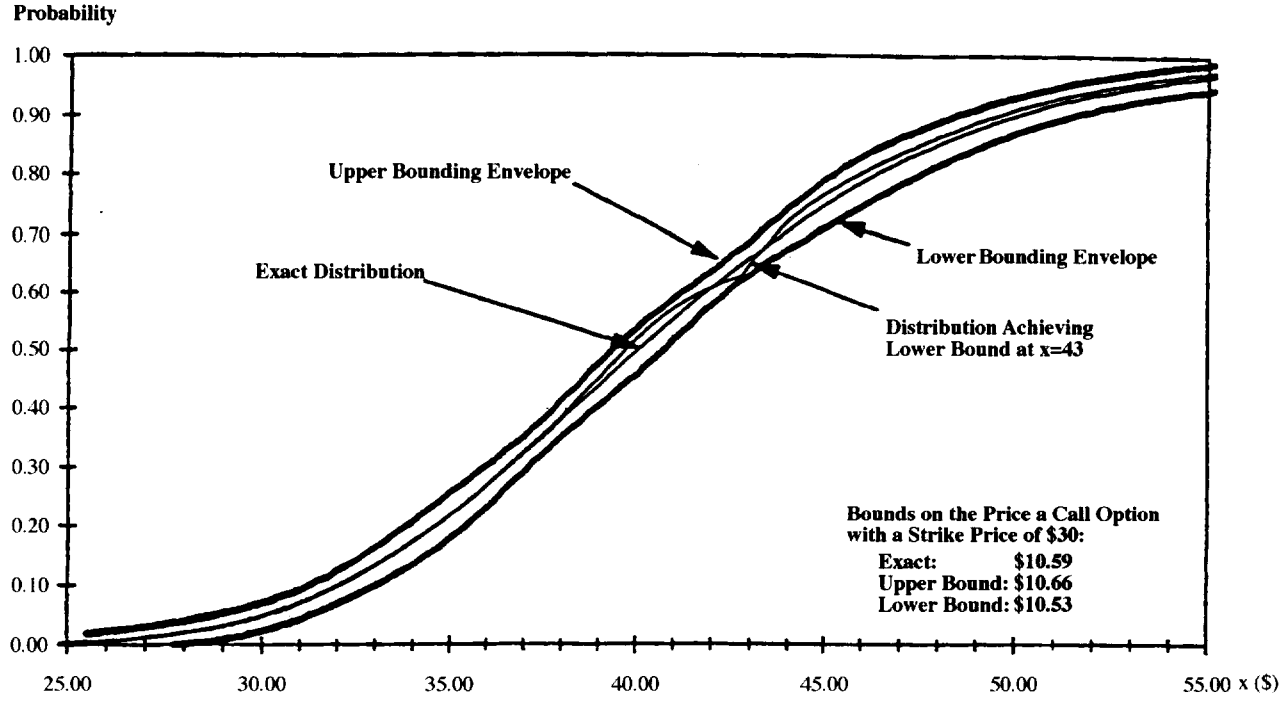
where $p(x)$ is the Radon-Nikodym derivative of P_1 with respect to P_2. If $P_1=P_2$, then $I(P_1, P_2)=0$, and $I(P_1, P_2)>0$ if $P_1 \neq P_2$. The minimum information distance estimate of P_2 subject to the constraint set is called the MDI estimate, and is discussed in detail in Brockett (1991).

In the above option pricing example, we are given prices for an asset and three call options on the asset, and we seek bounds on the underlying risk-neutral distribution and the price of call option with a $30 strike price. The results for this example are summarized in Figure 2. Here we have taken the prior P_2 to be a log-normal distribution (as assumed in the standard Black-Scholes model) with parameters consistent with the given asset and call prices. In this case, the minimum possible information distance is 0, and we take our entropy or information cutoff to be 0.02 (so that the unknown risk-neutral distribution is somewhat "close" to lognormal. Here, we see that the bounds on the risk-neutral distribution are much tighter than they were in the unrestricted case (compare Figures 1 and 2), and the bounds on the price of a call option with a $30 strike price are tighter as well: [10.53, 10.66] versus [10.50, 11.08]. (See Smith [1995] for a discussion of how to compute these values.)

Conclusions and Implications for Future Research

This paper has shown how to use the information available from market prices to determine bounds on the value of derivative instruments like options on insurance futures without completely specifying all

FIGURE 2
RESULTS FOR THE OPTION PRICING EXAMPLE WITH ENTROPY CONSTRAINTS



information about the distributions involved. That is, unlike the Black-Scholes option pricing formulae which require a lognormal probability distribution to obtain exact values for all options, this method assumes no knowledge of the probability distribution other than the values of observed market prices which are then viewed as "moments" relative to this unknown probability distribution. For a specific option under investigation, bounds on the value of the option are constructed using solutions to the moment problem. For catastrophic insurance futures, the assumption of lognormality is questionable, and the extent to which the Black-Scholes formulas give reasonable answers is a subject for further research. Also to be investigated are other insurance futures and options markets such as the crop insurance futures and options markets that have recently been created.

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II

Crosshedging of Insurance Portfolios

by Hans Bühlmann

Motivation

New tools for managing insurance risk are being created everywhere. With a special emphasis on catastrophe insurance and reinsurance, we want to better understand what these tools are basically trying to achieve.

It is important to note that the concept of *crosshedging* gives us the key to the actuarial understanding of most new tools arising in the area of alternative risk transfer mechanisms. *Securitization* of insurance, which is the theme of this symposium can, as I see it, almost always be seen as an exercise in *crosshedging*.

The *practical* motivation for my presentation theme comes from catastrophe insurance and reinsurance. I am therefore first going to model this important branch of insurance. The model will then serve as a guide throughout the whole lecture.

The Catastrophe Risk Model

Assume that your insurance company has written a portfolio of homeowners policies in a clearly defined geographic area. The total claims originating from this portfolio for one specific time interval (year, quarter) is called S.

As standard in nonlife actuarial techniques, assume

1. $S = \Sigma_j^N = 1\ Y_j \sim$ doubly stochastic sum (1.1)
   number of claims : $N \sim$ Poisson distributed

   amount of claim j: $Y_j \sim$identically distributed with cumulative distribution factor $F(x)$.

   Also assume $N, Y_1, Y_2, \ldots, Y_j, \ldots$ independent, which leads to the standard *compound Poisson model.*

   The specific character of an *insurance portfolio exposed to catastrophe risk* is modeled in the parameter of the Poisson distribution$\sim$*parameter of frequency of claims.*

   Think of this parameter as follows

2. $\mu + \lambda$ (1.2)
   where μ is fixed and λ is a random variable of shocks (due to catastrophes).

   It is instructive to look at the following numerical example:

3. $E[Y] = 1 \sim$ our unit is the average claim size
   $E[Y^2] = 3 \sim$ which is a rather high value for property claims (1.3)
   $\mu = 100$
   $\lambda = 0$ with probability 9/10 and
   $\lambda = 100$ with probability 1/10.

We have just constructed a model for a portfolio in which, on average, once in ten years the expected number of claims is doubled. Whenever this happens, common language will say that a catastrophe (windstorm, flood, earthquake) has happened.

Let us compute

$$\begin{aligned} E[S] &= (\mu + E[\lambda]) * E[Y] \\ &= (100 + 10) * 1 \\ &= 110 \end{aligned} \tag{1.4}$$

$$\mathrm{Var}[S] = \underbrace{(\mu + E[\lambda]) * E[Y^2]}_{A} + \underbrace{\mathrm{Var}\,[\lambda] * (E[Y])^2}_{B}$$

$$= (100 + 10) * 3 + 900 * 1 = 1230. \tag{1.5}$$

We observe two things:

1. The *variance* is big compared to the *mean*.
2. The *variance component B* gives us particular trouble.

What can we do to *reduce B*, which is the variation due to λ (in practical language: the variation due to the catastrophic risk)?

The Idea of Crosshedging

Assume that there is a second portfolio T covering risks in the same area (T could also be the aggregate portfolio written by all insurance companies other than yours in the same area). You are invited to participate in a quotashare of the insurance benefits of T.

Let us summarize:

You are given the opportunity to change the random variable

$$S \text{ into } S - \alpha * T \tag{2.1}$$

at the price of

$$\alpha * \Pi. \tag{2.2}$$

Clearly this leads to the following questions:

1. What is the optimal value of α? Call it α^*.
2. Which price Π is acceptable to you?

General Formulas for Crosshedging

The exercise:

$$\text{Find } \alpha^* \text{ such that } \mathrm{Var}[S - \alpha^* * T] = \min! \tag{3.1}$$

leads to

$$\alpha^* = \frac{\mathrm{Cov}(S,T)}{\mathrm{Var}[T]} \tag{3.2}$$

and with

$$S^* = S - \alpha^* * T \tag{3.3}$$

we find

$$\mathrm{Var}[S^*] = \mathrm{Var}[S] - \frac{(\mathrm{Cov}(S,T))^2}{\mathrm{Var}[T]}. \tag{3.4}$$

Remark: These formulas are derived in any introductory text on linear forecasting. It is actually the most elementary case which can be easily generalized to a higher dimension (using matrix notation).

As simple as these formulas are, the finance literature still fails to draw a lot of conclusions from them. This is due to the fact that, for example, in connection with catastrophe derivatives, nobody seems to have applied the crosshedging technique to the actuarial structure of the random variables S and T (assumed compound Poisson in this paper).

A Successful Area of Crosshedging in Insurance: Credibility Theory

Credibility can indeed be understood as *crosshedging the individual risk* against *its own past.*

Consider:

S~claims generated by the individual risk in *the year of study*

T~sum of the claims of the same risk in the past:

The credibility premium $P=M+\alpha^*T$ charged to the insured leads then to the net payment $S-\alpha^*T-M$ of the insurance company.

As

$$\mathrm{Var}[S - \alpha * T - M] = \mathrm{Var}[S - \alpha * T] \tag{4}$$

(you may omit the constant M),

we have reduced the credibility approach to our basic exercise in crosshedging!

The Fundamental Difference between the Basic Models in Credibility Theory and in Catastrophe Risk Crosshedging

The Credibility Model

The total claims of the individual risk in *one time period* are described by the random variable

$$S = \sum_{j=1}^{N} Y_j \sim \text{Compound Poisson}$$
$$\text{with } N \sim \text{Poisson}(\lambda). \tag{5.1}$$

The essential feature now relates to the fact that the *past variables* S_i, $i=1, 2, \ldots, n$, of the same risk are assumed compound Poisson *with the same* λ.

Hence, we can use the variable $T=\sum_{i=1}^{n} S_i$ to

1. *learn* about the value of λ~credibility language and
2. *reduce* the variance component due to the variability of λ~crosshedging language.

The Catastrophe Risk Model

S is defined as in (1.1)–(1.5):

1. $S=\sum_{j=1}^{N} Y_j$
2. $N \sim Poisson(\lambda+\mu)$

where

μ is fixed and
λ is a random variable of shocks.

Contrary to the credibility case, there is no use in looking at the past variables S_i, $i=1, 2, \ldots, n$. *Their* λ *parameter value has no relation whatsoever* to the λ value *of the recent random variable* S! Typically these values are drawn independently in each period.

But as catastrophes occur in whole areas, competing companies in the same area will also suffer from them. Hence, the increased value of λ *should also be effective in their portfolios*! This is the basic idea of crosshedging for catastrophe risks.

We, therefore, take T as total claims from a big competing company which is active in the same area (you can also think of T as representing the aggregate portfolio of all companies other than yours).

Let

$$T = \sum_{j=1}^{M} W_j \sim \text{Compound Poisson} \tag{5.2}$$

$$M \sim \text{Poisson}(\mu_T+\lambda_T). \tag{5.3}$$

Assume again μ_T is constant.

The *essential assumption* is, however, that

$$\lambda_T = A_T * \lambda \tag{5.4}$$

That is, we postulate that the expected frequency due to catastrophe events λ_T is a *multiple* of the corresponding λ in the distribution of S. In the following we explore two cases:

1. A_T is a deterministic factor, and
2. A_T is a random factor.

For simplicity we follow case 1 here, but will return to case 2 later.

Alternative Notation

From an intuitive point, one might prefer to write (5.4) as (5.4a):

$$\lambda = \frac{1}{A_T} * \lambda_T \quad \text{and then try to model } \frac{1}{A_T}. \tag{5.4a}$$

Of course for the case of deterministic A_T, this is just a change of notation, for the stochastic case one might however easily have $1/A_T=0$ with positive probability.

Remarks

1. We should think of T as substantially big relative to S; hence A_T is a big factor, and μ_T is big relative to μ.
2. For reasons of simplicity, we have assumed that S and T stem from different portfolios, which is contrary to some practical applications where T stands for the sum over all portfolios of all insurance companies in a given area. In such a case write

$$S - \alpha * T = S - \alpha * S - \alpha * (T - S).$$

S and $\tilde{T} = T - S$ stem now from disjoint portfolios.

Hence,

$$S - \alpha * T = (1 - \alpha) * S - \alpha * \tilde{T}$$

shows that crosshedging now implies a combination of crosshedging between distinct portfolios with a classical quotashare reinsurance.

Numerical Example

In order to do explicit calculations, let us also give numerical values for the parameters of the distribution of T.

$$E[W] = 1.5$$
$$E[W^2] = 6 \tag{5.5a}$$

$$\mu_T = 10000$$
$$A_T = 80 \tag{5.5b}$$

$$\lambda \text{ distributed as in (1.2) and (1.3).} \tag{5.5c}$$

Finally let us express the assumption that S and T stem from disjoint portfolios by the property:

$$S \text{ and } T \text{ are conditionally independent given } \lambda \quad (5.6)$$

Applying General Formulas for Crosshedging to the Microstructure of Variables S and T

For the covariance, we have

$$\begin{aligned}\mathrm{Cov}(S,T) &= E[\mathrm{Cov}(S,T)|\lambda] + \mathrm{Cov}[E[S|\lambda], E[T|\lambda]] \\ &= \mathrm{Cov}[\lambda * E[Y], A_T * \lambda * E[W]] \\ &= A_T * E[W] * E[Y] * \mathrm{Var}(\lambda).\end{aligned}$$

Similarly for the variances, (6.1)

$$\mathrm{Var}[S] = E[\lambda + \mu] * E[Y^2] + \mathrm{Var}(\lambda) * (E[Y])^2 \quad (6.2)$$

(see 1.5) and

$$\begin{aligned}\mathrm{Var}(T) &= \underbrace{E[A_T * \lambda + \mu_T]} * E[W^2] \\ &\quad + \mathrm{Var}[A_T * \lambda] * (E[W])^2 \\ &= (\mu_T + A_T * E[\lambda]) * E[W^2] \\ &\quad + A_T^2 * \mathrm{Var}(\lambda) * (E[W])^2. \qquad (6.3)\end{aligned}$$

From (3.2) we then get

$$\alpha^* = \frac{A_T * E[W] * E[Y] * \mathrm{Var}[\lambda]}{A_T^2 * (E[W])^2 * \mathrm{Var}[\lambda] + (\mu_T + A_T * E[\lambda]) * E[W^2]}$$

$$\alpha^* = \frac{E[Y]}{E[W]} * \frac{1}{A_T + \dfrac{E[W^2]}{(E[W])^2} * \dfrac{\mu_T + A_T * E[\lambda]}{A_T * \mathrm{Var}[\lambda]}} \qquad (6.4)$$

Using the numerical values of 1 and 5, we obtain

$$\alpha^* = \frac{1}{1.5} * \frac{1}{80 + \dfrac{6}{2.25} * \dfrac{10000 + 80 * 10}{80 * 90}} = \frac{1}{1.5} * \frac{1}{84}$$

For later purposes, we are introducing here the *reduction factor* η

$$\eta = \frac{A_T}{A_T + \dfrac{E[W^2]}{(E[W])^2} * \dfrac{\mu_T + A_T E[\lambda]}{A_T * \mathrm{Var}[\lambda]}}. \qquad (6.5)$$

We must comment on formula (6.4).

1. There are two critical values: μ_T and A_T.

2. Crosshedging is the more useful (i.e. α^* becomes bigger) *if* μ_T *becomes smaller.*

The conclusion is that the portfolio T used for crosshedging should (as much as possible) contain no other claims than those originating from catastrophes. Ideally we should have $\mu_T=0$. We call such a portfolio a *pure catastrophe portfolio.*

3. More critical is A_T, which we have assumed to be constant. If A_T is stochastic (and independent of λ), then the first A_T in the denominator of (6.4) has to be replaced by

$$\frac{E[A_T^2] * \mathrm{Var}[\lambda] + (E[\lambda])^2 * \mathrm{Var}[A_T]}{E[A_T] * \mathrm{Var}[\lambda]},$$

all others by $E[A_T]$.

The changes in formula (6.5) follow from those in (6.4).

For illustrative purposes, assume

$$\begin{aligned}A_T &= 40 \text{ with probability } 1/2 \\ &= 120 \text{ with probability } 1/2\end{aligned}$$

which leads to

$$\begin{aligned}E[A_T] &= 80 \qquad \text{as in the deterministic case} \\ E[A_T^2] &= 8000 \\ \mathrm{Var}[A_T] &= 1600.\end{aligned}$$

The figure 80 in our numerical example then jumps up to 102. The reduction factor η reduces from 0.95 to 0.75.

4. Nevertheless we see from the numerical exercise that the formula (6.4) derived from the deterministic case for A_T gives the right order of magnitude for α^*. Observe however that the pragmatic argument for

$$\alpha^* = \frac{E[Y]}{E[W]} * \frac{1}{A_T}$$

which is advocated in practice typically leads to values of α^* which are too high!

Safety and Cost

So far we have not addressed the question of whether our crosshedging exercise increases the safety of the insurance carrier. This question is obviously also related to the cost of the envisaged operation. Hence, we should ask *at which cost level* crosshedging can actually be used such that it contributes toward improving the safety of the insurance carrier. These are the questions that we are going to address now.

Assume the insurance carrier is collecting the premium P as its total of income for covering its obligations S. We propose to measure the safety of the insurance carrier by the *adjustment coefficient* κ.

Definition

$\kappa>0$ is called adjustment coefficient of a portfolio at premium income P if

$$E[e^{-\kappa(P-S)}]=1. \tag{7.1a}$$

If $P>E[S]$, such a $\kappa>0$ exists (under suitable mathematical assumptions, i.e., if the moment generating function of S is finite).

Remarks

1. Any text on risk theory will explain how to relate the adjustment coefficient κ to the probability of ruin criterion.

2. Our interpretation is going to be:

The *higher* κ, the higher the *safety* of S at premium P.

You can rewrite (7.1a) in the form

$$P = \frac{1}{\kappa} * \ln E[e^{\kappa S}], \tag{7.1b}$$

and by Taylor expansion of the right side we obtain the approximation

$$P = E[S] + \frac{\kappa}{2} * \mathrm{Var}[S]. \tag{7.2}$$

Hence, measuring safety means studying

$$\kappa = 2 * \frac{P - E[S]}{\mathrm{Var}[S]} = 2 * \frac{\text{Loading}}{\text{Variance}} \tag{7.3}$$

The factor 2 is, of course, irrelevant if we want just to *compare* different levels of safety.

Applying the Safety Concept to the Microstructure of S and S*

We first need for $S^*=S-\alpha^**T$

$$\mathrm{Var}[S^*] = \mathrm{Var}[S] - \frac{(\mathrm{Cov}(S,T))^2}{\mathrm{Var}[T]}.$$

Using the formulas (6.1), (6.2), and (6.3) we obtain

$$\mathrm{Var}[S^*] = E[\lambda + \mu] * E[Y^2] + (E[Y])^2 * \mathrm{Var}[\lambda] * \{1 - \eta\} \tag{8}$$

with the *reduction factor* η as defined in (6.5).

Remarks

1. We have indeed achieved our goal, namely a reduction of the variance component B as introduced in the section titled Catastrophe Risk Model.

2. Observe that in our numerical example, we improve from

$\mathrm{Var}[S] = 1230$ Variance component B: 900 to

$\mathrm{Var}[S^*] = 373$ Variance component B: 43.

3. The reduction in the variance component B is excellent. But recall our discussion of formula (6.4). If A_T is no longer assumed to be deterministic (the assumption indeed is not realistic), then the reduction factor η is reduced. In our explicit calculations it dropped from 0.95 to 0.75 which would lead to $\mathrm{Var}[S^*]=555$, variance component B: 225.

We conclude from this exercise that also under more realistic assumptions the crosshedging exercise gives us a considerable reduction of our variance. Again, the deterministic A_T reflects the general case but it overstates the effect.

Can We Afford the Crosshedging Operation?

All operations of securitization need to be seen in relation to their cost. As in all forms of reinsurance (securitization is one of them) cost is identical to the *amount of loading* which is transferred.

We define

$L=P-E[S]\sim$Amount of loading in the original portfolio at premium P; $L_T=\Pi-E[T]\sim$Amount of loading in the portfolio used for crosshedging, Π being the premium (e.g. futures price) charged for receiving the benefits of T.

Now compare the situation before and after crosshedging:

Before crosshedging

$$\frac{\kappa}{2} = \frac{P - E[S]}{\mathrm{Var}[S]} = \frac{L}{\mathrm{Var}[S]}. \tag{9.1}$$

After crosshedging

$$\frac{\kappa^*}{2} = \frac{P - E[S] - \alpha^*[\Pi - E[T]]}{\mathrm{Var}[S^*]}$$
$$= \frac{L - \alpha^* * L_T}{\mathrm{Var}[S^*]}. \qquad (9.2)$$

The crosshedging operation improves our safety exactly if $\kappa^* > \kappa$.

For the case $\kappa^* = \kappa$, we obtain the upper limit for all the premiums Π that we can afford. Call this limit Π^* (the corresponding loading L_T^*). From (9.1) and (9.2), we conclude

$$\frac{L}{\mathrm{Var}[S]} = \frac{L - \alpha^* * L_T^*}{\mathrm{Var}[S^*]} \Rightarrow \alpha^* * L_T^*$$
$$= L * \frac{\mathrm{Var}[S] - Var[S^*]}{\mathrm{Var}[S]}.$$

From (6.2) and (8) we have

$$\frac{\mathrm{Var}[S] - \mathrm{Var}[S^*]}{\mathrm{Var}[S]} = \frac{(E[Y])^2 * \mathrm{Var}[\lambda] * \eta}{(E[\lambda] + \mu) * E[Y^2] + (E[Y])^2 * \mathrm{Var}[\lambda]}.$$

Hence,

$$\alpha^* * L_T^* = L * \frac{1}{1 + \dfrac{E[Y^2] * E[\lambda + \mu]}{(E[Y])^2 * \mathrm{Var}[\lambda]}} * \eta. \qquad (9.3)$$

With our numerical values as used before, the middle factor turns out to be 0.7317; hence we conclude that in order to increase the safety of the portfolio S by crosshedging, we must have: Transferred loading < Collected loading $*0.7317*\eta$ (in money units).

The critical quantity is again the reduction factor η.

Epilogue

The approach chosen for this presentation is truly an actuarial one:

1. the construction of a sufficiently structured model,
2. the insistence that data must be available to estimate the relevant parameters in the model, and
3. the comparison of results in a practical situation with expected results from the model.

Of course, we have only been able to give some ideas for the first point. The ideas for points two and three are obvious and follow.

The main point, however, is that we have taken a truly scientific route and hope that this scientific attitude also will be accepted ultimately by the financial community. Let me take this opportunity to remind you that the methods used by the life actuary were in existence for more than a century before they were recognized as the fundamental basis for the life insurance industry. The nonlife actuary is still struggling for this recognition. It is certainly high time to take up the struggle for the Actuary of the Third Kind as well.

III

Pricing Insurance Derivatives: The Case of CAT Futures

by Paul Embrechts and Steffen Meister

Abstract

Since their appearance on the market, catastrophe insurance futures have triggered a considerable interest from both practitioners as well as academics. As one example of a securitized (re)insurance risk, its pricing and hedging contains many of the key problems to be addressed in the analysis of more general insurance derivatives. In the present paper we review the main methodological questions underlying the theoretical pricing of such products. We discuss utility maximization pricing more in detail. A key methodological feature is the theory of incomplete markets. Our paper follows closely the exposition given in Meister (1995).

Catastrophe Insurance Futures

Introduction

In recent years the magnitude of catastrophic losses has been staggering. Catastrophic events put significant financial demands on society. In response to market conditions, the Chicago Board of Trade (CBOT) introduced catastrophe insurance futures and options in 1992. The catastrophe insurance (CAT) futures are based upon a loss index consisting of specified losses reported by insurance companies. If an insurance company's insurance portfolio is highly correlated to the loss index, then the company may achieve a reinsurance by the purchase of catastrophe insurance futures or options. Therefore, the catastrophe insurance futures and options can be considered as standardized reinsurance instruments. Similar products are also under consideration in Europe.

The exact specifications of the catastrophe insurance futures contracts are the subject of many recent papers. A complete description is given by the CBOT in its *Catastrophe Insurance Reference Guide* (1995) and *The Management of Catastrophe Losses using CBOT Insurance Options* (1994). Data information is to be found in the CBOT's *Catastrophe Insurance Background Report* (1995). A very good introduction on the use of CAT futures is due to Albrecht, König, and Schradin (1994), which also contains a useful list of references. In this introduction, we only recall some of the basics underlying CAT futures.

Four different catastrophe insurance futures are traded at the CBOT: eastern, midwestern, western, and national catastrophe insurance futures (from now on: insurance futures). Several options of the American type on each contract are also traded. At the moment, the trading volume of the insurance futures options is larger than the one of the insurance futures proper. A reason is certainly that the options tend to hedge non-proportional reinsurance contracts rather than proportional ones, and proportional reinsurance contracts are not very common in the context of catastrophe insurance. In the following, we will discuss the actual insurance futures, rather than the corresponding options. Since the exact specifications of the four insurance futures are the same with exception of the states concerned, we will not distinguish between them. An excellent overall introduction to futures in general is Duffie (1989).

Insurance futures trade in quarterly cycles with contract months March, June, September, and December. For example, the June contract covers losses from events occurring during the first quarter of the same year as reported by the end of June. Since the settlement value is based on losses incurred (paid plus

estimated unpaid), a third quarter allows for loss settlement. Trading ends on the fifth day of the fourth month following the contract month. Therefore, the June contract settlement takes place on October 5. The settlement value of the contract is determined by a loss index.

The Loss Index

The loss index is the underlying instrument of the future's final value. It consists of losses reported to the Insurance Service Office (ISO). Losses are reported to the ISO by approximately 100 companies, but the ISO selects only some of the data on the basis of size, diversity of business and quality. As the selected losses should be representative for the different lines of insurance and states, they are replaced by weighted losses. The weights correspond to the percentages of (estimated) premiums received by the selected companies and total (estimated) premiums earned per line and state. Both the list of reporting companies included in the pool (selected companies) and the estimated premium volume are announced by the CBOT prior to the beginning of the trading period for that contract.

The different lines of insurance include homeowners, commercial multiple peril, earthquake, automobile physical damage, fire, allied lines, farmowners, and commercial inland marine. Reported losses can arise from the perils of windstorm, hail, earthquake, riot, and flood.

Now let L_T denote the sum of selected weighted losses incurred during the quarter corresponding to the contract and reported at the end of the following quarter. Let Π denote the announced premiums earned during the three months exposure period. Then, the insurance future's settlement value F_T is given by

$$F_T = \$25{,}000 \times \text{Min}\left(\frac{L_T}{\Pi}, 2\right). \qquad (1)$$

A convenient way of rewriting (1) is as follows

$$F_T = \$25{,}000 \times \left(\frac{L_T}{\Pi} - \text{Max}\left(\frac{L_T}{\Pi} - 2, 0\right)\right),$$

so that in finance terminology, a catastrophe future (or more precisely its settlement value) is equivalent with a long position in the loss ratio and a short position in a European call option with maturity T, strike price 2, with an underlying loss ratio. Depending on the assumptions on the process (L_t), results from general mathematical finance may be used.

The Loss Process (L_t)

The goal of this section is to develop a plausible stochastic model for the process $(L_t)_{0\le t\le T}$ of losses reported to the selected insurance companies until time t (T being the end of the reporting period). Basically, for $0\le t\le T$,

$$L_t = \sum_{k=1}^{7} O_{k,t} + \sum_{\ell=1}^{4} C_{\ell,t} \equiv L_t^{(1)} + L_t^{(2)}$$

where losses are subdivided in seven classes of ordinary losses $(O_{k,t})$ $(k=1,\ldots,7)$, arising from allied lines, automobile physical damage, commercial multiple peril, farmowners, fire, homeowners, and inland marine. There are four classes of catastrophic losses $(C_{\ell,t})$ $(\ell=1,\ldots,4)$ arising from earthquakes, wind/hail/flood, hurricanes, and riot. The latter classes of losses are defined as losses exceeding a certain high threshold. For example, in their interesting analysis Huygues-Beaufond and Partrat (1992) classified losses due to hurricanes exceeding \$30 million and those due to wind/hail/flood above \$7.5 million as catastrophic. Both statistical analysis (as in Huygues-Beaufond and Partrat [1992]) and theoretical considerations (superposition and thinning of point processes as in Daley and Vere-Jones [1988], Propositions 9.2 VI and 9.3.I.) lead to Poisson-type of assumptions on both $L^{(1)}$ and $L^{(2)}$. Under the assumption that the processes $L^{(1)}$ and $L^{(2)}$ are only weakly correlated or indeed uncorrelated, again from a theoretical point of view one may safely assume that (L_t) obeys one of the following nested assumptions.

Basic Assumptions

1. (L_t) follows a compound Poisson process.
2. (L_t) follows a mixed compound Poisson process.
3. (L_t) follows a doubly stochastic (or Cox) compound Poisson process.

For a definition of these processes, see the section immediately following.

Any insurance derivative involving claim payments will fall within one of the above categories, or at least be an important modeling component. See, for instance, the already mentioned work of Huygues-Beaufond and Partrat (1992), Panjer and Willmont (1992), Grandell (1991), and the references therein. As noted by Sondermann (1991), over the decades, actuaries have collected bulks of statistical information which

enable them to specify such processes one and two with considerable accuracy, so that one can safely say that a lot more is known about risk processes than about security price processes. For a model combining one of the above jump models together with a diffusion component, see Cummins and Geman (1994, 1995).

We assume the loss process $(L_t)_{0\le t\le T}$ to be defined on a basic probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$, where P is the so-called physical probability measure and $(\mathcal{F}_t)_{0\le t\le T}$ is an increasing family of sub-σ-algebras of $\mathcal{F}$ so that for all t, L_t is $\mathcal{F}_t$-measurable (i.e., (L_t) and is $(\mathcal{F}_t)$-adapted). We interpret $\mathcal{F}_t$ as containing all basic information up to and including time t, often (though not always) one assumes that $(\mathcal{F}_t)$ is the natural filtration belonging to (L_t), $\mathcal{F}_t=\sigma\ (L_s,\ 0\le s\le t)$. This essentially exposes a weakness in the trading of CAT futures: though L_t is the main information needed to determine the future's price F_t at time t, (L_t) is only known on two days. An interim report publishes the value of L_t at the fourth day after the reporting period; the final index value is known at maturity. A further uncertainty is due to the actual quality or type of reporting. In this way, a CAT future is fundamentally different from a common (finance) future which is based on a regularly (in many cases very frequently or even tick-by-tick) reported spot price. This incomplete reporting adds an extra component to the overall volatility of insurance futures.

Some Basic Notations and Definitions

Consider a probability space $(\Omega, \mathcal{F}, P)$ and a second probability measure Q on $(\Omega, \mathcal{F})$. Let X and Y be random variables on $(\Omega, \mathcal{F}, (P, Q))$. $P_X(s)$: $=P[X\le s]$ denotes the distribution of the random variable X under the measure P; $E_P[X|Y]$ denotes the conditional expectation of X given the σ-algebra $\sigma(Y)$ with respect to the measure P. If $A\in\mathcal{F}$ and $s\in Y(\Omega)$, then $P_{Y=s}[A]$ is a second notation for $E_P\ [1_A|Y=s]$. We call a function $f: R \to R$ measurable if it is measurable with respect to the Borel σ-algebra $\mathcal{B}(R)$. $\mathcal{B}(A)$ always denotes the Borel sets of A (where A itself is a Borel set). A cadlag function on R is a right continuous function such that the limits from the left always exist. $Q|_{\mathcal{F}}\sim P|_{\mathcal{F}}$ is the notation for the equivalence of the measures P and Q on the σ-algebra $\mathcal{F}$. $Q|_{\mathcal{F}} \perp P|_{\mathcal{F}}$ means that P and Q are mutually singular on $\mathcal{F}$; $Q_X\sim P_X$ means that $Q|_{\sigma(X)}\sim P|_{\sigma(X)}$ and $Q_X \perp P_X$ means that $Q|_{\sigma(X)} \perp P|_{\sigma(X)}$.

A doubly stochastic compound Poisson or compound Cox process $(S_t)_{t\ge 0}$ can be written as follows:

$$S_t = \sum_{k=1}^{N_t} X_k$$

where $(X_1, X_2, X_3, \ldots)$ are strictly positive, i.i.d. random variables on $(\Omega, \mathcal{F}, P)$, and (N_t) is an increasing point process, independent of (X_k), starting at zero, with

$$N_{s+t} - N_t \sim \text{Poisson}\ (\Lambda(s + t) - \Lambda(t))\ (s > 0)$$

where $(\Lambda(t))$ itself is a strictly increasing stochastic process on R^+, also defined on the probability space $(\Omega, \mathcal{F}, P)$. For a precise definition, see Grandell (1991).

A mixed compound Poisson process is a doubly stochastic compound Poisson process with

$$\Lambda(t) \equiv \Lambda \times t$$

and Λ being a strictly positive random variable on $(\Omega, \mathcal{F}, P)$. If the random variable Λ is constant, almost surely we call the resulting process a compound Poisson process. In this case, (N_t) is an homogeneous Poisson process with constant intensity $\lambda>0$. In this paper we always assume that $P[X_1>0]=1$, $X_1\in L^2(\Omega, \mathcal{F}, P)$, $P[\Lambda>0]=1$, $\Lambda\in L^2(\Omega, \mathcal{F}, P)$, and $\forall t\in R$: $\Lambda(t)\in L^2(\Omega, \mathcal{F}, P)$.

Recall that two measures P and Q on (Ω, P) are equivalent if they have the same nullsets. Finally, the integrable, adapted process (X_t) is an $(\mathcal{F}_t)$-Martingale with respect to Q if

$$\forall 0 \le s \le t \le T : E_Q\ (X_t - X_s \mid \mathcal{F}_s) = 0,$$
$$Q\text{-almost surely.}$$

Pricing by "No-Arbitrage"

Introduction

In order to highlight the fundamental differences between CAT futures (or indeed more general insurance derivatives) and traditional finance derivatives, let us look at the pricing problem for the latter in the context of no arbitrage. This introduction is based on the excellent paper by Föllmer (1990). Further basic references are Harrison and Kreps (1979), Harrison and Pliska (1981), and the more recent Delbaen and Schachermayer (1995). An excellent textbook treatment (in French) is Lamberton and Lapeyre (1991). For an English language edition, see Lamberton and Lapeyre (1996).

Consider a price process $(X_t)_{0\le t\le T}$ on $(\Omega, \mathcal{F}, P)$ and a random cash flow H before or at time T. H, as an $\mathcal{F}_T$-measurable random variable on $(\Omega, \mathcal{F}, P)$, is called a contingent claim. Typical examples are:

1. A European call option written on (X_t) with maturity T and strike price K,
$$H = (X_T - K)^+,$$
where $x^+=\max(x, 0)$;
2. If (X_t) describes a loss process and H is the final cash flow arising from a nonproportional reinsurance contract covering losses in a layer (K_1, K_2) at time T,
$$H = \min(X_T, K_2) - \min(X_T, K_1);$$
3. The settlement of a CAT future,
$$H = \$25{,}000 \times \min\left(\frac{X_T}{\Pi}, 2\right), \text{ and}$$
4. An Asian option with strike price K and maturity T,
$$H = \left(\frac{1}{T}\int_0^T X_u\, du - K\right)^+.$$

Recall that an arbitrage opportunity is the possibility of making a riskless sure profit. In a no-arbitrage market, such opportunities do not exist. The (fair) pricing of contingent claims in such a market standardly starts with the following assumption (construction): "Let Q be an equivalent Martingale measure to P, i.e. $Q \sim P$ and (X_t) is an $(\mathcal{F}_t)$-Q-Martingale." *It is precisely this assumption which crucially depends on the stochastic properties of the underlying process* (X_t). In standard finance markets like the Cox-Ross-Rubinstein binomial tree model, the Bachelier Brownian motion model or the Markovitz-Black-Scholes geometric Brownian motion model, one can show that not only such a Q exists but is moreover unique (as we shall discuss later, the latter is related to the notion of market completeness and is *crucially different in insurance markets*). However, let us for the moment assume that we have such a unique Q. Recall that H is $\mathcal{F}_T$ ($=\sigma(X_s : 0\le s\le T)$, say)-measurable; then in the above standard finance cases one can write H as an *Itô representation*:

$$H = H_0 + \int_0^T \xi_s\, dX_s \qquad (2)$$

for some H_0 and predictable (think left continuous) process (ξ_t). The representation (2) leads to a portfolio strategy replicating (riskless) the claim H if the premium H_0 is suitably chosen (and here Q will enter!). Indeed at time t we hold the amount ξ_t in the risky asset X_t, and the amount

$$\eta_t \equiv \left(H_0 + \int_0^t \xi_s\, dX_s\right) - \xi_t X_t$$

in the riskless asset ("money in the bank") given by the constant 1 (think of discounted amounts). The value V_t of this portfolio at time t is $V_t=\xi_t X_t+\eta_t$, and hence *by construction* $V_T=H$ (we neglect transaction costs!). This will all work if we can calculate the initial investment H_0 $(=V_0)$ and the process (ξ_t). Unfortunately, Itô's representation (2) is mostly only a nonconstructive existence theorem. Using the notion of self-financing strategies and Itô's lemma, one can derive a constructive solution (involving PDEs). Here is the final trick to calculate H_0:

1. (X_t) is a Q-Martingale (not necessarily a P-Martingale);
2. (ξ_s) is predictable and "nice," whence $(I_t\equiv\int_0^t\xi_s dX_s)_{0\le t\le T}$ is a Q-Martingale;
3. $E_Q(I_t)=E_Q(I_0)=0$; therefore $E_Q(H)=H_0$; since we know H and Q we have found the fair premium H_0.

Though admittedly we have left out various details, the above discussion clearly gives us a way in which to price and hedge insurance derivatives based on a risk process (X_t):

Step 1 Investigate the relationship between no-arbitrage conditions and the existence of equivalent Martingale measures Q.

Step 2 What about uniqueness of Q (related to completeness)?

Step 3 Find the Itô representation of (X_t) and investigate the explicit construction of hedging portfolios.

Step 4 What if any of the above fail?

In the following sections, we show that for risk processes in general many equivalent Martingale measures (and consequently fair premiums) exist so that a key discussion will be devoted to Step 4. By way of an important example, we restrict our discussion to mixed compound Poisson processes.

Mixed Compound Poisson Processes and Change of Measure

For a given process (X_t) on $(\Omega, \mathcal{F}, P)$, in order to investigate the existence of equivalent Martingale

measures, one has to be able to characterize Radon-Nikodym derivatives dQ/dP. The following result is proven in Meister (1995). We use the notation discussed previously in Some Basic Notations and Definitions.

THEOREM 1 *Suppose $(S_t)_{t\geq 0}$ is a mixed compound Poisson process on $(\Omega, \mathcal{F})$ under P and Q. The following statements are equivalent:*

1. $\forall s\geq 0 : Q|_{\mathcal{F}_s} \sim P|_{\mathcal{F}_s}$;
2. $P_{X_1} \sim Q_{X_1}$, *and*
3. $\exists \gamma : R \to R$ *measurable with*

$E_{P_{X_1}}(\exp(\gamma(X_1)))=1$, $E_{P_{X_1}}(X_1^2 \exp(\gamma(X_1)))<\infty$, and

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_s} = \exp\left(\sum_{i=1}^{N_s} \gamma(X_i)\right) \frac{E_{Q_\Lambda}(\Lambda^{N_s} \exp(-\Lambda s))}{E_{P_\Lambda}(\Lambda^{N_s} \exp(-\Lambda s))}.$$

Theorem 1 allows us to explicitly calculate Radon-Nikodym derivatives for the most important insurance risk processes.

Example 1 (Compound Poisson case)

Let P and Q be probability measures such that $(S_t)_{t\geq 0}$ is a P- and Q-compound Poisson process, and let $P_{X_1} \sim Q_{X_1}$, $P_\Lambda = \delta_{\lambda_1}$, $Q_\Lambda = \delta_{\lambda_2}$, $\lambda_1, \lambda_2 > 0$. Then Theorem 1 immediately yields

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_s} = \exp\left(\sum_{i=1}^{N_s} \gamma(X_i)\right) \left(\frac{\lambda_2}{\lambda_1}\right)^{N_s} e^{-(\lambda_2-\lambda_1)s}$$

for some measurable function γ.

Example 2 (Gamma mixed compound Poisson case)

Let P and Q be probability measures such that $(S_t)_{t\geq 0}$ is a P- and Q-gamma mixed compound Poisson process and $P_{X_1} \sim Q_{X_1}$. Consider $\gamma_1, \gamma_2, c_1, c_2 > 0$ to be the parameters describing the distribution of Λ, i.e.

$$P_\Lambda \sim \Gamma(\gamma_1, c_1),\ Q_\Lambda \sim \Gamma(\gamma_2, c_2),$$

where $\Gamma(\gamma_i, c_i)$ denotes the gamma distribution with parameters γ_i, c_i.

Then again using Theorem 1, for some measurable function γ,

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_s} = \exp\left(\sum_{i=1}^{N_s} \gamma(X_i)\right) \frac{\Gamma(N_s+\gamma_2)}{\Gamma(N_s+\gamma_1)} \left(\frac{(c_1+s)}{(c_2+s)}\right)^{N_s} \theta$$

where θ is given by

$$\theta = \frac{c_2^{\gamma_2}}{c_1^{\gamma_1}} \frac{(c_1+s)^{\gamma_1}}{(c_2+s)^{\gamma_2}} \frac{\Gamma(\gamma_1)}{\Gamma(\gamma_2)}.$$

The above model is also called the compound negative binomial process.

Example 3 (General inverse Gaussian mixed compound Poisson case)

Let P and Q be probability measures such that $(S_t)_{t\geq 0}$ is a P- and Q-general inverse Gaussian mixed compound Poisson process and $P_{X_1} \sim Q_{X_1}$. Consider $\mu_1, \mu_2, \beta_1, \beta_2 > 0$ and $\lambda_1, \lambda_2 \in R$ to be the parameters determining the distribution of Λ under the two measures (λ_1, λ_2 are constants here and not realizations of the random variable Λ), i.e.,

$$P_\Lambda \sim GIG(\mu_1, \beta_1, \lambda_1),\ Q_\Lambda \sim GIG(\mu_2, \beta_2, \lambda_2).$$

For further details on the processes, see Panjer and Willmot (1992). In this case, we obtain

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_s} = \theta \exp\left(\sum_{i=1}^{N_s} \gamma(X_i)\right) \left(\frac{\mu_2}{\mu_1}\right)^{N_s} \left(\frac{(1+2\beta_1 s)}{(1+2\beta_2 s)}\right)^{\frac{N_s}{2}} \frac{k_{\lambda_2+N_s}(\mu_2\beta_2^{-1}(1+2\beta_2 s)^{1/2})}{k_{\lambda_1+N_s}(\mu_1\beta_1^{-1}(1+2\beta_1 s)^{1/2})}$$

with

$$\theta = \frac{(1+2\beta_1 s)^{\lambda_1/2}}{(1+2\beta_2 s)^{\lambda_2/2}} \frac{k_{\lambda_1}(\mu_1\beta_1^{-1})}{k_{\lambda_2}(\mu_2\beta_2^{-1})}.$$

The above model is also called the compound Sichel process.

As explained in Meister (1995, Proposition 2.9), the above results can immediately be used to show that after a change of measure, the process $(S_t)_{t\geq 0}$ remains a mixed compound Poisson process if the Radon-Nikodym derivatives

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_s}$$

have the right structure.

Mixed Compound Poisson Processes and Martingales

Consider a mixed compound Poisson process $(S_t)_{t\geq 0}$ under P, and a premium density $p>0$. The following

result shows how equivalent Martingale measures for the process $(S_t - pt)_{t\geq 0}$ can be obtained. A proof of the result can be found in Meister (1995, Proposition 2.11). For more details on premium densities and actuarial premium calculation principles, see Delbaen and Schachermayer (1995).

THEOREM 2 *Suppose $(S_t)_{t\geq 0}$ as above. $(S_t - pt)_{t\geq 0}$ is an $(\mathcal{F}_t)$-Martingale under Q and for all $s\geq 0$, $Q|_{\mathcal{F}_s} \sim P|_{\mathcal{F}_s}$ if and only if*

1. $\exists \lambda > 0$, $\beta : R^+ \to R$ *measurable with* $E_P(e^{\beta(X_1)}) = \lambda$ *and* $E_P(X_1^2 e^{\beta(X_1)}) < \infty$ *such that*

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_s} = \exp\left(\sum_{i=1}^{N_s} \beta(X_i) - \lambda s\right)(E_P(\Lambda^{N_s} e^{-\Lambda s}))^{-1},$$

and

2. $p = E_P(X_1 e^{\beta(X_1)})$.

An immediate consequence from the above result is that if the jumpsizes of $(S_t)_{t\geq 0}$ are not constant under the measure P (i.e., nonconstant claimsizes), then an equivalent Martingale measure for $(S_t - pt)_{t\geq 0}$ *cannot be unique*! In the less interesting case of constant claims, we have uniqueness.

A Word about Completeness

Completeness essentially means that the underlying process $(X_t)_{t\geq 0}$ is so that every contingent claim can be replicated by a (self-financing) strategy. Not surprisingly is this notion linked to Itô representations.

DEFINITION 1 *The market* $((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, Q), (X_t)_{0\leq t\leq T})$ *is complete if every contingent claim* $H \in L^2(\Omega, \mathcal{F}_T, Q)$ *admits an Itô representation* (2) *with respect to the process* (X_t).

We basically know that the no-arbitrage condition is "equivalent" with the existence of an equivalent Martingale measure. For a mathematically precise statement see Delbaen and Schachermayer (1995) or Stricker (1990). Sondermann (1991) showed that a similar result holds in reinsurance markets. Besides proving the existence of replicating strategies in various no-arbitrage models, completeness implies (or is indeed equivalent with) uniqueness of equivalent Martingales. The following models are known to be complete:

1. One-dimensional Brownian motion (Itô, 1951),
2. Multidimensional Brownian motion and some special types of diffusions (Jacod, 1979),
3. $(N_t - \lambda t)_{t\geq 0}$ with (N_t) a homogeneous Poisson process (Kunita and Watanabe, 1967), and
4. Square integrable point process Martingales $(N_t - \int_0^t \lambda_s ds)_{t\geq 0}$ (Brémaud, 1981).

As soon as we move to compound processes based on homogeneous, mixed, or doubly stochastic Poisson processes, even when completeness is present, *we may lose the uniqueness of the equivalent Martingale property and therefore the unique pricing property.* For further details see Meister (1995, Section 3.3). We introduced the notion of completeness via the existence of hedge portfolios (Itô representation). In various markets completeness turns out to imply the uniqueness of an equivalent Martingale measure. These cases are finite probability spaces or continuous price processes (Jacod, 1979), of a price process for which the natural filtration $(\mathcal{F}_t = \sigma(X_s : s\leq t))$ is strictly left-continuous (Pratelli, 1994). Unfortunately in an insurance market framework in general, none of these cases apply. *Clearly further research on this topic is called for.*

Before embarking on the pricing problem for catastrophe insurance futures, we quote some words of warning concerning incompleteness and insurance pricing.

1. There is no "right" price of insurance; there is simply *the transacted market price* which is high enough to bring forth sellers, and low enough to induce buyers (Lane, 1995).
2. In incomplete markets, exact replication is impossible and holding an option is a genuinely risky business, meaning that *no preference independent pricing formula is possible.* If, however, option pricing is imbedded in a utility maximization framework, i.e., the potential option purchaser's attitude to risk is specified, then a unique measure emerges in a very natural way (Davis and Robeau, 1994).
3. Arbitrage-based pricing theories are theories about *relative prices* and do *not* attempt to explain *why* the prices of a particular stock reached their observed level. Only the *interrelationship between prices* is explained (Jensen and Nielsen, 1994).

Back to CAT Futures

The key questions concerning catastrophe insurance futures markets are:

1. Do strategies of taking long and short positions in insurance futures contracts exist which yield a sure profit?

2. What are necessary and sufficient conditions to exclude the opportunities of a sure profit?

Recall that we denote the process of total losses flowing into the insurance futures index by (L_t). Also we assume (L_t) to follow either a compound Poisson process, a mixed compound Poisson process or a doubly stochastic compound Poisson process, defined on some probability space $(\Omega, \mathcal{F}, P)$. We denote the insurance future's price process by $(F_t)_{0\le t\le T}$. A short position at time t means the commitment to pay the random amount $F_T - F_t$ at time T. A holder of a long position will receive $F_T - F_t$ at time T. There are no cashflows before T. This is a theoretical assumption as the CBOT clearing system requires certain payments from agents before T. Furthermore, the insurance future contract has the starting value zero, and settlement takes place at time T. Consider the market to be liquid and the contracts to be divisible in the sense that any agent can buy or sell any fraction of a contract at any time. We consider deterministic interest rates, though our results also hold for stochastic interest rate models. For expository purposes, we assume interest rates to be zero; because there are no cashflows between 0 and T, we can think of all cashflows to be discounted to their value at time zero. We, furthermore, recall that F_T is determined by \$25,000 min

$$\left(\frac{L_T}{\Pi}, 2\right),$$

hence the process (F_t) is bounded. Both (L_t) and (F_t) will be adapted with respect to the filtration $(\mathcal{F}_t)$, denoting the increasing information σ-algebras available through time. We have to give a new definition for the meaning of a strategy: a strategy is used by an agent who agrees at certain random times to several long or short positions in an insurance future contract, where the number of the positions only depends on the information which is available at those (random) times. To be more precise,

Definition 2 *The set* $(\xi)=\{n, \tau_1, \ldots, \tau_n, \xi_1, \ldots, \xi_n\}$, *where* $n\in N$, $\tau_i : \Omega \to [0, T]$, $i=1 \ldots n$ *are* $(\mathcal{F}_t)$*-stopping times*; $\xi_i : \Omega \to R$ *are* $\mathcal{F}_{\tau_i}$*-measurable, and square integrable is called a strategy. The final gain of trade* $G_T(\xi)$ *of a strategy* (ξ) *is given by*

$$G_T(\xi) = \sum_{k=1}^{n} \xi_k (F_T - F_{\tau_k}) \ (\in L^2(\Omega, \mathcal{F}_T, P)).$$

It may not seem to be obvious that we do not allow for continuous strategies. Note that usually a continuous strategy describes the amount held in some risky asset. The price fluctuation then determines the final gain of trade. The above context is slightly different: we do not hold a certain amount of an underlying asset, but at any time we can agree to an insurance future contract, which is settled at time T. As it stands, definition 2 allows for quite realistic strategies.

A strategy (ξ) allowing for a sure profit is called an *arbitrage strategy*, i.e., a strategy (ξ) satisfying:

$$G_T(\xi) \ge 0 \quad P\text{ - almost surely, and}$$

$$E_P\,[G_T(\xi)] > 0.$$

Under the natural assumption that (F_t) is a right-continuous process, the following theorem gives a necessary and sufficient condition for the absence of arbitrage strategies. The formulation turns out to be exactly the same as in the context of standard financial markets.

Theorem 3 *Consider the insurance futures market* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$, *with right-continuous price process* $(F_t)_{0\le t\le T}$. *The following are then equivalent*:

1. $(F_t)_{0\le t\le T}$ *does not allow arbitrage strategies, and*
2. *there exists an equivalent measure* Q *(i.e.* $P\sim Q$*) such that* (F_t) *is an* $(\mathcal{F}_t)$*-Martingale under* Q.

For a proof, see Meister (1995, Proposition 3.6, Theorem 3.7).

Example 4

A common premium principle in insurance mathematics is the Esscher principle, see for instance Gerber and Shiu (1995). Applied to the pricing of insurance futures, the Esscher principle leads to

$$F_t \equiv E_Q\,(F_T \mid \mathcal{F}_T), \qquad 0\le t\le T,$$

where for some $\alpha>0$

$$\frac{dQ}{dP} = \frac{e^{\alpha F_T}}{E_P\,(e^{\alpha F_T})}.$$

Obviously $Q\sim P$ and (F_t) is a Q-Martingale. If (L_t) follows a doubly stochastic compound Poisson process, then (F_t) is certainly right-continuous; hence, the Esscher model is arbitrage free. For a further discussion on the relevance of the Esscher principle in general finance, see Bühlmann, Embrechts, and Shiryaev (1996).

Pricing and Replication in the Insurance Futures Market

Introduction

An insurance market is often not complete, and even complete insurance markets do in general allow for many equivalent Martingale measures. Hence, there exists several possibilities to price contingent claims excluding arbitrage opportunities. In this context the common approach to pricing is a preference dependent model. Preferences are usually described by von Neumann-Morgenstern utility functions. We shall distinguish between prices calculated by individual agents and equilibrium prices over a whole market.

An individual agent's objective is to maximize expected utility of wealth at a certain fixed time. The agent, therefore, only agrees to a position in an insurance future contract if it is an attractive investment compared with other possible investments. Hence, the insurance future's price should only depend on the agent's preferences and investment opportunities. A market equilibrium is the situation where by exchanging risks all agents can maximize their expected utility at the same time. Equilibrium prices are derived by changing the measure and taking the corresponding expectations under the new measure. In this case the insurance future's price should only depend on all the agents' utilities and investment opportunities.

In some situations, an agent wants to replicate a contingent claim by an engagement in the corresponding market. In an incomplete market, there exists claims which cannot be replicated in the sense that they do not admit an Itô representation. Thus, there remains some uncertainty about the replication cost.

Pricing of Insurance Futures in a Utility Maximization Framework

A recent study on the subject of option pricing in a utility maximization framework is due to Davis and Robeau (1994). Although one cannot apply the results to an insurance market context in a straightforward manner, we find it important to discuss these basic ideas. An interesting paper giving an easy introduction to utility functions in an insurance context is Gerber (1987).

An investor with utility u and a certain initial endowment x forms a dynamic portfolio. To determine the portfolio, he or she can make the choice of a strategy π out of the set $\mathcal{S}$ of possible strategies. The cash value of the portfolio at time t is $X_x^\pi(t)$. The objective is to maximize expected utility of wealth $X_x^\pi(T)$ at a fixed final time T. The investor asks the question whether the maximum utility can be increased by the purchase or short-selling of a European option whose cash value at time T is some nonnegative random variable, the purchase price at time $t=0$ being p. Thus, from the investor's point of view, p is a fair price for the option if diverting a little of the funds into it at time $t=0$ has a neutral effect on the investor's achievable utility.

This "marginal rate of substitution" argument leads then (under additional assumptions) to a general option pricing formula basically dependent on the set of strategies $\mathcal{S}$, the initial endowment x, and the utility function u.

Let the investor now be an insurance company. The company holds a portfolio of insurance policies for which it receives premiums, but also has to pay for occurred losses. Let $(P_t)_{0\le t\le T}$ denote the total value of premiums received up to time t, and $(Y_t)_{0\le t\le T}$ be the total value of claims occurring up to time t, both processes defined on some probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ where $\mathcal{F}_t=\sigma(P_s, Y_s, s\le t)$. We assume the existence of a liquid reinsurance market, i.e. at any time $t\le T$ the insurance company can decide to sell any fraction of the remaining risk $(Y_s)_{t\le s\le T}$, based on the information available at time t. To be more precise,

Definition 3 *If $t\in[0, T]$, a reinsurance strategy $(\xi_s)_{t\le s\le T}$ is a predictable stochastic process on $(\Omega, \mathcal{F}, (\mathcal{F}_s), P)$ with*

$$0 \le \xi_s \le 1 \text{ for all } s \in [t, T].$$

$\mathcal{H}_t$ denotes the set of all reinsurance strategies which "start" at time t.

Again, for expository purposes, we assume interest rates to be deterministic, described by the function $(r(t))_{0\le t\le T}$, where $r(t)$ denotes the value at time T of a cashflow arising from an investment of 1 at time t. We introduce, furthermore, the process $(X_t)_{0\le t\le T}$ given by

$$X_t = r(t)\,(P_t - Y_t),\ 0 \le t \le T,$$

denoting the inflated net earnings from the insurance business up to time t.

If the insurance company at time t chooses some reinsurance strategy $(\xi_s)\in\mathcal{H}_t$, then the company's final gain at time T (positive or negative) is given by

$$G_T(\xi) = \int_t^T \xi_s dX_s,$$

where we assume the reinsurance companies to receive primary insurer premiums for their engagement.

Assume the insurance company to have a utility function u, denoting the company's preferences. We assume u to be a C^2-function on R with $u'>0$, and $u''\leq 0$. The insurer's objective is to maximize expected utility of the final gain at time T by using the information $\mathcal{F}_t$. Let

$$V = \sup_{\xi \in \mathcal{A}_t} E_P\,[u(G_T(\xi))|\mathcal{F}_t]$$

be the maximum expected utility of the (inflated) final gain, and $\mathcal{A}_t$ be some subset of $\mathcal{H}_t$. The insurance company now asks the question whether its maximum utility V could be increased by the agreement to a long or a short position in an insurance future contract. The "marginal rate of substitution" argument is used as follows: F_t is a fair price for the insurance future at time t if "agreeing a little" into a contract has a neutral effect on the company's achievable utility. For the purpose of giving a mathematical formulation, we define for any δ and F_t:

$$W(\delta, F_t) = \sup_{\xi \in \mathcal{A}_t} E_P\,[u(G_T(\xi) + \delta(F_T - F_t))|\mathcal{F}_t]$$

and give the following:

Definition 4 *Suppose that for each F_t the function*

$$\delta \to W(\delta, F_t)$$

is differentiable at $\delta=0$, and there is a unique solution $\hat{F}_t$ of the equation

$$\frac{\partial W}{\partial \delta}(0, F_t) = 0.$$

Then $\hat{F}_t$ is the fair price (in the above sense) for the insurance future at time t.

In the following theorem, we can give *a* pricing formula for insurance futures:

Theorem 4 *Suppose that there exists $\xi \in \mathcal{A}_t$ such that*

$$V = E_P\,[u(G_T(\hat{\xi})) \mid \mathcal{F}_t]$$

and the function

$$\delta \to W(\delta, F_t)$$

is differentiable at $\delta=0$ for each F_t. Then the fair insurance future price at time t is given by

$$\hat{F}_t = \frac{E_P\,[u'(G_T(\hat{\xi}))\,F_T \mid \mathcal{F}_t]}{E_P\,[u'(G_T(\hat{\xi})) \mid \mathcal{F}_t]}.$$

The proof of this result is given in Meister (1995, Theorem 4.3). This result should also be compared with formula (17) in Gerber (1987).

Example 5

Consider the insurance company to have an exponential utility $u(x)=(1-e^{-\alpha x})$ with risk averseness $\alpha>0$ when it decides not to reinsure its claims, i.e., $\mathcal{A}_t=\{(1)_{t\leq s\leq T}\}$. Then we have

$$\hat{F}_t = \frac{E_P\,[e^{-\alpha(X_T-X_t)}\,F_T \mid \mathcal{F}_t]}{E_P\,[e^{-\alpha(X_T-X_t)} \mid \mathcal{F}_t]}.$$

Assume, furthermore, the premium process (P_t) to be deterministic. Then, using that Y_t is $\mathcal{F}_t$-measurable and $r(T)=1$, it follows that

$$\hat{F}_t = \frac{E_P\,[e^{\alpha(r(T)Y_T-r(t)Y_t)}\,F_T \mid \mathcal{F}_t]}{E_P\,[e^{\alpha(r(T)Y_T-r(t)Y_t)} \mid \mathcal{F}_t]} = \frac{E_P\,[e^{\alpha Y_T}\,F_T \mid \mathcal{F}_t]}{E_P\,[e^{\alpha Y_T} \mid \mathcal{F}_t]}.$$

Now replace the insurance company by the insurance market holding those policies which lead to losses flowing into the insurance future's index, i.e., for all $0\leq t\leq T$, $Y_t=L_t$. The fair insurance future price (from the insurance market's point of view), therefore, is ($c=\$25{,}000/\Pi$):

$$\hat{F}_t = \frac{E_P\,[e^{\alpha L_T}\,c\,(L_T \wedge 2\Pi) \mid \mathcal{F}_t]}{E_P\,[e^{\alpha L_T} \mid \mathcal{F}_t]}. \tag{3}$$

Showing that in this case the utility maximization approach essentially leads to the Esscher principle.

In order to work out formula (3), one needs to impose specific conditions on $(L_t)_{0\leq t\leq T}$, like the three basic assumptions in the loss process (L_t) section. For instance, under assumption one (compound Poisson case) one obtains

$$\hat{F}_t = c\,e^{\lambda(t-T)E_P(e^{\alpha X_1})} \sum_{k=0}^{\infty} \frac{(\lambda(T-t))^k}{k!}\,\eta_k$$

where

$$\eta_k = \int_0^{(2\Pi-L_t)^+} e^{\alpha s}(s + L_t)\,dF^{*k}(s) + 2\Pi(1 - F^{*k}(2\Pi - L_t)^+).$$

Here $F(x)=P(X_1 \leq x)$ is the jump (or claim) distribution, and F^{*k} denotes the kth convolution of F.

In the uncapped case,

$$F_T = \$25{,}000 \frac{L_T}{\Pi} \equiv cL_T$$

and gamma distributed claim amounts $X_1 \sim \Gamma(n, \mu)$, the above formula reduces to the very easy

$$\hat{F}_t = c\left\{L_t + \lambda(T-t)\frac{n\mu^n}{(\mu-\alpha)^{n+1}}\right\}$$

which explicitly shows that $\hat{F}_t$ increases with L_t, λ, and α; $\hat{F}_t$ decreases as time to maturity T becomes shorter. For generalizations and further analysis see Aase (1994) and Meister (1995).

Remarks

1. As we have remarked in the section titled, Pricing By No-Arbitrage, insurance markets are often not complete; hence, there exist contingent claims which cannot be perfectly hedged in the sense that they do not admit an Itô representation. However, one can show that under rather weak assumptions, there exist "best" risk minimizing reinsurance strategies in the sense of Föllmer and Sondermann (1986), Föllmer and Schweizer (1989), and Schweizer (1990, 1994). Details on this approach concerning CAT futures are to be found in Meister (1995, Section 4.4). An excellent paper using equilibrium pricing is Aase (1994).
2. Whenever pricing formulas concerning a financial instrument are to be worked out in an incomplete market (allowing for various equivalent Martingale measures), it is useful to estimate the distribution of the contingent claim under the physical measure P and apply standard loading techniques as is often done in an insurance context. The latter is mostly better than a blindfold application of some (nonunique) Martingale pricing formula. Such an analysis has been given by Klüppelberg and Mikosch (1995). In the latter paper it is assumed that $(N_t)_{t\geq 0}$ is either a renewal process or a family of Poisson rv's with intensities $(\lambda(t))_{t\geq 0}$, such that $E(N_t)\equiv\lambda(t) \to \infty$, $t \to \infty$. The claim process equals $S_t=\Sigma_{i=1}^{N_t} X_i$, where $(X_n)_{n\in N}$ is a sequence of i.i.d. rv's having distribution function F and mean μ. In the presence of large (catastrophic) claims, a natural condition on F is of the type $\bar{F}(x)\equiv 1-F(x)\sim x^{-\alpha}L(x)$ for $x \to \infty$, where $\alpha>1$ and L is slowly varying, i.e., for all $t>0$,

$$\lim_{x\to\infty}\frac{L(xt)}{L(x)}=1.$$

We denote this condition by $\bar{F}\in\mathcal{R}(-\alpha)$. See Bingham, Goldie, and Teugels (1987) for a detailed discussion on the latter conditions. For a discussion on the modelling of catastrophic claims, see Embrechts, Klüppelberg, and Mikosch (1997). Motivated by CAT futures, Klüppelberg and Mikosch (1997) estimate for large t, the distributional behavior of

$$V(t) \equiv \$25{,}000 \times \min\left(\frac{S_t}{c\lambda(t)\mu}, 2\right)$$

for some safety loading $c>0$.

Theorem 5 *Under the above assumptions, the following asymptotic estimates hold.*

1. *If* $\alpha>1$, *then as* $t \to \infty$,

$$E_P(V(t)) = \$25{,}000\,\frac{1}{c}\left(1 + (1+o(1))\frac{(2c-1)\lambda(t)}{\alpha-1}\bar{F}((2c-1)\mu\lambda(t))\right).$$

2. *If* $\alpha>2$, *then as* $t \to \infty$,

$$\mathrm{Var}_P(V(t)) = (\$25{,}000)^2\,\frac{1}{c^2}\left(\frac{E(X_1^2)}{\mu^2\lambda(t)} + (1+o(1))\frac{2(2c-1)^2\lambda(t)}{(\alpha-2)}\bar{F}((2c-1)\mu\lambda(t))\right).$$

The key problem in obtaining results of the above type can be seen as follows:

$$E_P\left(\frac{S_t}{c\mu(t)} - K\right)^+ = \frac{1}{c\mu(t)}\int_{\gamma\mu(t)}^{\infty} P(S_t - \mu(t) > x)\,dx, \quad (4)$$

where we assume that $\gamma\equiv Kc-1>0$. Letting $t \to \infty$ in (4), we need estimates on $P(S_t-\mu(t)>x)$ for $x=x(t) \to \infty$. This leads to the well-known area of large deviation results, however, under the nonstandard (heavy-tail) condition $\bar{F}\in\mathcal{R}(-\alpha)$. This is exactly the theory worked out in Klüppelberg and Mikosch (1995).

3. Throughout this paper, we have concentrated mostly on the pricing of the CAT futures themselves; clearly the same theory may be used to price derivatives on the CAT futures like options and

call-option spreads. See, for instance, Aase (1994) and Meister (1995).

4. The final word on the pricing of insurance derivatives has not yet been said. The present paper provides some insight into the underlying mathematical methodology. More and more, insurance products are coming onto the market containing a financial component of some sort. Both finance experts as well as actuaries will have to get to know the other expert's field better. We hope that our paper contributes toward closing the existing gap so that with the right methodology at hand, we can seriously start tackling risk securitization.

Acknowledgment

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IV

The Perfume of the Premium. . . or Pricing Insurance Derivatives

by John Finn and Morton Lane

Q. "What is the right price for underwriting an insurance risk?"
A. "It is where the perfume of the premium overcomes the pong of the peril."

This old saw sums up what savvy traders and underwriters know instinctively about markets: that there is no "right" price for a piece of insurance. There is simply the transacted market price, which is high enough to bring forth sellers and low enough to induce buyers. Supply and demand determine price in all markets, including sophisticated risk-transfer markets such as futures, options, and insurance.

That said, any set of transacted insurance prices (premiums) contain within them an implicit assessment of the underlying risks (perils). Our objective in this paper is to make those *implicit* risk assessments *explicit*. We do this by examining the prices paid for catastrophic futures and options (CATs)* at the Chicago Board of Trade (CBOT) and by deriving their corresponding "Implied Loss Distributions."

This original approach allows us to (1) generate theoretical prices for different CAT covers, (2) establish cheapness and dearness among the alternatives, and (3) compare the prices paid for "CAT layers" with certain types of insurance "event" covers. To illustrate our approach, we have extracted and compared third quarter eastern implied loss distributions (ILDs) for 1993, 1994, and 1995 and examined the behavior of the 1994 ILD as that hurricane season progressed.

*The analysis contained in "The Perfume of the Premium" is based upon the CBOT's ISO contracts which have been replaced with contracts based on indices provided by Property Claims Services (PCS). Accordingly, Sedgwick Lane Financial has updated and extended this analysis in a paper titled "The Perfume of the Premium II."

As of yet, there is no uniform or generally accepted method of analyzing CATs. CBOT traders approach CATs as conventional financial options. Underwriters approach CATs as a particular form of catastrophic insurance. The two approaches need to be reconciled. We begin, then, with a brief discussion of insurance versus options.

Insurance versus Options

A buyer of insurance purchases the right to be reimbursed by the insurance writer for specified losses over and above the deductible or retention. For this right, he pays a premium.

A buyer of call options purchases the right to buy the underlying instrument from the seller (writer) to capture its increased value above the strike price. For this right, he pays a premium.

Clearly the two instruments are very similar. Indeed, some have referred to insurance contracts as options with fuzzy strikes and fuzzy pay-offs. Some of the fuzziness comes from a very important principle of insurance: indemnity. Insurance pays off only if the insured has an insurable interest *and* suffers a loss. Options holders, by contrast, do not need to have an "optionable" interest to buy the option and are paid off based on the value of the underlying instrument independently of losses or gains suffered elsewhere.

For a contract to qualify as insurance, the purchaser cannot experience a net gain. This is not true of CATs. CATs are not insurance: they are an insurance surrogate. Buyers of CAT options will experience a recovery that depends on the size of industry catastrophic losses (as determined by the Insurance Service Office [ISO]) whether they experience a loss themselves. Thus, while recoveries from insurance are insured specific,

recoveries from CATs are industry specific. A conventional financial option, then, is like insurance without the principle of indemnity, and conversely, an insurance cover without the principle of indemnity is like a conventional financial option.

Pricing Insurance

Textbooks on insurance refer to pricing insurance, and more particularly reinsurance, by one of four basic methods: experience rating (rate-on-line, payback), comparable cover, Pareto, or benchmark theory. None is very precise, and nearly all are based on a rearview mirror approach which involves looking at an insured's past loss experience and assuming that it will continue into the future. A gross-up factor is included in the pricing structure, and it is assumed that this will produce profitable underwriting over time.

Catastrophic reinsurance is more difficult to price because of the low frequency and high severity of catastrophic losses. The history of these occurrences is sparse. Nevertheless, the above approaches are applied to loss records stretching back many years to gain a better sense of the insurer's risk and the appropriate pricing. For example, events which have occurred, or are likely to occur, every five years must be priced in excess of a 20% rate-on-line to be profitable.

Catastrophic losses are often assumed to conform to a particular statistical distribution, such as the Pareto, compound-Poisson, or gamma distribution. The shape of the distribution is then fitted to the historical record of catastrophic losses and remains relatively fixed into the future for pricing purposes. While this approach results in consistent pricing of premiums, it does not take into account changes in factors affecting the magnitude of catastrophic losses, changes in market perceptions about catastrophe frequencies, or changes in the supply and demand of risk capital.

Our approach reverses this process. It takes traded premiums and works backwards to derive the statistical distribution which best explains these prices. This process is familiar to options traders.

Pricing Options

In 1968, Fischer Black and Myron Scholes designed a closed-form model for pricing financial options. Their model figuratively rocked the financial world. Exchanges exclusively dedicated to trading options were set up in the U.S., so that now options are traded on a wide variety of underlying securities in the over-the-counter market and on nearly every futures and securities exchange throughout the world. These days, cheap, fast computing has made the closed-form model less important, but it still lies at the core of most options pricing analyses.

Volatility is the key ingredient for the Black-Scholes model; it measures how variable the option's underlying instrument will be over a specific period of time. In the early days of options trading, traders used past prices of the underlying instrument to calculate an "historical volatility" which they plugged into the Black-Scholes model to derive the option's theoretical price.

Naturally, not everybody views history the same way. The option price traded in the market often was and is different from the price derived from a particular historical analysis of prices.

As options markets evolved, participants developed a shorthand way of pricing options that relied on the "invertibility" of the Black-Scholes formula. Portfolio managers and traders could enter the market price of a particular option into the model which would then calculate the volatility "implied" by the market. This implied volatility could then be compared directly with historical volatility to gain a sense of the cheapness or dearness of the option.

Note that neither historical nor implied volatility is the correct volatility. The only correct volatility is that which actually transpires, namely "realized volatility."

A diagram illustrating this relationship is as follows:

Historical Volatility	Implied Volatility	Realized Volatility
Based on the past	Based on current market prices	To be determined in the future (will determine if option is profitable)

So it is with our analysis of insurance pricing.

Experience Rating	Implied Loss Distribution	Realized Value of Insurance
Based on past losses	Based on current market prices	To be determined by future loss

The missing ingredients for inferring the market's underlying assumption of catastrophic losses are price

transparency and standardization. CAT contracts provide these ingredients, allowing us to derive the implied loss distribution (ILD) and thereby making implicit loss assumptions explicit.

Implied Loss Distributions

Conventional option pricing models assume that percentage changes in the price of the underlying instrument are normally distributed around the mean, which is the same as saying that the prices themselves are log-normally distributed around the current price. Based on this distribution assumption, the price of a call option is simply the discounted expected value of all outcomes above the strike price. For discrete price changes, the generalized formula is:

options price = *PV* of the sum of the [value$_i$ × probability$_i$]

for all i above the strike price where the *value*$_i$ is the *ith* price minus the strike price, and the *probability*$_i$ is the appropriate log-normally distributed probability of that outcome.

For catastrophic losses, which are characterized by a low frequency and high severity, neither the normal nor the log-normal distribution is appropriate. Neither has a sufficiently long tail measure to take account of the small but significant probability of huge losses, such as those caused by Hurricane Andrew. Its insured losses were eleven times the annual Florida premium—hardly an outcome which can be ignored. Gamma distributions do have sufficiently long tails, provided their parameters are appropriately set. We make the assumption that aggregate CAT losses are distributed according to a gamma distribution.

Once we have fitted a gamma distribution to *current market* prices, we can examine the traded price of CAT options and combinations, such as spreads, butterflies, and condors and compare them with their theoretical values. As with conventional options, the theoretical price of CATs is the discounted expected value of the option assuming the various levels of loss are gamma distributed. The difference between the market price and theoretical price is a measure of cheapness or dearness.

If the market is efficient and if the assumption of a gamma distribution is reasonable, the sum of these cheapness and dearness measures will be minimal.

Lane Financial uses a proprietary algorithm to search the family of gamma distributions and select the one which best explains all transacted prices and all bids and offers currently outstanding. That distribution is referred to as the ILD.

Like implied volatility for conventional options, ILDs are not based on history or extrapolated expectations. An ILD is simply the distribution which best explains current market prices. It does not look at all possible distributions or nonstandard distributions (although it could). It confines itself to gamma distributions in the same way that Black-Scholes implied volatility confines itself to log-normal distributions with a mean of zero.

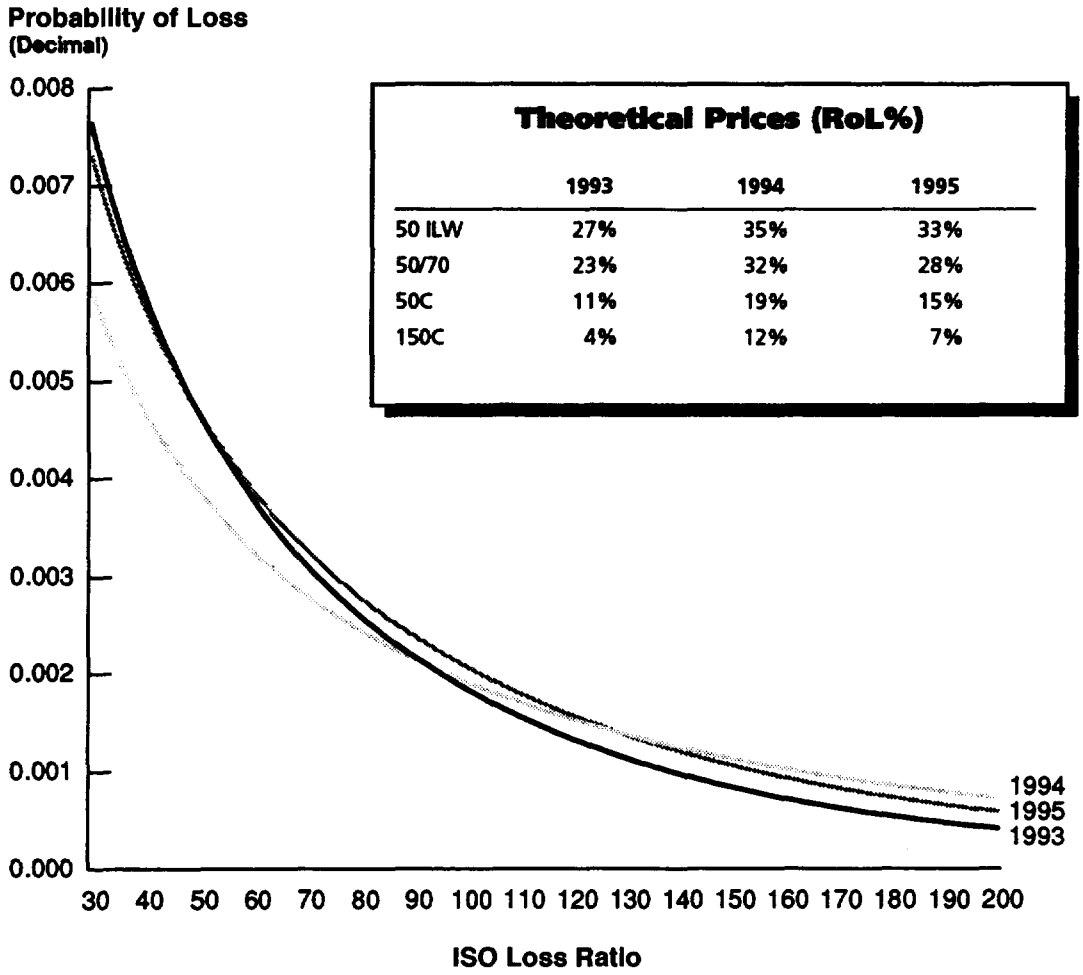
ILDs for 1993, 1994, and 1995

Given the methodology described above, Lane Financial has derived ILDs for third quarter eastern CATs (where most of the trading has taken place) using midseason 1993 prices, early season 1994 prices, and the current preseason 1995 prices. These distributions are shown in Figure 1. There are significant differences between the years, which can best be seen by comparing the implied probability of loss at a high level of attachment with the implied probability of loss at a lower attachment.

1993 was the first and "thinnest" year of CAT trading. Spread trading (simultaneously buying and selling options) did not begin until August 1993, midway through the hurricane season. The prices that did trade, however, implied that the perceived probability of high aggregate ISO losses occurring was low relative to the perceived probability of lower levels of aggregate losses. In 1994, in contrast, the market perceived and/or feared that large events were more probable (or what is the same thing, demanded and paid very high prices for covers with high attachment points) relative to lower down events. Preseason 1995 prices (trading as of February 1995) have struck a balance between the preceding years.

One way to see these interyear differences is to compare the theoretical value of a lower struck CAT call with that of a higher struck CAT call. Since CAT futures and options are capped at a 200% loss ratio, we can standardize the comparisons using rates-on-line, which is the ratio of premium to exposure. Table 1 compares the theoretical ROLs of the 50 and 150 calls for each of the three years traded.

FIGURE 1
IMPLIED LOSS DISTRIBUTIONS (THIRD QUARTER EASTERN)



Theoretical Prices (RoL%)

	1993	1994	1995
50 ILW	27%	35%	33%
50/70	23%	32%	28%
50C	11%	19%	15%
150C	4%	12%	7%

TABLE 1
THEORETICAL PRICES (ROL%)

	1993	1994	1995
50 Call	11%	19%	15%
150 Call	4	12	7

Now the difference in the relative market prices becomes apparent. Prices have changed dramatically during the past few years as supply and demand in the reinsurance market and at the CBOT have changed.

Franchise Covers versus Layers

In Table 1, a buyer would only receive full recovery from his or her CAT cover if third quarter eastern aggregate loss ratios equalled or exceeded 200%. One can think of CAT covers as proportionate covers, but more specifically, they are proportionate by industry loss, not insured loss. For example, in 1995, $1 of full cover attaching at a 50% ISO loss ratio and achieving full recovery at 200% ISO would cost 15¢.

Most trading in CAT covers has actually occurred in the 50/70 call spread. In this spread, recovery attaches at 50% ISO and is fully received above 70% ISO. Naturally, since this type of cover pays out in full sooner, it is more expensive. In 1995, the 50/70's theoretical rate-on-line is approximately 29%.

Press clippings over the past two years reveal a great deal of debate about whether the price for this layer is too expensive (and therefore should not be bought) or too cheap (and therefore should not be sold). Some of these comments are self-serving excuses for inaction. Beyond this, however, comparing the 50/70 directly with the underlying cash market is difficult. What exactly is being compared? Traditional catastrophic reinsurance programs are reinsured, not industry, specific. Industry loss warranties (a.k.a. franchise covers or original market loss warranties) usually pay out in full once the industry trigger is reached, not proportionately over a layer of industry events. Also, most ILWs are not covers for aggregate losses; they are single-event covers with reinstatement provisions. Finally, not all policies cover the same causes of loss or reporting times as the CBOT contract.

Lane Financial's theoretical model cannot resolve all these differences but can make a comparison between covers which pay out fully at a trigger level and those which pay out proportionately over a layer. For example, assume an ILW is based on a 50% ISO trigger, covers third quarter east coast (ISO) aggregate losses, and has no reinstatement provisions. The theoretical price of this warranty cover is shown in comparison with the 50/70 layer in Table 2.

TABLE 2
THEORETICAL PRICES (ROL%)

	1993	1994	1995
50% ILW	27%	35%	33%
50/70 Layer	23	32	28

What the table shows is that instant-pay warrants should be priced some 20% or more higher than the equivalent-attachment 20-point CAT layers. (It is implicitly assumed that with the ILW the reinsured will always have losses sufficient to make the claim. To the extent that this is not true, the ILW price should be discounted).

A buyer of the above 1995 50% ILW at a 35% ROL should be willing to pay up to 29% for the 50/70 CAT layer. Similarly, sellers prepared to sell the ILW should be indifferent between selling it at 36% and selling the 50/70 layer at 29%. In 1994, the seller would have been indifferent between a 50% ILW at 40% ROL and the 50/70 layer at 32% ROL.

With the above discussion, we have started an explanation of the prices at which different covers should trade. In a future paper, we will look at the implied cost of reinstatements versus aggregate cover.

The Effect of Time

The passage of time is an important factor in trading conventional financial options and can best be seen when the price of the underlying remains steady. These option prices do not decline linearly; they actually "decay" more rapidly as expiration approaches. What about insurance? What is the value of an insurance policy during its last few months if there have been no claims? Does its value decline linearly?

Third quarter eastern CAT trading over the 1994 hurricane season has provided the first insight into the effect of the passage of time on the value of insurance options as shown in Table 3. Because the 1994 season was the quietest season since 1925, it is an ideal period to gauge the effect of the passage of time. Figure 2 shows how the market assessed the risk (in probability terms) for the final three months, two months, and two weeks of the loss period. Corresponding points from the ILDs reveal that, as with financial options, prices seem to decline at an accelerating rate.

TABLE 3
THEORETICAL LEVELS (ROLS)

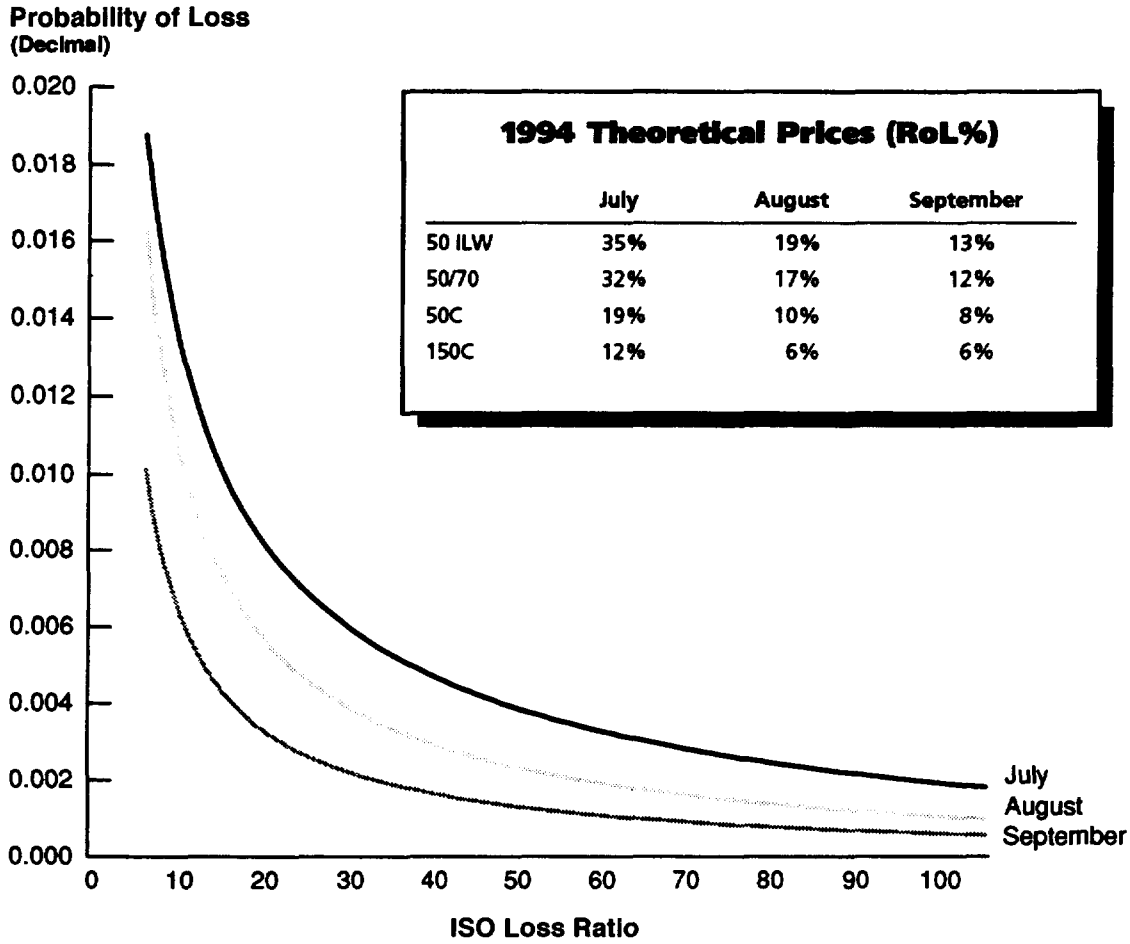
	July	Aug	Sept
50/70:	32%	17%	12%

In active years, end-of season prices are likely to be highly volatile just like financial options.

Cheapness and Dearness

This paper has mainly dealt with using implied loss distributions derived from existing market prices to value insurance derivatives. As research on these instruments (and their new cousins—PCS, CATs and crop insurance contracts) develops, more theory and analyses will occur. For the present, however, a theoretical model produces a most important framework within which to trade. Certainly Lane Financial has made successful use of these models for its own

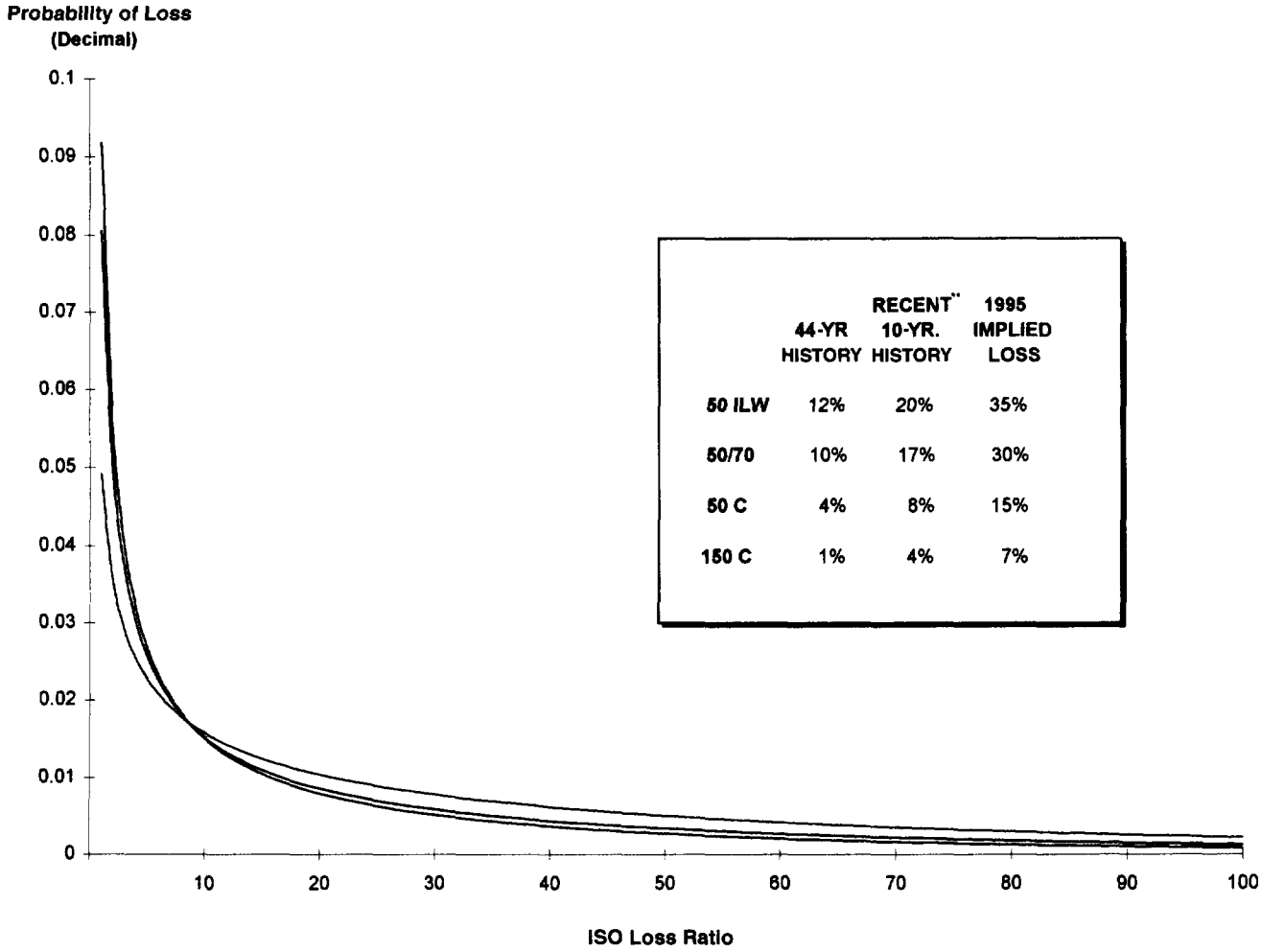
FIGURE 2
IMPLIED LOSS DISTRIBUTIONS (1994 THIRD QUARTER EASTERN)



1994 Theoretical Prices (RoL%)

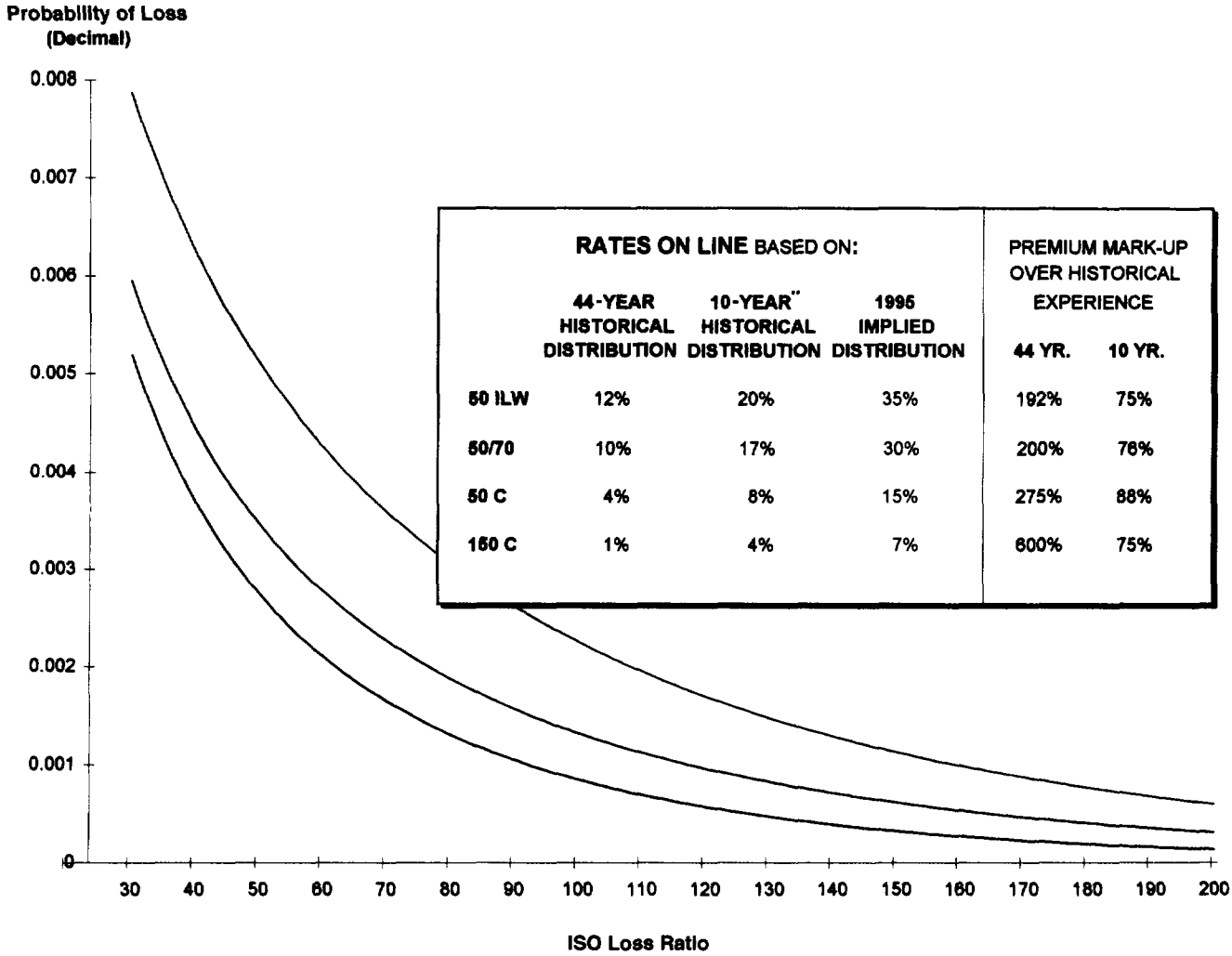
	July	August	September
50 ILW	35%	19%	13%
50/70	32%	17%	12%
50C	19%	10%	8%
150C	12%	6%	6%

FIGURE 3
HISTORICAL LOSS EXPERIENCE VERSUS IMPLIED LOSS DISTRIBUTION EARLY MAY, 1995



	44-YR HISTORY	RECENT** 10-YR. HISTORY	1995 IMPLIED LOSS
50 ILW	12%	20%	35%
50/70	10%	17%	30%
50 C	4%	8%	15%
150 C	1%	4%	7%

FIGURE 4
HISTORICAL LOSS EXPERIENCE VERSUS 1995 IMPLIED LOSS DISTRIBUTION EARLY MAY, 1995



	RATES ON LINE BASED ON:			PREMIUM MARK-UP OVER HISTORICAL EXPERIENCE	
	44-YEAR HISTORICAL DISTRIBUTION	10-YEAR** HISTORICAL DISTRIBUTION	1995 IMPLIED DISTRIBUTION	44 YR.	10 YR.
50 ILW	12%	20%	35%	192%	75%
50/70	10%	17%	30%	200%	76%
50 C	4%	8%	15%	275%	88%
150 C	1%	4%	7%	600%	75%

*Based on PCS data prior to 1969 and ISO data since 1969, back-trended at 10% per year by the CBOT. **1983 through 1992

proprietary trading and market making. Using Table 4 as an illustration, consider these prices and their theoretical values as of February 15, 1995:

TABLE 4
ACTUAL VERSUS THEORETICAL PRICES
EASTERN CATASTROPHE CONTRACTS
THIRD QUARTER 1995

Instrument	Bid/Ask	Theoretical Price
Future		47.7
Calls		
100	8.5/12.0	10.0
150	4.0/ 5.5	3.6
190	0.2/ 0.9	0.6
Call Spreads		
35/55	6.2/ 7.5	7.1
45/65	6.0/ 6.5	6.1
50/70	5.3/ 5.5	5.7
60/80	4.0/ 5.5	5.0
100/120		3.0
140/160	1.5/ 2.0	1.9
160/180		1.5
Butterflies		
20/55/90	7.0/12.5	5.5
50/100/150	/ 6.5	5.5
Condors		
40/60 100/120	1.5/	3.5
50/60 70/80	0.4/	0.7

Clearly, the theoretical model shows a number of interesting things. First, it indicates that the futures theoretical price is 46.8 (i.e., an ROL of 23.4%).

Second, of the outright calls, the 150 call seems particularly overbid at 4.0 (ROL of 8%) versus its theoretical price of 3.5. It would make a good sale.

Third, of the call spreads, the 45/65 is well bid (with its bid close to its theoretical prices), and the 50/70 is well offered at 5.5 (it is two ticks below its theoretical price). The 140/160 also provides a good offer at 2.0.

Fourth, the 20/55/90 butterfly was at one time bid at 7.0. According to the model, it should have sold because its theoretical value was only 5.1.

In the exotica corner, of the two condors where bids have been shown, the 50/60 and 70/80 are reasonably priced to theory, whereas the 40/60 and 100/120 are cheaply bid.

Finally, we have shown two prices for the 100/120 and 160/180 where no market has presently been shown. The well-prepared market-maker (who accepted the analysis proposed here) would be prepared to bid or offer around 3.1 and 1.5, respectively.

Concluding Remarks

In this paper we have discussed the differences and similarities between insurance and options pricing, derived implied loss distributions from market prices transacted at the CBOT, and used these ILDs to contribute to the debate on whether traded contracts (surrogate insurance) are cheaper or more expensive than real insurance. We have also shown how the ILDs can be used to identify cheap and expensive alternatives. Although this paper has covered a lot of ground, it has barely scratched the surface of the possibilities that lie ahead.

V

The Emerging Asset Class: Insurance Risk*

Ten years ago, the insurance industry never talked in terms of a $10 billion loss; today, the industry is concerned about a potential earthquake or hurricane loss of $50 to $100 billion. Catastrophic risks can be much larger as a result of greater concentrations of people and developed property, as well as higher exposed values than previously believed.

Reinsurance has always provided a level of surplus protection to insurance companies; however, as the stakes get higher, traditional reinsurance capacity is unable to handle these new enormous potential losses.

By establishing insurance risk as an asset class, participants (insurance and reinsurance companies, reinsurance intermediaries, capital market institutions, and investors) will provide the insurance industry with vast pools of capital to mitigate catastrophic risk through the use of new nontraditional reinsurance products. These products look like financial instruments to investors and reinsurance contracts to insurance companies. Investors can now invest directly and solely in insurance risk through these instruments without becoming a reinsurance company.

Catastrophe Exposure Coverage

The High Cost of Catastrophe Insurance

Financial innovation is changing the property-casualty insurance industry. The driving forces are the size of the industry and the economic inefficiencies surfacing in some areas of the insurance/reinsurance infrastructure. U.S. property and casualty (P&C) premiums totaled $264 billion in 1994. Of this, the most rapidly growing area is that of catastrophe (commonly referred to as CAT) exposure coverage, an area which currently accounts for anywhere between $10 and $20 billion of premiums annually. These flows are by no means small in comparison with other major financial cash flows. For example, dividends paid in 1994 by all firms listed on the NYSE were $130 billion. The sheer size of insurance markets means there are substantial rewards for value-added innovation.

High cost is one factor that attracts innovation. P&C costs are high for several reasons. First, there are limited pools of capital dedicated to P&C exposures, with the limitations being particularly severe in certain segments. A paucity of risk-takers implies poor diversification, resulting in high costs of insurance capital.

Second, innovators attempting to capture profit associated with new forms of risk distribution will have a window of opportunity to earn large returns, and, at the same time, substantially reduce system-wide costs.

Third, the information and analysis used to identify and price risk—particularly CAT—has lacked sophistication. Inadequate information generates unwillingness to

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supply insurance. It also dampens consumer demand for insurance products, as lack of hard exposure analysis makes it difficult to tell consumers a straight story justifying price and credit quality.

Fourth, widely accepted standardized packages of aggregated risks are just beginning to become accepted and used. The presence of this type of aggregated risk packaging is essential for promoting liquidity and low-cost transferability of CAT risks.

Today, these high cost barriers are eroding rapidly. With existing technology, it is possible to spread undiversifiable insurance risks better, make distribution activities more efficient, provide better identification and analysis of risks for product pricing, and create standardized packages of risks which can be priced in a broader, more liquid marketplace.

When measured by these technological capabilities, today's institutional mechanisms for financing insurance have fostered an environment for rapid evolution. Innovative approaches are springing up.

The Long and Short of CAT Exposures

Although the route that innovation will take is complex and unsure, its destination is clear: some insurance risks will be traded among investors and issuers like the securities of any other asset class. Here we focus in on the most immediately securitizable of these insurance risks: CAT exposures.

CAT exposures constitute the largest unacknowledged asset class today. However, this lack of recognition is about to change. While in the past people viewed CAT risks primarily from the perspective of an insurance customer, it will soon be commonplace that the investor perspective will be of equal importance. The job of the financial system is to redistribute CAT risks from insurance customers to investors.[1]

There is only one way to accomplish this redistribution: make insurance investors out of insurance customers. That is, people who buy insurance for themselves as customers must sell insurance to others as investors. It is as though each person is born excessively "long" for his own risks and excessively "short" for the risks of others. To undo this, financial intermediation is required. New forms of intermediation must provide the bundling, repackaging, and unbundling needed to transform reliable customer liabilities into liquid investor assets. In practice, this means that customers will obtain high-credit quality, low-cost insurance for their homes by having their pension portfolios underwrite a portion of those risks.

The notion that insurance exposures constitute an asset class might at first seem lofty and impossibly distant. Most market participants today would not consider CAT exposures as a potential asset class. However, it is hard to overemphasize how such widely-held notions are changeable. A useful example comes from that of foreign currencies.

Foreign exchange certainly meets the major requirements for an asset class. It is a large risk class (from a dollar-based investor's perspective, over half of the world portfolio is exposed to currency risk); a durable source of risk; and a distinctive exposure. It is also highly liquid, with a daily world trading volume of over $900 billion. Foreign exchange has today become widely accepted as an asset class. However, it has not always been thus.

Can Insurance Be an Asset Class?

In the early 1990s, a common theme at asset management conferences was: "Is currency an asset class?" Only four years ago, most practitioners were in agreement that foreign exchange was not an asset class on par with stocks, bonds, and real estate.

As for insurance risks, whatever the past preconceptions of investors might be, insurance exposures are clearly large enough, durable enough, and distinctive enough to be considered a separate asset class.

Some observers argue that there is no point distributing insurance risks across more investors. They maintain that certain of these risks—say a $100 billion hurricane—are too large and sudden to be managed effectively by the capital markets. This line of argument, however, is flawed. There is plenty of liquidity to finance these risks in an inexpensive and orderly way, if and when the risks are widely distributed.

While a CAT loss of $100 billion would surely be large by historical standards, it is small compared to what happens in the capital markets every day. To see this, note that a recent study puts U.S. wealth in stocks and bonds at approximately $13 trillion.[2] If one includes real estate (an illiquid but nevertheless risky asset class), U.S. financial wealth is probably at least $19 trillion. The daily standard deviation of this portfolio—which is one way to measure an average day's

fluctuation—is on the order of 70 basis points, or about $133 billion.[3]

In other words, a large and infrequently experienced CAT exposure is smaller than the average daily change in total financial wealth.

The Impact of CAT Risk on Financial Portfolios

As with any asset class, diversified investor holdings can only be achieved through the creation of low cost, standardized portfolio investments that can be purchased by pension funds, mutual funds, endowments, and high net-worth individuals. As with any asset class, the rationale for holding these instruments must lie in their portfolio properties. Too much exposure to CAT risk in a specific portfolio is clearly bad, as is too little.

As we will see, moderate amounts of exposure to CAT risk have little effect on portfolio returns while decreasing portfolio risk.

Portfolio Aspects of CAT Risks

To see the impact of CAT risks on portfolios, we need to examine their return characteristics. In the past, this would not have been possible because hard information on returns from underwriting such risks was not available.

However, in this paper, we use new pricing and claims data from actual reinsurance contracts—more than 2,000 of them—brokered by Guy Carpenter from 1970 to 1994. For much of this sample period, the data cover a substantial portion of the CAT risks reinsured by large U.S. insurance carriers. There is coverage of regional insurers in the sample as well. Most of the contracts carry CAT risks that last one year.

Our returns are those an investor would have earned by providing capital directly to fund reinsurance contracts. All of our contracts are "excess-of-loss," which means that there is a prespecified maximum amount, or limit, at risk. Each contract "layer" represents a limit in excess of a deductible. However, many of the contracts also include reinstatement commitments which require that a fresh contract of the same limit be struck should an event occur prior to the end of the contract year. Because of this reinstatement feature, it is theoretically possible that, if two large events occur during the contract period, an investor could lose twice the limit.

Calculating Returns

To calculate return, we act as though the investor puts up an amount at the beginning of the contract year equal to two times the limit, thus accounting for a potential limit reinstatement. Insurers contribute the reinsurance premium and reinstatement premium (if any) into the same dollar pool. We assume that all these funds are invested in U.S. Treasury bills, until and unless there is a drawdown due to the occurrence of an event. At the end of the year, the investor takes home all funds remaining in the account minus 1% of the limit, which we assume goes toward transaction fees.[4] Unless otherwise mentioned, we calculate the resulting investor's return as the excess above what an equal-sized investment in one-year Treasury bills would have returned.

The above return calculation is conservative, in that it assumes investors have "fully collateralized" their risk exposure. That is, there is no way that they can lose more than they have put up, and indeed, are guaranteed to get at least something back at year's end. In practice, investors tend to "lever" such investments, often putting up only a fraction of the maximum potential loss. However, with leverage, there is at least a chance that they will have to pay out more later, much like Lloyd's names have recently done. In any event, leverage has the effect of increasing, by equal factors, the magnitude of excess returns and risk.

Return on Investment

Table 1 presents summary data from the fully collateralized returns in excess of the risk-free rate. By investing in a portfolio of CAT reinsurance contracts (weighted by limit), an investor would have earned 200 basis points above the Treasury bill rate. If insurance companies were to prepay the full premiums, investors would need to put up less of their own money initially. The investment, therefore, could be fully collateralized using less investor-provided funds. As a result, the return on the same portfolio of reinsurance contracts rises to 224 basis points[5] above the Treasury bill rate.

In the best and worst years from 1970 to 1994, the excess return would have been 7.5% and −22.1%,

TABLE 1
RETURNS IN EXCESS OF RISK-FREE RATE FROM INVESTMENTS IN CAT REINSURANCE[1,2]

	Mean	Maximum	Minimum	Standard Deviation (%)	Observations
	Basis Points				
1970–1994: All Layers Limit Weighted	200	753	−2211	5.8	25
National	210	879	−2561	6.6	25
Regional	44	253	−245	1.1	25
1970–1994: Upper Layers Limit Weighted	137	599	−1843	4.3	25
National	142	726	−2258	5.2	25
Regional	43	173	−3	0.5	24
1970–1994: Lower Layers Limit Weighted	232	1198	−2970	10.8	25
National	249	1290	−3257	11.7	25
Regional	78	469	−733	3.2	25
1970–1994: All Layers Equally Weighted	237	595	−1182	3.9	23
National	248	823	−2764	7.3	25
Regional	83	364	−241	1.3	25

[1] National refers to a company that writes policies throughout the United States. Regional refers to a company that writes policies in a sub-part of the United States. A third classification, International, refers to a company whose underwriting extends beyond the United States. International contains only a few observations, so it is not shown separately, but it is included in the totals.
[2] CAT returns are net of expenses of 100 basis points of limit.

respectively. Higher average returns were earned by the lower layers, which are the excess-of-loss contracts that are most frequently impacted by loss. Also, the returns on national exposures were generally higher than returns on regional exposures. However, they were more volatile too, which is expected given national companies are, on average, more exposed to high risk areas (such as Florida and California).

Figure 1 shows that CAT investments under-performed domestic stocks, but over-performed domestic bonds.[6] It is evident from Figure 2 that the returns on CAT risks are less volatile than either stocks or bonds. Even the largest of these losses (due to Hurricane Andrew in 1992) is smaller than some of the losses in stocks.[7]

Taken together, Table 1 and Figures 1 and 2 suggest that CAT risk investments on average earned returns substantially in excess of Treasury bills during this period, much like investments in stocks and bonds. CAT investments appear somewhat less volatile than these other asset classes, but still have large downsides should a serious event occur.[8]

Zero Beta Asset

The second major characteristic of CAT exposures is that they are uncorrelated with the returns on other major asset classes, such as stocks and bonds. This point is made in Table 2. The table shows the correlations between returns on CAT exposures and other major asset classes: foreign stocks and bonds, in addition to the domestic stocks and bonds shown above. The results suggest that CAT risks are basically uncorrelated with these other asset classes. The estimated correlation coefficients between CAT exposures and other asset classes range from a low of −0.13 to a high of 0.21. None are statistically distinguishable from zero.

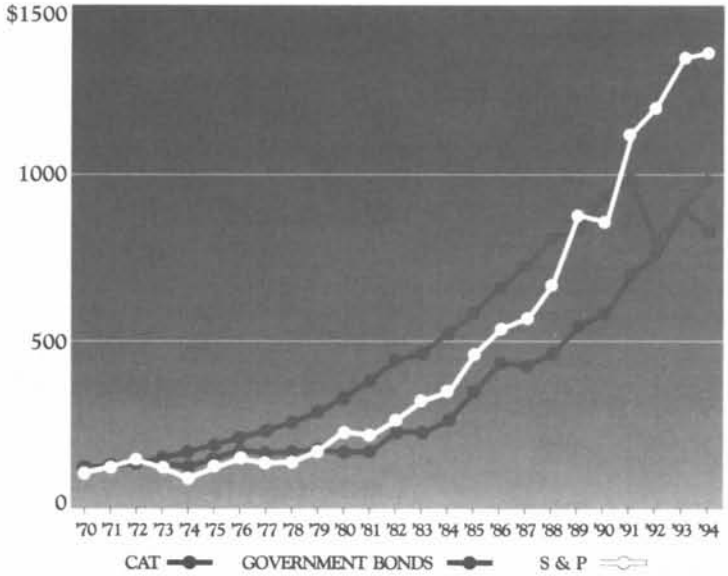
By comparison, the correlation between U.S. stocks and bonds is estimated to be much higher, at 0.40; between international stocks and international bonds at 0.45; and between international stocks and domestic stocks at 0.58. All of these latter correlations are statistically positive at standard levels of significance.

The fact that CAT risks are uncorrelated with returns on other asset classes suggests that the "fair" return on CAT risks are low. In particular, expected returns on CAT risk are likely to be lower than fair returns on stocks, which are commonly thought to be about 6% in excess of Treasury bills. The lack of correlation with all other asset classes and the small relative size of CAT exposures compared to U.S. liquid financial wealth, point toward a fair return for CAT risks that is approximately equal to the Treasury bill rate.

Portfolio Enhancements

Our findings that CAT risks have provided average returns above Treasury bills, with no correlation with

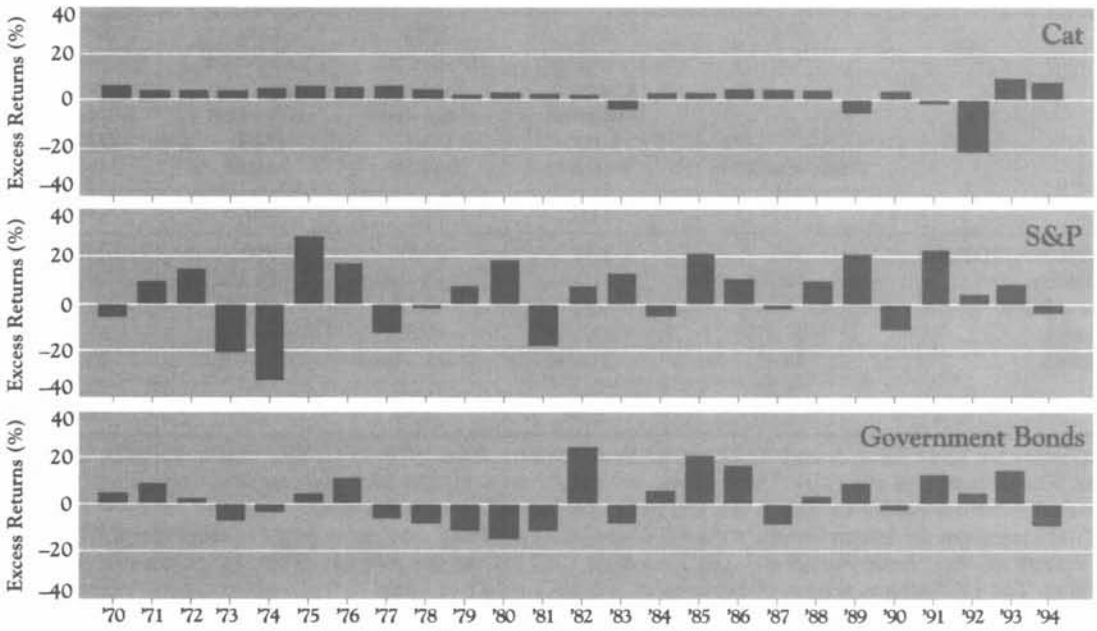
FIGURE 1
CUMULATIVE RETURNS ON ALTERNATIVE ASSET CLASSES



Cumulative return is based on an initial investment of $100 at the beginning of the period.

Cat return is based on an expense loading of 100 basis points of limit.

FIGURE 2
EXCESS RETURNS OF THREE ASSET CLASSES



Cat excess return includes an expense loading of 100 basis points of limit.

TABLE 2
RETURN CORRELATION MATRIX OF EXCESS RETURNS FOR FIVE ASSET CLASSES

	CAT Bonds	S&P 500	Government Bonds	EAFE[1]	International Bonds
CAT Bonds					
S&P 500	−0.13				
Government Bonds	−0.07	0.40			
EAFE[1]	0.21	0.58	0.25		
International Bonds	0.21	−0.03	0.24	0.45	

[1] EAFE is an index constructed from a portfolio of European, Asian and Far East stock indices. *Source:* Goldman Sachs

other major asset classes, have an important implication for portfolios: adding CAT risk products improves overall portfolio performance.

Table 3 shows what happens when CAT risk investments are mixed in with a diversified portfolio of domestic and international stocks and bonds.[9] Note that, with additional increments of CAT risks, overall portfolio return falls somewhat while portfolio volatility falls substantially. This can be seen by comparing their ratio, commonly termed the "reward-to-risk" ratio (computed by calculating the realized return minus the risk-free return divided by the standard deviation of the portfolio). This ratio rises strongly as CAT risks are included in the portfolio. Indeed, it keeps rising after the CAT risk level in the mix rises all the way to 25%. Note, that if Treasury bills are sold when CAT notes are purchased, the net effect on overall portfolio returns is more positive, and the increase in the reward-to-risk ratio is retained.

All this occurs because CAT risks help to significantly diversify the portfolio. Greater portfolio diversification and lower volatility allow the investor to take on additional high-return, high-risk investments without a substantial net increase in portfolio risk.

Conclusion

The characteristics of CAT exposures make the linkage with capital markets very desirable. The benefits to investors are:

- the improvement of the overall behavior of their portfolios,

TABLE 3
PORTFOLIO ENHANCEMENT EFFECTS OF CAT BONDS[1]

	Basic Portfolio	5% Unlevered Expense Adjusted CAT Bonds[2]	10% Unlevered Expense Adjusted CAT Bonds[2]	15% Unlevered Expense Adjusted CAT Bonds[2]	20% Unlevered Expense Adjusted CAT Bonds[2]	25% Unlevered Expense Adjusted CAT Bonds[2]
	(1)	(2)	(3)	(4)	(5)	(6)
Average Excess Return (b.p.)	313	306	298	290	282	275
Portfolio Volatility	11.8%	11.3%	10.7%	10.1%	9.5%	9.0%
Reward to Risk Ratio	0.26	0.27	0.28	0.29	0.30	0.30
Minimum Return	−22.5%	−21.1%	−19.8%	−18.5%	−17.1%	−15.8%
Maximum Return	24.4%	23.3%	22.2%	21.1%	20.0%	19.0%
Skewness[3]	−0.34	−0.32	−0.30	−0.27	−0.25	−0.22
Kurtosis[4]	2.56	2.54	2.51	2.47	2.42	2.37

[1] Summary statistics are given for a dataset of 23 annual excess returns from 1970–1992. The "Basic Portfolio" is composed of 70% domestic and 30% international securities. The domestic securities consist of 70% S&P 500 and 30% long-term U.S. government bonds. The international securities consist of 70% EAFE and 30% international government debt.

[2] Columns (2)–(6) represent the excess returns a modified portfolio containing varying amounts of unlevered CAT bonds and the complimentary amount of the "Basic Portfolio." The 23 annual CAT returns are average values for policies in that year weighted by occurrence limit. The CAT returns are net of expenses of 100 basis points of limit.

[3] Skewness is an indicator of asymmetry in returns. A negative skewness coefficient indicates that the distribution of returns "tails off" to the left. The skewness of the portfolio moves closer to 0 as the proportion of CAT bonds in the portfolio increases, a consequence of the fact that CAT returns are negatively correlated with other elements in the portfolio.

[4] Kurtosis is a measure of the peakedness of a frequency distribution. The normal distribution has a kurtosis coefficient of 3. For a given standard deviation, a distribution with high kurtosis may have more outlying points farther from the expected value (the mean).

- the opportunity to add different investments without increasing portfolio risk, and
- the creation of a larger reservoir from which to support more efficient and more reliable customer insurance products.

The historical evidence suggests the addition of CAT exposures to investment portfolios is the equivalent to a free lunch for investors and insurance consumers alike.

END NOTES

1. For the distinction between customers and investors as claimants of financial firms, see Robert C. Merton, *Continuous Time Finance*, Basil Blackwell (Cambridge), 1990.
2. This study, by the U.S. General Accounting Office, puts the value of world liquid bonds and stocks at $18.6 and $13.7 trillion, respectively. The U.S. share of the $32.2 trillion total is about $13 trillion. This number leaves out not only real estate and cash, but also illiquid financial holdings such as letter stock private-company holdings.
3. The volatility of stock returns over the last two decades has been approximately 17% per annum, or 108 basis points per day. Average daily bond-return volatilities are about 70 basis points. Daily real estate volatilities cannot be measured directly. However, real estate returns (inclusive of mortgage borrowing) display volatility just below that of stocks. For the purpose of this report, we have used the more conservative 70-basis point figure in determining average daily fluctuation.
4. We make no attempt here to reconstruct actual costs. Given the arguments above, the costs of distributing insurance exposures can be expected to be quite different going forward than in the past. One percent of limit seemed to us a sensible level of expenses, but readers can make their own judgments—and adjustments—accordingly.
5. The tables and figures assume (conservatively) that the premium is *not* fully pre-paid, so that investors put up two times the limit of their own funds.
6. Domestic stock returns are measured here as the return on the S&P 500, while bond returns are measured by the Ibbotson long-term government bond index. Returns on these asset classes have not been reduced for expenses.
7. Our annual CAT return series pools together different contract inception dates, whereas stock and bond returns represent January 1 through December 31 results. Thus, Figures 1 and 2 and Table 2 do not contain exact figures and are intended only to be indicative of CAT return behavior and correlation.
8. Of course, the average returns reported here are very sensitive to the frequency with which large storms, such as Hurricane Andrew, or earthquakes happen to occur in the sample.
9. In this exercise, we use a base portfolio of 70% domestic (of which 70% is stocks and 30% is bonds) and 30% foreign (of which 70% is stocks and 30% is bonds). To the extent that the base portfolio is less heavily invested in U.S. stocks, CAT risk provides an even more attractive addition to the portfolio than reported in Table 3. Average holdings of U.S. stocks versus bonds are considerably less than the 70%/30% assumed above.

VI

An Actuarial Bridge to Option Pricing

by Hans U. Gerber and Elias S.W. Shiu

1. Introduction

Actuaries measure, model, and manage risks. Risk associated with the investment function is a major uncertainty faced by many insurance companies. Actuaries should have knowledge of the asset side of the balance sheet of an insurance company and how it relates to the liability side. Such knowledge includes the operation of financial markets, the instruments available to the insurance companies, the options imbedded in these instruments, and the methods of pricing such options and derivative securities.

In this paper we study the pricing of financial options and derivative securities, and their synthetic replication by means of the primitive securities. We show that a time-honored concept in actuarial science, the Esscher transform, is an efficient tool for pricing many options and contingent claims if the logarithms of the prices of the primitive securities are certain stochastic processes with stationary and independent increments. The Swedish actuary F. Esscher (1932) suggested that the Edgeworth approximation (a refinement of the normal approximation) yields better results, if it is applied to a modification or transformation of the original distribution of aggregate claims. Here, this Esscher transform is defined more generally as a change of measure for certain stochastic processes. An Esscher transform of such a process induces an equivalent probability measure on the process. The Esscher parameter or parameter vector is determined so that the discounted price of each primitive security is a martingale under the new probability measure. A derivative security or contingent claim is valued as the expectation, with respect to this equivalent martingale measure, of the discounted payoffs.

Although there may be more than one equivalent martingale measure, in general, the risk-neutral Esscher measure provides a unique and transparent answer, which can be justified if there is a representative investor maximizing his or her expected utility. The option price is unique whenever a self-financing replicating portfolio can be constructed. This is the case in the multidimensional geometric Brownian motion model and also in the multidimensional geometric shifted compound Poisson process model. The latter is at the same time simpler (in view of its sample paths) and richer (the former can be retrieved as a limit). The Esscher method can be extended to pricing the derivative securities of (possibly) dividend-paying stocks.

We show that, in the case of a multidimensional geometric Brownian motion model, the price of a European option does not depend on the interest rate, provided that the payoff is a homogeneous function of degree one with respect to the stock prices. Moreover, with the aid of Esscher transforms, a change of numeraire can be discussed in a concise way.

2. The Esscher Transform of a Single Random Variable

Let Y be a given random variable and h a nonzero real number for which the expectation

$$E[e^{hY}]$$

exists. The positive random variable

$$\frac{e^{hY}}{E[e^{hY}]} \tag{2.1}$$

can be used (as the Radon-Nikodym derivative) to define a new probability measure, which is *equivalent* to the old measure in the sense that they both have the same null sets (sets of measure zero). In other words, the old and new measures are mutually absolutely continuous. For a measurable function ψ, the expectation of the random variable $\psi(Y)$ with respect to the new measure is

$$E[\psi(Y);\, h] = \frac{E[\psi(Y)e^{hY}]}{E[e^{hY}]}. \tag{2.2}$$

We call this new measure the *Esscher measure* of parameter h. The corresponding distribution is usually called the *Esscher transform* in the actuarial literature (Esscher, 1932; Philipson, 1963; Jensen, 1991). In some statistical literature, the term *exponential tilting* is used to describe this change of measure.

The method of Esscher transforms was developed to approximate the aggregate claim amount distribution around a point of interest, y_0, by applying an analytic approximation (the first few terms of the Edgeworth series) to the transformed distribution with the parameter $h=h_0$ chosen such that the new mean is equal to y_0. Let

$$c(h) = \ln(E[e^{hY}]) \tag{2.3}$$

be the cumulant-generating function. Its first- and second-order derivatives are

$$c'(h) = \frac{E[Ye^{hY}]}{E[e^{hY}]} = E[Y;\, h] \tag{2.4}$$

and

$$c''(h) = \frac{E[Y^2e^{hY}]}{E[e^{hY}]} - \left(\frac{E[Ye^{hY}]}{E[e^{hY}]}\right)^2 = \mathrm{Var}[Y;\, h]. \tag{2.5}$$

Since $\mathrm{Var}[Y;\, h]>0$ for a nondegenerate random variable Y, the function $c'(h)$ is strictly increasing; thus the number h_0 for which

$$y_0 = c'(h_0) = E[Y;\, h_0]$$

is unique. In using the Esscher transform to calculate a stop-loss premium, the parameter h_0 is usually chosen such that the mean of the transformed distribution is the retention limit.

3. Discrete-Time Stock-Price Models

A purpose of this paper is to show that the concept of Esscher measures is an effective tool for pricing stock options and other derivative securities. We need to extend the change of measure for a single random variable to that for a stochastic process. In this section we consider the simpler case of discrete-time stochastic processes.

For $j=0, 1, 2, \ldots$, let $S(j)$ denote the price of a stock a time j. Assume that there is a sequence of independent (but not necessarily identically distributed) random variables $\{Y_k\}$ such that

$$S(j) = S(0)\exp(Y_1 + Y_2 + \cdots + Y_j), \quad j = 1, 2, 3, \ldots \tag{3.1}$$

Assume that the moment generating function for each Y_i exists, and write

$$M_{Y_i}(h) = E[e^{hY_i}]. \tag{3.2}$$

For a sequence of real numbers $\{h_k\}$, define

$$Z_j = \exp\Big(\sum_{k\le j} h_kY_k\Big)\Big/\prod_{k\le j} M_{Y_k}(h_k). \tag{3.3}$$

Then $\{Z_j\}$ is a positive martingale which can be used to define a change of measure for the stock-price process. For a positive integer m, let $\psi(m)$ be a random variable that is a function of $Y_1, \ldots, Y_m$,

$$\psi(m) = \psi(Y_1, \ldots, Y_m). \tag{3.4}$$

The expected value of $\psi(m)$, with respect to the new measure, is

$$E[\psi(m)\, Z_m]. \tag{3.5}$$

In (3.5) the random variable Z_m can be replaced by Z_j, $j\ge m$, because of the martingale property.

We assume that the risk-free interest rate is constant through time and the stock pays no dividends. Let δ denote the risk-free force of interest. The *risk-neutral Esscher measure* is the measure, defined by the sequence of numbers $\{h_k^*\}$, with respect to which

$$\{e^{-\delta j}S(j);\, j = 0, 1, 2, \ldots\} \tag{3.6}$$

is a martingale. This leads to

$$e^{\delta} = M_{Y_k}(1 + h_k^*)/M_{Y_k}(h_k^*), \quad k = 1, 2, 3, \ldots \tag{3.7}$$

As we pointed out at the end of the last section, the numbers $\{h_k^*\}$ are unique.

Suppose that each Y_k is a Bernoulli random variable, i.e., it takes on two distinct values, a_k and b_k, only. Then there is only one risk-neutral measure, given by

$$Pr^*(Y_k = a_k) = \frac{e^{\delta} - e^{b_k}}{e^{a_k} - e^{b_k}} \tag{3.8}$$

and

$$Pr^*(Y_k = b_k) = \frac{e^{\delta} - e^{a_k}}{e^{b_k} - e^{a_k}}. \tag{3.9}$$

(To rule out arbitrage opportunities we assume that the force of interest δ is between a_k and b_k for each k.)

If we assume that the random variables $\{Y_k\}$ are identically distributed in addition to being independent, then all h_k^* are the same number. This points to an approach to extend the change of measure to certain continuous-time models, as we shall see in Section 5. On the other hand, the risk-neutral Esscher measure can also be defined for dependent random variables $\{Y_k\}$. In this more general situation, each h_k^* is a function of $Y_1, Y_2, \ldots, Y_{k-1}$ and thus a random variable itself.

4. Fundamental Theorem of Asset Pricing

In this paper we assume that the market is frictionless and trading is continuous. There are no taxes, no transaction costs, and no restriction on borrowing or short sales. All securities are perfectly divisible. It is now understood that, in such a security market model, the absence of arbitrage is "essentially" equivalent to the existence of a *risk-neutral measure* or an *equivalent martingale measure*, with respect to which the *price* of a random payment is the expected discounted value. Dybvig and Ross (1987) call this result the *Fundamental Theorem of Asset Pricing*. In general, there may be more than one equivalent martingale measure. A merit of the risk-neutral Esscher measure is that it provides a general, transparent, and unambiguous solution.

That the condition of no arbitrage is intimately related to the existence of an equivalent martingale measure was first pointed out in Harrison and Kreps (1979) and Harrison and Pliska (1981, 1983). Their results are rooted in the idea of the risk-neutral valuation of Cox and Ross (1976). In a finite discrete-time model, the absence of arbitrage opportunities is equivalent to the existence of an equivalent martingale measure (Dalang, Morton, and Willinger 1990; Schachermayer 1992b). In a more general setting the characterization is delicate, and we have to replace the term "equivalent to" by "essentially equivalent to." It is beyond the scope of the present paper to discuss the details. Some recent papers are Artzner and Heath (1995), Back (1991), Back and Pliska (1991), Christopeit and Musiela (1994), Cox and Huang (1989), Delbaen (1992), Delbaen and Schachermayer (1994a, 1994b), Frittelli and Lakner (1994), Müller (1989), Schachermayer (1992a, 1994), Schweizer (1992), and Stricker (1993).

We note that the idea of changing the probability measure to obtain a consistent positive linear pricing rule has appeared in the actuarial literature in the context of equilibrium reinsurance markets. See Borch (1960, 1990), Bühlmann (1980, 1984), Deprez and Gerber (1985), Lienhard (1986), Gerber (1987), Sonderman (1991), Aase (1993a, 1993b), and Chevallier and Müller (1994).

5. Continuous-Time Stock-Price Models

In the rest of the paper, we consider continuous-time stock-price models. For $t \geq 0$, let $S(t)$ denote the price at time t of a nondividend-paying stock. We assume that there is a stochastic process $\{X(t)\}$ with independent and stationary increments such that

$$S(t) = S(0)\, e^{X(t)}, \quad t \geq 0. \tag{5.1}$$

For a theoretical "justification" that stock prices should be modeled with such processes, see Samuelson (1965) or Parkinson (1977). (Some authors call $\{X(t)\}$ a *Lévy process*.) To rule out arbitrage opportunities, we need the condition that $X(t) - \delta t$ assumes positive and negative values. If this were not the case, for example, if $X(t) - \delta t \geq 0$ for all t, we would have

$$S(0)e^{\delta t} \leq S(0)e^{X(t)};$$

thus, by borrowing $S(0)$ and investing it in the stock, one could make a sure profit (unless $X(t) \equiv \delta t$). This condition is analogous to the condition in the Bernoulli example at the end of Section 3 that δ is between a_k and b_k.

We assume that the moment generating function of $X(t)$,

$$M(h, t) = E[e^{hX(t)}],$$

exists and that

$$M(h, t) = M(h, 1)^t. \tag{5.2}$$

The process

$$\{e^{hX(t)} M(h, 1)^{-t}\} \tag{5.3}$$

is a positive martingale and can be used to define a change of probability measure, i.e., it can be used to define the Radon-Nikodym derivative dQ/dP, where P is the original probability measure and Q is the Esscher measure of parameter h. The *risk-neutral Esscher measure* is the Esscher measure of parameter $h=h^*$ such that the process

$$\{e^{-\delta t} S(t)\} \tag{5.4}$$

is a martingale.

The condition

$$E[e^{-\delta t}S(t); h^*] = e^{-\delta 0}S(0) = S(0)$$

yields

$$\begin{aligned} e^{\delta t} &= E[e^{(1+h^*)X(t)} M(h^*, 1)^{-t}] \\ &= [M(1 + h^*, 1)/M(h^*, 1)]^t, \end{aligned}$$

or

$$e^{\delta} = M(1 + h^*, 1)/M(h^*, 1), \tag{5.5}$$

which is analogous to (3.7) with $\{Y_k\}$ being identically distributed. The parameter h^* is unique. There may be many other equivalent martingale measures.

Because, for $t \geq 0$,

$$e^{hX(t)} M(h, 1)^{-t} = \frac{e^{hX(t)}}{E[e^{hX(t)}]} = \frac{S(t)^h}{E[S(t)^h]}, \tag{5.6}$$

we have the following: Let g be a measurable function and h, k, and t be real numbers, $t \geq 0$; then

$$\begin{aligned} &E[S(t)^k g(S(t)); h] \\ &= E[S(t)^k g(S(t)) e^{hX(t)} M(h, 1)^{-t}] \\ &= \frac{E[S(t)^{h+k} g(S(t))]}{E[S(t)^h]} \\ &= \frac{E[S(t)^{h+k}]}{E[S(t)^h]} \frac{E[S(t)^{h+k} g(S(t))]}{E[S(t)^{h+k}]} \\ &= E[S(t)^k; h] E[g(S(t)); h + k]. \end{aligned} \tag{5.7}$$

This *factorization formula* simplifies many calculations and is a main reason why the method of Esscher measures is an efficient device for valuing certain derivative securities. For example, applying (5.7) with $k=1$, $h=h^*$ and $g(x)=I(x>K)$ [where $I(A)$ denotes the indicator random variable of an event A], we obtain

$$\begin{aligned} &E[S(\tau) I(S(\tau) > K); h^*] \\ &\quad = E[S(\tau); h^*] E[I(S(\tau) > K); h^* + 1] \\ &\quad = E[S(\tau); h^*] Pr[S(\tau) > K; h^* + 1] \\ &\quad = S(0)e^{\delta\tau} Pr[S(\tau) > K; h^* + 1]. \end{aligned} \tag{5.8}$$

The last equality holds because (5.4) is a martingale with respect to the risk-neutral Esscher measure. Thus we have a pricing formula for a European call option on a nondividend-paying stock,

$$\begin{aligned} &E[e^{-\delta\tau} (S(\tau) - K)_+; h^*] \\ &\quad = E[e^{-\delta\tau} (S(\tau) - K) I(S(\tau) > K); h^*] \\ &\quad = e^{-\delta\tau}\{E[S(\tau) I(S(\tau) > K); h^*] \\ &\qquad - KE[I(S(\tau) > K); h^*]\} \\ &\quad = S(0)Pr[S(\tau) > K; h^* + 1] \\ &\qquad - Ke^{-\delta\tau}Pr[S(\tau) > K; h^*]. \end{aligned} \tag{5.9}$$

For $\{X(t)\}$ being a Wiener process, (5.9) is the celebrated Black-Scholes formula; see also (10.20) below.

6. Representative Investor with Power Utility Function

When there is more than one equivalent martingale measure, why should the option price be the expectation, with respect to the risk-neutral Esscher measure, of the discounted payoff? This particular choice may be justified within a utility function framework. Consider a simple economy with only a stock and a risk-free bond and their derivative securities. There is a representative investor who owns m shares of the stock and bases his or her decisions on a risk-averse utility function $u(x)$. Consider a derivative security that provides a payment of $\pi(\tau)$ at time τ, $\tau>0$; $\pi(\tau)$ is a function of the stock price process until time τ. What is the investor's *price* for the derivative security, such that it is optimal for him or her not to buy or sell any fraction or multiple of it? Let $V(0)$ denote this price. Then, mathematically, this is the condition that the function

$$\phi(\eta) = E[u(mS(\tau) + \eta[\pi(\tau) - e^{\delta\tau}V(0)])] \tag{6.1}$$

is maximal for $\eta=0$. From

$$\phi'(0) = 0,$$

we obtain

$$V(0) = e^{-\delta\tau} \frac{E[\pi(\tau)u'(mS(\tau))]}{E[u'(mS(\tau))]} \tag{6.2}$$

(as a necessary and sufficient condition, since $\phi''(\eta)<0$ if $u''(x)<0$). In the particular case of a power utility function with parameter $c>0$,

$$u(x) = \begin{cases} \dfrac{x^{1-c}}{1-c} & \text{if } c \neq 1 \\ \ln x & \text{if } c = 1 \end{cases}, \tag{6.3}$$

we have $u'(x)=x^{-c}$, and

$$\begin{aligned} V(0) &= e^{-\delta\tau} \frac{E[\pi(\tau)[mS(\tau)]^{-c}]}{E[[mS(\tau)]^{-c}]} \\ &= e^{-\delta\tau} \frac{E[\pi(\tau)S(\tau)^{-c}]}{E[S(\tau)^{-c}]}. \end{aligned} \tag{6.4}$$

Formula (6.4) must hold for all derivative securities. For $\pi(\tau)=S(\tau)$ and therefore $V(0)=S(0)$, (6.4) becomes

$$\begin{aligned} S(0) &= e^{-\delta\tau} \frac{E[S(\tau)^{1-c}]}{E[S(\tau)^{-c}]} \\ &= e^{-\delta\tau} S(0) \frac{M(1-c, \tau)}{M(-c, \tau)}, \end{aligned}$$

or

$$e^{\delta} = \frac{M(1-c, 1)}{M(-c, 1)}. \tag{6.5}$$

On comparing (6.5) with (5.5), we see that the value of the parameter c is $-h^*$. Hence $V(0)$ is indeed the discounted expectation of the payoff $\pi(\tau)$, calculated with respect to the Esscher measure of parameter $h^*=-c$.

By considering different points in time τ, we get a consistency requirement. This is satisfied if the investor has a power utility function. We conjecture that it is violated for any other risk-averse utility function, which implies that the pricing of an option by the risk-neutral Esscher measure is a consequence of the consistency requirement. Some related papers are Rubinstein (1976), Bick (1987, 1990), Constantinides (1989), Naik and Lee (1990), Stapleton and Subrahmanyam (1990), He and Leland (1993), Heston (1993), and Wang (1993).

7. Logarithm of Stock Price as a Shifted Poisson Process

Here we consider the so-called pure jump model. The assumption is

$$X(t) = kN(t) - ct, \tag{7.1}$$

where $\{N(t)\}$ is a Poisson process with parameter λ, and k and c are constants with $k\neq 0$. Then the price of the nondividend-paying stock is modeled as

$$S(t) = S(0)e^{kN(t)-ct}. \tag{7.2}$$

The condition that $X(t)-\delta t$ assumes positive and negative values is that k and $c+\delta$ have the same sign. This model contains the classical Wiener process model as a limiting case. Note that

$$E[X(1)] = k\lambda - c \tag{7.3}$$

and

$$\mathrm{Var}[X(1)] = k^2\lambda. \tag{7.4}$$

Suppose that we vary k, λ, and c so that (7.3) and (7.4) remain the constant values μ and σ^2, respectively, i.e., we set

$$\lambda = (\sigma/k)^2 \tag{7.5}$$

and

$$c = (\sigma^2/k) - \mu. \tag{7.6}$$

In the limit as $k\to 0$, $\{X(t)\}$ has continuous sample paths, and hence it is a Wiener process with drift μ and infinitesimal variance σ^2. This is illustrated in the two graphs in Figure 1. We note that the discontinuities of the second sample path are not recognizable and that it appears to be a sample path of a Wiener process.

We now determine the risk-neutral Esscher measure according to Section 5. Since

$$E[e^{zN(t)}] = \exp[\lambda t(e^z - 1)],$$

we have

$$\begin{aligned} M(z, t) &= E[e^{zX(t)}] \\ &= E(e^{z[kN(t)-ct]}) \\ &= \exp([\lambda(e^{zk} - 1) - zc]t). \end{aligned} \tag{7.7}$$

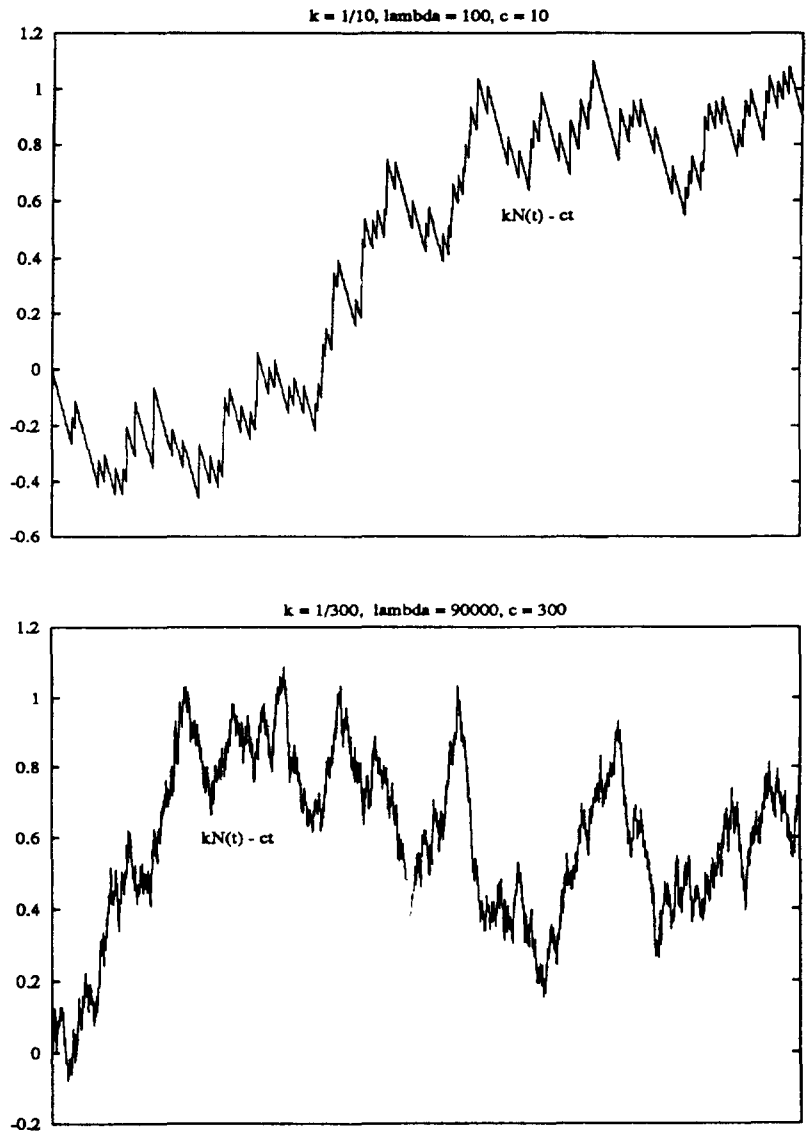
Because

$$\begin{aligned} E[e^{zX(t)}; h] &= \frac{M(z+h, t)}{M(h, t)} \\ &= \exp([\lambda e^{hk}(e^{zk} - 1) - zc]t), \end{aligned}$$

we see that, under the Esscher measure of parameter h, the process $\{X(t)\}$ remains a shifted Poisson process, but with modified Poisson parameter λe^{hk}. Formula (5.5) is the condition that

$$\delta = \lambda e^{h^*k}(e^k - 1) - c. \tag{7.8}$$

FIGURE 1
SAMPLES PATHS OF TWO SHIFTED POISSON PROCESSES WITH $\mu = 0$ AND $\sigma^2 = 1$



Hence, the risk-neutral Esscher measure is the measure with respect to which $\{N(t)\}$ becomes a Poisson process with parameter

$$\begin{aligned}\lambda^* &= \lambda e^{h^* k}. \\ &= (\delta + c)/(e^k - 1). \qquad (7.9)\end{aligned}$$

The ratio in (7.9) is indeed a positive number because we have imposed the condition that k and $\delta+c$ have the same sign.

Consider a European option or contingent claim with exercise date τ and payoff function $\Pi(s)$. [At time τ the option owner receives $\Pi(S(\tau))$.] The method of the Esscher transforms prices the option as the expected discounted payoff, with the expectation taken with respect to the risk-neutral Esscher measure. That is, for $t \leq \tau$, the *Esscher option price* at time t, with $S(t)=s$, is the conditional expectation.

$$\begin{aligned}V(s, t) &= E[e^{-\delta(\tau-t)}\Pi(S(\tau))|S(t) = s; h^*] \\ &= e^{-\delta(\tau-t)} \sum_{n=0}^{\infty} Pr[N(\tau) - N(t) = n; h^*]\, \Pi(se^{nk-c(\tau-t)}) \\ &= e^{-(\delta+\lambda^*)(\tau-t)} \sum_{n=0}^{\infty} \frac{[\lambda^*(\tau - t)]^n}{n!}\, \Pi(se^{nk-c(\tau-t)}). \qquad (7.10)\end{aligned}$$

In this stock price model, the option price is unique and given by (7.10). To see this, we construct a self-financing portfolio of the stock and risk-free bond whose value at time t is $V(S(t), t)$. The amounts invested in the stock and bond are dynamically adjusted. The term *self-financing* means that, once started, no funds are added or withdrawn from the portfolio until the option exercise date τ. As $t \to \tau$, $V(S(t), t) \to \Pi(S(\tau))$ with certainty; hence, at time τ, the value of the self-financing portfolio is equal to the payoff of the option. Consequently, the option price at any previous point in time must be identical to the portfolio value at that time, i.e., the option price is indeed the Esscher option price (7.10).

For $t<\tau$, let $\eta(S(t), t)$ denote the amount in the portfolio invested in the stock at time t; therefore the difference $V(S(t), t) - \eta(S(t), t)$ is the amount invested in the risk-free bond at time t. The crucial question is whether we can define $\eta(S(t), t)$ so that the portfolio is self-financing, i.e., that the investment gain of the portfolio is identical to the change of the portfolio value, as defined by V, in any time interval and in any situation. We have to examine two scenarios.

First, we consider the case where the stock price has a discontinuity at time t, jumping from $S(t)$ to $S(t)e^k$. Then the condition that the instantaneous investment gain is equal to the instantaneous change of the portfolio value yields the equation

$$\begin{aligned}&\eta(S(t), t)e^k - \eta(S(t), t) \\ &\qquad = V(S(t)e^k, t) - V(S(t), t), \qquad (7.11)\end{aligned}$$

resulting in the condition

$$\eta(S(t), t) = \frac{V(S(t)e^k, t) - V(S(t), t)}{e^k - 1}. \qquad (7.12)$$

Second, we consider the case where the stock price process does not have a jump in a time interval around a certain point t_0. Let $S(t_0)=s$. For t in the time interval, we have

$$S(t) = se^{-c(t-t_0)} \qquad (7.13)$$

and

$$\begin{aligned}&V(S(t), t) \\ &\quad = V(se^{-c(t-t_0)}, t) \\ &\quad = e^{-(\delta+\lambda^*)(\tau-t)} \sum_{n=0}^{\infty} \frac{[\lambda^*(\tau - t)]^n}{n!}\, \Pi(se^{nk-c(\tau-t_0)}). \qquad (7.14)\end{aligned}$$

Thus

$$\begin{aligned}&\frac{d}{dt} V(S(t), t) \\ &\qquad = (\delta + \lambda^*)V(S(t), t) - \lambda^* V(S(t)e^k, t), \qquad (7.15)\end{aligned}$$

and the instantaneous change of the portfolio value is

$$[(\delta + \lambda^*)V(S(t), t) - \lambda^* V(S(t)e^k, t)]dt. \qquad (7.16)$$

On the other hand, the instantaneous investment gain of the portfolio is

$$\eta(S(t), t)(-cdt) + [V(S(t), t) - \eta(S(t), t)](\delta dt). \qquad (7.17)$$

Here, the condition that the instantaneous investment gain is equal to the instantaneous change of the portfolio value yields the equation

$$\begin{aligned}&-c\eta(S(t), t) + \delta[V(S(t), t) - \eta(S(t), t)] \\ &\qquad = (\delta + \lambda^*)V(S(t), t) - \lambda^* V(S(t)e^k, t),\end{aligned}$$

or

$$\eta(S(t), t) = \frac{\lambda^*}{c + \delta}[V(S(t\, e^k, t) - V(S(t), t)]. \qquad (7.18)$$

Since λ^* is defined by (7.9), conditions (7.12) and (7.18) are equivalent, and with this choice of $\eta(S(t), t)$, the portfolio is indeed self-financing.

Observe that, in constructing the self-financing portfolio which replicates the option payoff, we did not need $\{N(t)\}$ to be a Poisson process. The self-financing portfolio can be constructed because in each infinitesimal time interval exactly two scenarios are possible: a jump with known magnitude, or no jumps. Thus $N(t)$ in (7.1) and (7.2) can be assumed to come from a more general class of counting processes; the equivalent martingale measure is the measure with respect to which $\{N(t)\}$ becomes a Poisson process with parameter λ^* given by (7.9).

It is of interest to consider the limiting case where $k \to 0$, with λ and c varying according to (7.5) and (7.6). With respect to the risk-neutral Esscher measure, the drift and infinitesimal variance of the process (7.1) are $k\lambda^* - c$ and $k^2\lambda^*$, respectively. Because of (7.6),

$$\lim_{k \to 0} kc = \sigma^2. \tag{7.19}$$

Hence

$$\begin{aligned} k\lambda^* - c &= k\frac{\delta + c}{e^k - 1} - c \\ &= \delta\frac{k}{e^k - 1} - c\frac{e^k - 1 - k}{e^k - 1} \\ &\to \delta - \frac{\sigma^2}{2} \end{aligned} \tag{7.20}$$

and

$$\begin{aligned} k^2\lambda^* &= k^2\frac{\delta + c}{e^k - 1} \\ &\to \sigma^2 \end{aligned} \tag{7.21}$$

as $k \to 0$. Thus, in the limit the risk-neutral Esscher measure corresponds to the Wiener process with drift (7.20) and infinitesimal variance (7.21).

Furthermore, in the limit ($k \to 0$), formula (7.12) becomes

$$\eta(S(t), t) = S(t)V_s(S(t), t), \tag{7.22}$$

showing that the number of shares in the replicating portfolio at time t, $\eta(S(t),t)/S(t)$, is given by the partial derivative $V_s(S(t),t)$, which is usually called *delta* in the option literature. Also, by means of the Taylor expansion, we have

$$\begin{aligned} &\lambda^*[V(S(t)e^k, t) - V(S(t), t)] \\ &\quad = \lambda^*\{(e^k - 1)S(t)V_s(S(t), t) \\ &\qquad + [(e^k - 1)S(t)]^2V_{ss}(S(t), t)/2 + \ldots\} \\ &\quad = (\delta + c)S(t)V_s(S(t), t) \\ &\qquad + \sigma^2S(t)^2V_{ss}(S(t), t)/2 + \ldots \end{aligned} \tag{7.23}$$

Substituting (7.23) in the right-hand side of (7.15) and

$$\frac{d}{dt}V(S(t), t) = V_s(S(t), t)[-cS(t)] + V_t(S(t), t) \tag{7.24}$$

in its left-hand side, canceling the cSV_s terms, and letting k tend to 0 yields the equation

$$\begin{aligned} V_t(S(t), t) &= \delta V(S(t), t) - \delta S(t)V_s(S(t), t) \\ &\quad - \frac{\sigma^2}{2}S(t)^2V_{ss}(S(t), t), \end{aligned} \tag{7.25}$$

which was first derived by Black and Scholes (1973) with a replicating portfolio argument.

8. Extension to Multiple Assets

In this section we extend the model in the last section to more than one nondividend-paying stock. For $j = 1, 2, \ldots, n$, let $S_j(t)$ denote the price of stock j at time t, $t \geq 0$, and write

$$X_j(t) = ln[S_j(t)/S_j(0)]. \tag{8.1}$$

Generalizing (7.1) we model the processes $\{X_1(t)\}$, $\{X_2(t)\}, \ldots, \{X_n(t)\}$ as shifted compound Poisson processes with

$$E[X_j(t)] = \mu_j t \tag{8.2}$$

and

$$\mathrm{Cov}(X_i(t), X_j(t)) = \sigma_{ij}t, \tag{8.3}$$

where μ_j and σ_{ij}, $1 \leq i, j \leq n$, are constants with the n-by-n matrix

$$(\sigma_{ij}) \tag{8.4}$$

being positive definite. Let $k \neq 0$, and let $\{N_1(t)\}$, $\{N_2(t)\}, \ldots, \{N_n(t)\}$ be n independent Poisson processes, each with the same parameter value

$$\lambda = k^{-2}. \tag{8.5}$$

Let $\{a_{ij};\ 1 \leq i, j \leq n\}$ be n^2 numbers such that

$$\sum_{j=1}^{n} a_{ij}\, a_{hj} = \sigma_{ih}, \quad 1 \leq i, h \leq n. \tag{8.6}$$

(The numbers a_{ij} are not unique. One way to obtain them is the Choleski factorization algorithm. As n-by-n matrices, they are related to each other by left

multiplication with orthogonal matrices.) For $i=1, 2, \ldots, n$, define

$$c_i = \frac{1}{k}\sum_{j=1}^{n} a_{ij} - \mu_i \tag{8.7}$$

and

$$X_i(t) = k\sum_{j=1}^{n} a_{ij}N_j(t) - c_i t. \tag{8.8}$$

Then (8.2) and (8.3) are satisfied. (To reconcile with the notation in this section, one would replace the k in the last section by $k\sigma$.)

Let

$$\mathbf{X}(t) = (X_1(t), X_2(t), \ldots, X_n(t))'. \tag{8.9}$$

The process $\{\mathbf{X}(t)\}$ has independent and stationary increments with drift vector

$$(\mu_1, \mu_2, \ldots, \mu_n)' \tag{8.10}$$

and covariance matrix per unit time (8.4). With μ_i and a_{ij} held fixed, as $k \to 0$ [λ and c_i varying according to (8.5) and (8.7)], the limiting stochastic process $\{\mathbf{X}(t)\}$ has continuous sample paths and is thus an n-dimensional Wiener process with drift vector (8.10) and diffusion matrix (8.4).

To rule out arbitrage opportunities in the model of the last section, we imposed the condition that k and $\delta + c$ have the same sign, which, in turn, guarantees that the new Poisson parameter as defined by (7.9),

$$\lambda^* = (\delta + c)/(e^k - 1),$$

is positive. Here, we need to generalize the condition to one on the parameters a_{ij}, c_i, k, and δ. We make the following assumption: if $\eta_1, \eta_2, \ldots, \eta_n$ are any n real numbers such that

$$\sum_{i=1}^{n} \eta_i(e^{a_{ij}k} - 1) \geq 0, j = 1, 2, \ldots, n, \tag{8.11}$$

with strict inequality for at least one j, then this implies that

$$\sum_{i=1}^{n} \eta_i(c_i + \delta) > 0. \tag{8.12}$$

This assumption can be justified by an arbitrage argument. If it were violated, there would be n real numbers, $\eta_1, \eta_2, \ldots, \eta_n$, satisfying (8.11), with strict inequality for at least one j, and such that

$$\sum_{i=1}^{n} \eta_i(c_i + \delta) \leq 0. \tag{8.13}$$

Inequality (8.13) is equivalent to

$$\sum_{i=1}^{n} \eta_i + \left[\sum_{i=1}^{n} \eta_i(-c_i)\right]\bar{s}_{\overline{t}|} \geq \left(\sum_{i=1}^{n} \eta_i\right)e^{\delta t}, \quad t \geq 0, \tag{8.14}$$

both sides of which have economic interpretations. If the amount of

$$\sum_{i=1}^{n} \eta_i$$

is invested in the risk-free bond at time 0, the expression on the right-hand side is the (accumulated) value of the investment at time t. An alternative, more sophisticated investment strategy is to invest the amount of η_i in stock i at time 0 and keep this amount fixed at all subsequent times by investing all gains (or losses) in the risk-free bond, $i=1, 2, \ldots, n$. The expression on the left-hand side of (8.14) is the value at time t of this investment portfolio if no jumps have yet occurred. When the first jump occurs, say, due to Poisson process j, the instantaneous change of portfolio value is

$$\sum_{i=1}^{n} \eta_i(e^{a_{ij}k} - 1). \tag{8.15}$$

Because (8.15) is nonnegative for all j and positive for at least one, we see how a risk-free profit can be made: by selling a bond of the amount

$$\sum_{i=1}^{n} \eta_i$$

and investing this amount in the n stocks according to the strategy described above.

We assume that the n-by-n matrix $(e^{a_{ij}k}-1)$ is nonsingular. [This is a relatively weak assumption: since the matrix (a_{ij}) is nonsingular and $k \neq 0$, it is satisfied if $|k|$ is sufficiently small or if (a_{ij}) is triangular.] Let $\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*$ be the solution of the system of equations

$$\sum_{j=1}^{n} \lambda_j^*(e^{a_{ij}k} - 1) = c_i + \delta, \quad i = 1, 2, \ldots, n. \tag{8.16}$$

Analogous to λ^* defined by (7.9), each λ_j^* is positive. To see this, let $\eta_1, \eta_2, \ldots, \eta_n$ be the solution of the system of equations

$$\sum_{i=1}^{n} \eta_i(e^{a_{ij}k} - 1) = 1$$

$$\sum_{i=1}^{n} \eta_i(e^{a_{im}k} - 1) = 0,\ m \neq j,\ m = 1, 2, \ldots, n. \quad (8.17)$$

Hence, (8.11) is satisfied with one strict inequality, and according to (8.12) we have

$$\begin{aligned} 0 &< \sum_{i=1}^{n} \eta_i(c_i + \delta) \\ &= \sum_{i=1}^{n} \eta_i \sum_{m=1}^{n} \lambda^*_m (e^{a_{im}k} - 1) \\ &= \sum_{m=1}^{n} \lambda^*_m \sum_{i=1}^{n} \eta_i(e^{a_{im}k} - 1) \\ &= \lambda^*_j. \end{aligned} \quad (8.18)$$

Extending the self-financing portfolio argument in the last section, we show that the price of a European option or contingent claim of the n stocks is the expectation of its discounted payoff, with the expectation taken with respect to a certain modified probability measure, which is unique. We introduce the function $V(s_1, s_2, \ldots, s_n, t)$, $t \leq \tau$, defined as the discounted conditional expectation

$$\begin{aligned} &V(s_1, s_2, \ldots, s_n, t) \\ &\quad = e^{-\delta(\tau - t)} E^*[\Pi(S_1(\tau), S_2(\tau), \ldots, S_n(\tau)) \\ &\qquad |S_1(t) = s_1, S_2(t) = s_2, \ldots, S_n(t) = s_n], \end{aligned} \quad (8.19)$$

where the expectation is to be taken with respect to the new Poisson parameters $\lambda^*_1, \lambda^*_2, \ldots, \lambda^*_n$ defined by (8.16). [For $j = 1, 2, \ldots, n$, the Poisson process $\{N_j(t)\}$ is to have the new Poisson parameter λ^*_j.] The option price at time t, $t \leq \tau$, is necessarily

$$V(S_1(t), S_2(t), \ldots, S_n(t), t). \quad (8.20)$$

To prove this, we note that (8.20) converges to $\Pi(S_1(\tau), S_2(\tau), \ldots, S_n(\tau))$ with certainty for $t \to \tau$, and we construct a self-financing portfolio of the stocks and risk-free bond whose value at time t is precisely given by (8.20). In this portfolio, let

$$\eta_j(S_1(t), S_2(t), \ldots, S_n(t), t)$$

be the amount invested in stock j at time t; therefore the difference

$$V(S_1(t), \ldots, S_n(t), t) - \sum_{j=1}^{n} \eta_j(S_1(t), \ldots, S_n(t), t) \quad (8.21)$$

is the amount invested in the risk-free bond at time t. We have to show that it is possible to choose the quantities $\{\eta_j(S_1(t), S_2(t), \ldots, S_n(t), t), j = 1, 2, \ldots, n\}$ such that the portfolio is self-financing, i.e., that the change of the portfolio value is equal to the investment gain in any time interval under each scenario.

We have to examine $n+1$ scenarios. (In an infinitesimally small time interval, exactly one of $n+1$ events will take place: either none of the n independent Poisson processes has a jump, or else exactly one of them has a jump.) If Poisson process j has a jump at time t, the price of stock i jumps from $S_i(t)$ to $S_i(t)e^{a_{ij}k}$, $i = 1, 2, \ldots, n$, and the portfolio value changes from $V(S_1(t), S_2(t), \ldots, S_n(t), t)$ to

$$V(S_1(t)e^{a_{1j}k}, S_2(t)e^{a_{2j}k}, \ldots, S_n(t)e^{a_{nj}k}, t).$$

For the portfolio to be self-financing, the change must be identical to the investment gain, yielding the equation

$$\begin{aligned} &V(S_1(t)e^{a_{1j}k}, \ldots, S_n(t)e^{a_{nj}k}, t) - V(S_1(t), \ldots, S_n(t), t) \\ &\qquad = \sum_{i=1}^{n} \eta_i(S_1(t), \ldots, S_n(t), t)\,(e^{a_{ij}k} - 1). \end{aligned} \quad (8.22)$$

There are n such equations, one for each of the Poisson processes ($j = 1, 2, \ldots, n$). The solution values $\{\eta_i(S_1(t), \ldots, S_n(t), t), i = 1, 2, \ldots, n\}$ of these n simultaneous equations are the amounts of stocks in the self-financing portfolio at time t.

Next, we examine the scenario of a time interval, say around t_0, in which none of the Poisson processes has a jump. For t in this interval,

$$S_i(t) = S_i(t_0)e^{-c_i(t - t_0)}, \quad i = 1, 2, \ldots, n. \quad (8.23)$$

Then, generalizing (7.15), we have

$$\begin{aligned} &\frac{d}{dt} V(S_1(t), \ldots, S_n(t), t) \\ &\quad = (\delta + \sum_{j=1}^{n} \lambda^*_j) V(S_1(t), \ldots, S_n(t), t) \\ &\qquad - \sum_{j=1}^{n} \lambda^*_j V(S_1(t)e^{a_{1j}k}, \ldots, S_n(t)e^{a_{nj}k}, t), \end{aligned} \quad (8.24)$$

and the instantaneous change of the portfolio value is

$$\begin{aligned} &[(\delta + \sum_{j=1}^{n} \lambda^*_j) V(S_1(t), \ldots, S_n(t), t) \\ &\qquad - \sum_{j=1}^{n} \lambda^*_j V(S_1(t)e^{a_{1j}k}, \ldots, S_n(t)e^{a_{nj}k}, t)]dt. \end{aligned}$$

On the other hand, the instantaneous investment gain of the portfolio is

$$\sum_{i=1}^{n} \eta_i\,(-c_i dt) + [V - \sum_{i=1}^{n} \eta_i](\delta dt).$$

[For simplicity we write η_i for $\eta_i(S_1(t), \ldots, S_n(t), t)$, and V for $V(S_1(t), \ldots, S_n(t), t)$.] Hence, the condition that the instantaneous change of the portfolio value is equal to the instantaneous investment gain is that

$$\begin{aligned}(\delta + \sum_{j=1}^{n} \lambda_j^*)V - \sum_{j=1}^{n} \lambda_j^* V(S_1(t)e^{a_{1j}k}, \ldots, S_n(t)e^{a_{nj}k}, t) \\ = -\sum_{i=1}^{n} c_i\eta_i + \delta\left[V - \sum_{i=1}^{n} \eta_i\right] \\ = \delta V - \sum_{i=1}^{n} \eta_i(\delta + c_i), \qquad (8.25)\end{aligned}$$

or

$$\begin{aligned}\sum_{i=1}^{n} \eta_i(\delta + c_i) = \sum_{j=1}^{n} \lambda_j^*[V(S_1e^{a_{1j}k}, \ldots, S_ne^{a_{nj}k}, t) \\ - V(S_1, \ldots, S_n, t)]. \qquad (8.26)\end{aligned}$$

It follows from (8.22) and (8.16) that the right-hand side of (8.26) is

$$\begin{aligned}\sum_{j=1}^{n} \lambda_j^* \left[\sum_{i=1}^{n} \eta_i(e^{a_{ij}k} - 1)\right] \\ = \sum_{i=1}^{n} \eta_i \left[\sum_{j=1}^{n} \lambda_j^* (e^{a_{ij}k} - 1)\right] \\ = \sum_{i=1}^{n} \eta_i\,(\delta + c_i), \qquad (8.27)\end{aligned}$$

which is the left-hand side. Hence, the portfolio so constructed is indeed self-financing, which completes the proof that the option price at time t is given by (8.20).

Let us now consider the limiting case where $k \to 0$ and λ and c_i vary according to (8.5) and (8.7). It follows from (8.7) that, for $i = 1, 2, \ldots, n$,

$$\lim_{k\to 0} kc_i = \sum_{j=1}^{n} a_{ij}. \qquad (8.28)$$

Expanding the exponential functions in (8.16) as a Maclaurin series yields

$$\delta + c_i = k \sum_{j=1}^{n} \lambda_j^* a_{ij} + \ldots, \qquad (8.29)$$

from which and (8.28) we obtain that, for $i = 1, 2, \ldots, n$,

$$\begin{aligned}\lim_{k\to 0} k^2 \sum_{j=1}^{n} \lambda_j^* a_{ij} = \lim_{k\to 0} k(\delta + c_i) \\ = \sum_{j=1}^{n} a_{ij}.\end{aligned}$$

Since the matrix (a_{ij}) is nonsingular, we have, for $j = 1, 2, \ldots, n$,

$$\lim_{k\to 0} k^2\lambda_j^* = 1. \qquad (8.30)$$

It now follows from (8.6) that

$$\begin{aligned}\lim_{k\to 0} k^2 \sum_{j=1}^{n} \lambda_j^* a_{ij} a_{hj} = \sum_{j=1}^{n} a_{ij} a_{hj} \\ = \sigma_{ih}. \qquad (8.31)\end{aligned}$$

Considering one more term in the Maclaurin series expansion of (8.29), we have

$$\begin{aligned}\delta + c_i = k \sum_{j=1}^{n} \lambda_j^* a_{ij} + \frac{k^2}{2} \sum_{j=1}^{n} \lambda_j^* a_{ij}^2 + O(k) \\ = k \sum_{j=1}^{n} \lambda_j^* a_{ij} + \frac{\sigma_{ii}}{2} + O(k), \qquad (8.32)\end{aligned}$$

by (8.31). [We write $f(k) = O(g(k))$ if $f(k)/g(k)$ is bounded as $k \to 0$.] Let E^* and Cov* denote the expectation and covariance operators with respect to the equivalent martingale measure [the probability measure such that the (independent) Poisson processes $\{N_1(t)\}$, $\{N_2(t)\}, \ldots, \{N_n(t)\}$ have parameters $\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*$, respectively]. Then

$$\begin{aligned}E^*[X_i(1)] = k \sum_{j=1}^{n} a_{ij}\, E^*[N_j(1)] - c_i \\ = k \sum_{j=1}^{n} a_{ij}\lambda_j^* - c_i \\ \to \delta - \frac{\sigma_{ii}}{2},\end{aligned} \qquad (8.33)$$

as $k \to 0$, by (8.32), and

$$\begin{aligned}\mathrm{Cov}^*(X_i(1), X_h(1)) \\ = \mathrm{Cov}^*(k \sum_{j=1}^{n} a_{ij}\, N_j(1), k \sum_{m=1}^{n} a_{hm}\, N_m(1)) \\ = k^2 \sum_{j=1}^{n} a_{ij}\, a_{hj}\, \lambda_j^* \\ \to \sigma_{ih}, \qquad (8.34)\end{aligned}$$

as $k \to 0$, by (8.31). In the limit the process (8.9) is an n-dimensional Wiener process, which, with respect to the equivalent martingale measure, has a drift vector given by (8.33) instead of (8.10), and unchanged diffusion matrix (8.4). See also (10.15) below.

Furthermore, in the limit as $k \to 0$,

$$\eta_i(s_1, \ldots, s_n, t) = s_i V_{s_i}(s_1, \ldots, s_n, t), \quad (8.35)$$

showing that the number of shares of stock i in the replicating portfolio at time t is simply the partial derivative

$$V_{s_i}(S_1(t), S_2(t), \ldots, S_n(t), t). \quad (8.36)$$

To derive (8.35), divide (8.22) [with $S_1(t)=s_1, \ldots, S_n(t)=s_n$] by k and let k tend to 0 to obtain

$$\sum_{i=1}^{n} V_{s_i} s_i a_{ij} = \sum_{i=1}^{n} \eta_i a_{ij}, \quad j = 1, 2, \ldots, n. \quad (8.37)$$

Since the matrix (a_{ij}) is nonsingular, we have (8.35). For another proof that, in a multidimensional Brownian motion model, the number of shares of stock i in the replicating portfolio is V_{s_i}, see Theorem 1 of Pedersen (1995).

If the payoff function Π is homogeneous of degree one in the stock-price variables, then the option-price function V is also homogeneous of degree one in the stock-price variables. By Euler's theorem for homogeneous functions,

$$V(s_1, \ldots, s_n, t) = \sum_{i=1}^{n} s_i V_{s_i}(s_1, \ldots, s_n, t). \quad (8.38)$$

It follows from (8.35) that, in the multidimensional geometric Brownian motion model,

$$V(s_1, \ldots, s_n, t) = \sum_{i=1}^{n} \eta_i(s_1, \ldots, s_n, t); \quad (8.39)$$

hence there is no bond component in the replicating portfolio and the option price V does not depend on the interest rate δ. In Section 10 we derive this independence of interest rate result in the context of change of numeraire. For a proof by means of differential equations, see Pedersen (1995).

To conclude this section, we derive the generalization of (7.25). It follows from (8.23) and an application of the chain rule that (8.24) can be rewritten as

$$-\sum_{i=1}^{n} c_i s_i V_{s_i} + V_t$$

$$= \delta V - \sum_{j=1}^{n} \lambda_j^* [V(s_1 e^{a_{1j}k}, \ldots, s_n e^{a_{nj}k}, t) - V]. \quad (8.40)$$

By the multivariate Taylor expansion formula,

$$V(s_1 e^{a_{1j}k}, \ldots, s_n e^{a_{nj}k}, t) - V(s_1, \ldots, s_n, t)$$

$$= \sum_{i=1}^{n} (e^{a_{ij}k} - 1) s_i V_{s_i}$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{h=1}^{n} (e^{a_{ij}k} - 1)(e^{a_{hj}k} - 1) s_i s_h V_{s_i s_h} + \ldots \quad (8.41)$$

Multiplying (8.41) with λ_j^* and summing over j gives the sum in the right-hand side of (8.40). It follows from (8.16) that

$$\sum_{j=1}^{n} \lambda_j^* \sum_{i=1}^{n} (e^{a_{ij}k} - 1) s_i V_{s_i}$$

$$= \sum_{i=1}^{n} \delta s_i V_{s_i} + \sum_{i=1}^{n} c_i s_i V_{s_i}, \quad (8.42)$$

the last sum of which cancels with the one on the left-hand side of (8.40). Because of (8.31),

$$\sum_{j=1}^{n} \lambda_j^* (e^{a_{ij}k} - 1)(e^{a_{hj}k} - 1)$$

$$= \sum_{j=1}^{n} \lambda_j^* k^2 a_{ij} a_{hj} + \ldots \to \sigma_{ih}, \quad (8.43)$$

as $k \to 0$. It follows from (8.42) and (8.43) that, in the limit as $k \to 0$, (8.40) becomes the parabolic differential equation

$$V_t = \delta V - \delta \sum_{i=1}^{n} s_i V_{s_i} - \frac{1}{2} \sum_{i=1}^{n} \sum_{h=1}^{n} \sigma_{ih} s_i s_h V_{s_i s_h}. \quad (8.44)$$

9. Extension to Dividend-Paying Stocks

The results in Section 5 can be extended to the case where the stock pays dividends continuously, at a rate proportional to its price. In other words, we assume that there is a nonnegative number ϕ such that the dividend paid between time t and $t+dt$ is

$$\phi\, S(t)\, dt. \quad (9.1)$$

(The number ϕ may be called the dividend-yield rate.) If all dividends are reinvested in the stock, each share of the stock at time 0 grows to $e^{\phi t}$ shares at time t. The risk-neutral Esscher measure is the Esscher measure of parameter $h=h^*$ such that the process

$$\{e^{-(\delta-\phi)t} S(t)\} \quad (9.2)$$

is a martingale. Condition (5.5) now becomes

$$e^{\delta-\phi} = M(1 + h^*, 1)/M(h^*, 1). \tag{9.3}$$

Since

$$E[S(\tau); h^*] = S(0)\, e^{(\delta-\phi)\tau}, \tag{9.4}$$

the European call option pricing formula (5.9) is generalized as

$$\begin{aligned} E[e^{-\delta\tau}\,(S(\tau) - K)_+; h^*] \\ = S(0)e^{-\phi\tau}\, Pr[S(\tau) > K; h^* + 1] \\ - Ke^{-\delta\tau}\, Pr[S(\tau) > K; h^*]. \end{aligned} \tag{9.5}$$

Formula (9.5) may also be used to price currency exchange options, with $S(\tau)$ denoting the spot exchange rate at time τ, δ the domestic force of interest, and ϕ the foreign force of interest. For $\{S(t)\}$ being a geometric Brownian motion, (9.5) is known as the *Garman-Kohlhagen formula*; see also (10.20) below.

We can extend the model to more than one dividend-paying stock. As in the last section, we let $S_j(t)$ denote the price of stock j at time t, $j=1, 2, \ldots, n$. For each j, we assume that there exists a nonnegative constant ϕ_j such that stock j pays dividends of amount

$$\phi_j\, S_j(t)\, dt$$

between time t and $t+dt$. Same as (8.1) and (8.9), we write

$$X_j(t) = ln[S_j(t)/S_j(0)], \quad j = 1, 2, \ldots, n, \tag{9.6}$$

and

$$\mathbf{X}(t) = (X_1(t), X_2(t), \ldots, X_n(t))'. \tag{9.7}$$

Let R^n denote the linear space of column vectors with n real entries, and

$$M(\mathbf{z}, t) = E[e^{\mathbf{z}'\mathbf{X}(t)}], \qquad \mathbf{z} \in R^n, \tag{9.8}$$

be the moment generating function of $\mathbf{X}(t)$. We assume that $\{\mathbf{X}(t)\}_{t\geq 0}$ is a stochastic process with independent and stationary increments and that

$$M(\mathbf{z}, t) = [M(\mathbf{z}, 1)]^t, \qquad t \geq 0. \tag{9.9}$$

Let $\mathbf{h}=(h_1, h_2, \ldots, h_n)' \in R^n$ for which $M(\mathbf{h}, 1)$ exists. The positive martingale

$$\{e^{\mathbf{h}'\mathbf{X}(t)}\, M(\mathbf{h}, 1)^{-t}\}_{t\geq 0} \tag{9.10}$$

can be used to define a new measure, the Esscher measure of parameter vector $\mathbf{h}$. The risk-neutral Esscher measure is the Esscher measure of parameter vector $\mathbf{h}=\mathbf{h}^*$ such that, for each j, $j=1, 2, \ldots, n$,

$$\{e^{-(\delta-\phi_j)t}\, S_j(t)\} \tag{9.11}$$

is a martingale. Condition (9.3) is generalized as n simultaneous conditions:

$$e^{\delta-\phi_j} = M(\mathbf{1}_j + \mathbf{h}^*, 1)/M(\mathbf{h}^*, 1), j = 1, \ldots, n. \tag{9.12}$$

Here,

$$\mathbf{1}_j = (0, \ldots, 0, 1, 0, \ldots, 0)', \tag{9.13}$$

where the 1 in the column vector $\mathbf{1}_j$ is in the j-th position.

As an illustration, let us consider the model in Section 8, where $\{\mathbf{X}(t)\}$ is defined by n independent Poisson processes $\{N_1(t)\}, \ldots, \{N_n(t)\}$, each with the same parameter value

$$\lambda = k^{-2}; \tag{9.14}$$

see (8.8). Because

$$\begin{aligned} E\left(\exp\left[\sum_{j=1}^n b_j N_j(t)\right]\right) &= \prod_{j=1}^n E(\exp[b_j N_j(t)]) \\ &= \prod_{j=1}^n \exp(t\lambda[e^{b_j} - 1]) \\ &= \exp\left(t\lambda \sum_{j=1}^n [e^{b_j} - 1]\right), \end{aligned}$$

we have

$$\begin{aligned} M(\mathbf{z}, 1) &= E[e^{\mathbf{z}'\mathbf{X}(t)}] \\ &= \exp\left(-\sum_{i=1}^n z_i c_i\right) E\left(\exp\left[k \sum_{j=1}^n \sum_{i=1}^n z_i a_{ij} N_j(t)\right]\right) \\ &= \exp\left(-\sum_{i=1}^n z_i c_i + \sum_{j=1}^n \lambda\left[\exp\left(k \sum_{i=1}^n z_i a_{ij}\right) - 1\right]\right). \end{aligned} \tag{9.15}$$

Hence,

$$\begin{aligned} M(\mathbf{z} + \mathbf{h}, 1)/M(\mathbf{h}, 1) \\ = \exp\Bigg\{ - \sum_{i=1}^n z_i c_i + \sum_{j=1}^n \lambda \exp\left[k \sum_{i=1}^n (z_i + h_i) a_{ij}\right] \\ - \sum_{j=1}^n \lambda \exp\left(k \sum_{i=1}^n h_i a_{ij}\right)\Bigg\} \\ = \exp\Bigg\{ - \sum_{i=1}^n z_i c_i + \sum_{j=1}^n \lambda \exp(k \sum_{i=1}^n h_i a_{ij}) \\ \times \left[\exp(k \sum_{i=1}^n z_i a_{ij}) - 1\right]\Bigg\}. \end{aligned} \tag{9.16}$$

On comparing (9.16) with (9.15), we see that the Esscher measure of parameter vector $\mathbf{h}$ is the probability

measure such that, for $j=1, 2, \ldots, n$, the Poisson process $\{N_j(t)\}$ has the parameter value

$$\lambda_j(\mathbf{h}) = \lambda\exp(k \sum_{i=1}^{n} h_i a_{ij})$$
$$= \frac{1}{k^2} \exp(k \sum_{i=1}^{n} h_i a_{ij}). \qquad (9.17)$$

It follows from (9.12) and (9.16) that the risk-neutral Esscher measure is determined by the equations

$$\delta - \phi_i = -c_i + \sum_{j=1}^{n} \lambda_j(\mathbf{h}^*)(e^{ka_{ij}} - 1),$$
$$i = 1, 2, \ldots, n. \qquad (9.18)$$

Not surprisingly, with $\phi_i=0$ and $\lambda_j(\mathbf{h}^*)=\lambda_j^*$, (9.18) is identical to (8.16).

It follows from (9.17) that, for $1\leq j, m\leq n$,

$$\lambda_j(\mathbf{h})e^{ka_{mj}} = \lambda_j(\mathbf{h} + \mathbf{1}_m). \qquad (9.19)$$

Hence, an interesting way to express (9.18) is

$$\phi_i - c_i + \sum_{j=1}^{n} \lambda_j(\mathbf{h}^* + \mathbf{1}_i) = \delta + \sum_{j=1}^{n} \lambda_j(\mathbf{h}^*),$$
$$i = 1, 2, \ldots, n, \qquad (9.20)$$

where the right-hand side is constant for all i.

For $\mathbf{k}=(k_1, \ldots, k_n)'$, write

$$\mathbf{S}(t)^{\mathbf{k}} = S_1(t)^{k_1} \ldots S_n(t)^{k_n}. \qquad (9.21)$$

Then,

$$E[\mathbf{S}(t)^{\mathbf{k}}g(\mathbf{S}(t)); \mathbf{h}]$$
$$= \frac{E[\mathbf{S}(t)^{\mathbf{k}}g(\mathbf{S}(t))\, e^{\mathbf{h}'\mathbf{X}(t)}]}{E[e^{\mathbf{h}'\mathbf{X}(t)}]}$$
$$= \frac{E[\mathbf{S}(t)^{\mathbf{k}}g(\mathbf{S}(t))\, \mathbf{S}(t)^{\mathbf{h}}]}{E[\mathbf{S}(t)^{\mathbf{h}}]}$$
$$= \frac{E[\mathbf{S}(t)^{\mathbf{k}+\mathbf{h}}]}{E[\mathbf{S}(t)^{\mathbf{h}}]} \frac{E[g(\mathbf{S}(t))\, \mathbf{S}(t)^{\mathbf{k}+\mathbf{h}}]}{E[\mathbf{S}(t)^{\mathbf{k}+\mathbf{h}}]}$$
$$= E[\mathbf{S}(t)^{\mathbf{k}}; \mathbf{h}]\, E[g(\mathbf{S}(t)); \mathbf{k} + \mathbf{h}], \qquad (9.22)$$

which generalizes the factorization formula (5.7). An immediate consequence of formula (9.22) and that (9.11) is a martingale under the risk-neutral Esscher measure is the formula:

$$E[e^{-\delta t}S_j(t)g(\mathbf{S}(t)); \mathbf{h}^*]$$
$$= E[e^{-\delta t}S_j(t); \mathbf{h}^*]\, E[g(\mathbf{S}(t)); \mathbf{h}^* + \mathbf{1}_j]$$
$$= S_j(0)\, e^{-\phi_j t}\, E[g(\mathbf{S}(t)); \mathbf{h}^* + \mathbf{1}_j]. \qquad (9.23)$$

The *Margrabe option* (Margrabe 1978) is the option to exchange one stock for another at the end of a stated period, say time τ, $\tau>0$. The payoff of this European option is

$$[S_1(\tau) - S_2(\tau)]_+. \qquad (9.24)$$

Its value at time 0, calculated with respect to the risk-neutral Esscher measure, is

$$E(e^{-\delta\tau}[S_1(\tau) - S_2(\tau)]_+ ; h^*). \qquad (9.25)$$

Since

$$(s_1 - s_2)_+ = s_1 I(s_1 > s_2) - s_2 I(s_1 > s_2),$$

it follows from (9.23) that

$$E(e^{-\delta\tau}[S_1(\tau) - S_2(\tau)]_+ ; \mathbf{h}^*)$$
$$= S_1(0)e^{-\phi_1\tau}\, E(I[S_1(\tau) > S_2(\tau)]; \mathbf{h}^* + \mathbf{1}_1)$$
$$- S_2(0)e^{-\phi_2\tau}\, E(I[S_1(\tau) > S_2(\tau)]; \mathbf{h}^* + \mathbf{1}_2)$$
$$= S_1(0)e^{-\phi_1\tau}\, Pr[S_1(\tau) > S_2(\tau); \mathbf{h}^* + \mathbf{1}_1]$$
$$- S_2(0)e^{-\phi_2\tau}\, Pr[S_1(\tau) > S_2(\tau); \mathbf{h}^* + \mathbf{1}_2]. \qquad (9.26)$$

A special case of (9.26) is (9.5).

10. Change of Numeraire and Homogeneous Payoff Function

Consider a European option or derivative security with exercise date τ and payoff

$$\Pi(S_1(\tau), \ldots, S_n(\tau)). \qquad (10.1)$$

Let $E_t[\cdot]$ denote the expectation conditional on all information up to time t. For $0\leq t\leq\tau$, let $V(t)=V(S_1(t), S_2(t), \ldots, S_n(t), t)$ denote the option price at time t, calculated with respect to the risk-neutral Esscher measure,

$$V(t) = E_t[e^{-\delta(\tau-t)}\, \Pi(S_1(\tau), \ldots, S_n(\tau)); \mathbf{h}^*]$$
$$= E_t[e^{-\delta(\tau-t)}\, S_j(\tau)\, \Pi(S_1(\tau), \ldots, S_n(\tau))/S_j(\tau); \mathbf{h}^*]$$
$$= E_t[e^{-\delta(\tau-t)}\, S_j(\tau); \mathbf{h}^*]$$
$$\times E_t[\Pi(S_1(\tau), \ldots, S_n(\tau))/S_j(\tau); \mathbf{h}^* + \mathbf{1}_j]$$
$$= e^{-\phi_j(\tau-t)}\, S_j(t)\, E_t[\Pi(S_1(\tau), \ldots, S_n(\tau))/S_j(\tau); \mathbf{h}^* + \mathbf{1}_j]. \qquad (10.2)$$

Thus,

$$\frac{V(t)}{e^{\phi_j t}\, S_j(t)} = E_t[\frac{1}{e^{\phi_j\tau}\, S_j(\tau)}\, \Pi(S_1(\tau), \ldots, S_n(\tau)); \mathbf{h}^* + 1_j], \qquad (10.3)$$

from which it follows that, with respect to the Esscher measure of parameter vector $\mathbf{h}^*+\mathbf{1}_j$, the process

$$\left\{\frac{V(t)}{e^{\phi_j t}\, S_j(t)};\ 0 \le t \le \tau\right\} \tag{10.4}$$

is a martingale. In particular, with respect to this measure, the processes

$$\left\{\frac{e^{\delta t}}{e^{\phi_j t}\, S_j(t)}\right\} \tag{10.5}$$

and

$$\left\{\frac{e^{\phi_k t}\, S_k(t)}{e^{\phi_j t}\, S_j(t)}\right\} \tag{10.6}$$

are martingales (and conversely these conditions determine the parameter vector $\mathbf{h}^*$). To explain the denominator $e^{\phi_j t}S_j(t)$, we consider stock j as a standard of value or a *numeraire*. We imagine that there is a mutual fund consisting of stock j only and all dividends are reinvested; all other securities are measured in terms of the value of this mutual fund. See also Geman, El Karoui, and Rochet (1995).

Now, we assume that the payoff function Π is homogeneous of degree one. It follows from

$$\begin{aligned}&\Pi(s_1, \ldots, s_n)\\ &= s_j\, \Pi(s_1/s_j, \ldots, s_{j-1}/s_j, 1, s_{j+1}/s_j, \ldots, s_n/s_j)\end{aligned} \tag{10.7}$$

that (10.3) becomes

$$\begin{aligned}&\frac{V(t)}{e^{\phi_j t}\, S_j(t)}\\ &= E_t\left[\Pi\left(\frac{S_1(\tau)}{e^{\phi_j \tau}\, S_j(\tau)}, \ldots, \frac{S_n(\tau)}{e^{\phi_j \tau}\, S_j(\tau)}\right);\ \mathbf{h}^* + \mathbf{1}_j\right].\end{aligned} \tag{10.8}$$

The right-hand side is a conditional expectation, with respect to the Esscher measure of parameter vector $\mathbf{h}^*+\mathbf{1}_j$, of a function of the $(n-1)$-dimensional random vector

$$\begin{aligned}&(X_1(\tau) - X_j(\tau), \ldots, X_{j-1}(\tau) - X_j(\tau),\\ &\qquad X_{j+1}(\tau) - X_j(\tau), \ldots, X_n(\tau) - X_j(\tau))'.\end{aligned} \tag{10.9}$$

Consider the special case that $\{\mathbf{X}(t)\}$ is an n-dimensional Wiener process, with $\boldsymbol{\mu}=(\mu_1, \mu_2, \ldots, \mu_n)'$ and $\mathbf{V}=(\sigma_{ij})$ denoting the mean vector and the covariance matrix of $\mathbf{X}(1)$, respectively. It is assumed that $\mathbf{V}$ is nonsingular. Because

$$M(\mathbf{z}, t) = \exp\left[t\left(\mathbf{z}'\boldsymbol{\mu} + \frac{1}{2}\mathbf{z}'\mathbf{V}\mathbf{z}\right)\right],\ \mathbf{z} \in R^n, \tag{10.10}$$

we have, for $\mathbf{h}\in R^n$,

$$\begin{aligned}&E[e^{\mathbf{z}'\mathbf{X}(t)};\ \mathbf{h}]\\ &= M(\mathbf{z} + \mathbf{h}, t)/M(\mathbf{h}, t)\\ &= \exp\left\{t\left[\mathbf{z}'(\boldsymbol{\mu} + \mathbf{V}\mathbf{h}) + \frac{1}{2}\mathbf{z}'\mathbf{V}\mathbf{z}\right]\right\},\ \mathbf{z} \in R^n,\end{aligned} \tag{10.11}$$

showing that, under the Esscher measure of parameter vector $\mathbf{h}$, $\{\mathbf{X}(t)\}$ remains an n-dimensional Wiener process with modified drift vector

$$\boldsymbol{\mu} + \mathbf{V}\mathbf{h}$$

and unchanged diffusion matrix $\mathbf{V}$. It follows from (9.12) that, for $k=1, 2, \ldots, n$,

$$\delta - \phi_k = \mathbf{1}_k'(\boldsymbol{\mu} + \mathbf{V}\mathbf{h}^*) + \frac{1}{2}\mathbf{1}_k'\mathbf{V}\mathbf{1}_k. \tag{10.12}$$

Thus,

$$\boldsymbol{\mu}^* = E[\mathbf{X}(1);\ \mathbf{h}^*] \tag{10.13}$$

$$= \boldsymbol{\mu} + \mathbf{V}\mathbf{h}^* \tag{10.14}$$

$$\begin{aligned}&= \delta\mathbf{1} - \left(\phi_1 + \frac{1}{2}\sigma_{11}, \phi_2 + \frac{1}{2}\sigma_{22},\right.\\ &\qquad \left.\ldots, \phi_n + \frac{1}{2}\sigma_{nn}\right)',\end{aligned} \tag{10.15}$$

where

$$\mathbf{1} = (1, 1, 1, \ldots, 1)'. \tag{10.16}$$

[Recall (8.33).] Also,

$$\begin{aligned}&E[\mathbf{X}(1);\ \mathbf{h}^* + \mathbf{1}_k]\\ &= \boldsymbol{\mu} + \mathbf{V}(\mathbf{h}^* + \mathbf{1}_k)\\ &= \boldsymbol{\mu}^* + \mathbf{V}\mathbf{1}_k\\ &= \delta\mathbf{1} - \left(\phi_1 - \sigma_{1k} + \frac{1}{2}\sigma_{11}, \phi_2 - \sigma_{2k} + \frac{1}{2}\sigma_{22},\right.\\ &\qquad \left.\ldots, \phi_n - \sigma_{nk} + \frac{1}{2}\sigma_{nn}\right)'.\end{aligned} \tag{10.17}$$

For an n-dimensional Wiener process $\{\mathbf{X}(t)\}$, (10.9) is a normal random vector under the Esscher measure of parameter vector $\mathbf{h}^*+\mathbf{1}_j$, and it follows from (10.17) that its mean does not involve the force of interest δ, and, of course, its $(n-1)$-dimensional covariance matrix, which is the same for all $\mathbf{h}$, does not depend on

δ. Thus $V(t)$, the price of a derivative security with a payoff function which is homogeneous of degree one, does not depend on δ. For example, consider the European Margrabe option, which has the payoff function

$$\Pi(s_1, s_2) = (s_1 - s_2)_+.$$

Let

$$\begin{aligned} \nu^2 &= \mathrm{Var}[X_1(1) - X_2(1)] \\ &= \sigma_{11} - 2\sigma_{12} + \sigma_{22}, \end{aligned} \tag{10.18}$$

$$\zeta(\tau) = \frac{1}{\nu\sqrt{\tau}} \ln\left(\frac{e^{-\phi_1\tau}S_1(0)}{e^{-\phi_2\tau}S_2(0)}\right), \tag{10.19}$$

and Φ denote the standardized normal distribution function. Then (9.26) becomes

$$\begin{aligned} &E(e^{-\delta\tau}[S_1(\tau) - S_2(\tau)]_+;\ \mathbf{h}^*) \\ &\quad = e^{-\phi_1\tau}S_1(0)\Phi\left(\zeta(\tau) + \frac{1}{2}\nu\sqrt{\tau}\right) \\ &\qquad - e^{-\phi_2\tau}S_2(0)\Phi\left(\zeta(\tau) - \frac{1}{2}\nu\sqrt{\tau}\right), \end{aligned} \tag{10.20}$$

which does not depend on δ. For nondividend-paying stocks ($\phi_1=\phi_2=0$), formula (10.20) has been given by Margrabe (1978). Fischer (1978) has also derived (10.20) with $\phi_1=0$ as a European call option formula; for him, $S_2(\tau)$ is the stochastic exercise price at time τ.

This independence of the interest rate is not valid in general. Consider the shifted compound Poisson model discussed earlier, where

$$\begin{aligned} &X_i(t) - X_m(t) \\ &\quad = k\sum_{j=1}^{n}(a_{ij} - a_{mj})N_j(t) - (c_i - c_m)t. \end{aligned} \tag{10.21}$$

It follows from (9.19) that, under the Esscher measure of parameter vector $\mathbf{h}^*+\mathbf{1}_m$, the process N_j has parameter value $\lambda_j^* e^{ka_{mj}}$. The risk-neutral parameter values $\{\lambda_j^*\}$, which are the solution of (9.18) [or (8.16)], depend on δ. To see how the interest rate δ gets canceled away as $k\to 0$ [with λ and c_i varying according to (8.5) and (8.7)], we apply (8.33) with the interest rate, replacing its difference with the dividend-yield rate and (8.34) to obtain

$$\begin{aligned} &E[X_i(1) - X_m(1);\ \mathbf{h}^* + \mathbf{1}_m] \\ &= k\sum_{j=1}^{n}(a_{ij} - a_{mj})\lambda_j^* e^{ka_{mj}} - (c_i - c_m) \\ &= k\sum_{j=1}^{n}(a_{ij} - a_{mj})\lambda_j^*(1 + ka_{mj} + \ldots) - (c_i - c_m) \\ &\to \left(\delta - \phi_i - \frac{\sigma_{ii}}{2} + \sigma_{im}\right) - \left(\delta - \phi_m - \frac{\sigma_{mm}}{2} + \sigma_{mm}\right). \end{aligned} \tag{10.22}$$

The last expression is identical to the one obtained from (10.17).

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VII

Insurance Futures: Examining the Context for Trading Insurance Risk

by Prakash Shimpi

Abstract

Catastrophic insurance futures are being traded, albeit thinly, at the Chicago Board of Trade. This paper examines the broad context for trading insurance risk. The term "insurance risk" is examined to provide a basis for understanding the concept of tradable insurance risk. The motivation for trading insurance risks, specifically to increase the capacity to cover risks, is discussed. Models for transforming insurance risk to a tradable form are considered, leading to a brief comment on the development of an insurance-specific index. The perspectives of market participants such as an insurer, reinsurer, investor, and speculator are offered. Two examples of trading insurance risk are described.

Introduction

The fundamental principles of insurance include the transfer of risk, the diversification of risk, and the pooling of risks. These elements contribute to our understanding of the risks that can be covered and how insurance companies can cover these risks.

With the innovations in the capital markets and the forces of change surrounding the insurance industry, this is an appropriate juncture to examine the context for trading insurance risk. Can insurance risk be traded? Can this trading activity provide a means for increasing capital to cover insurance risks? If so, how will the insurance industry participate in this activity? Only a few of the issues raised by these questions are examined in this paper.

Most of the literature surrounding the trading of insurance risk has focused on the mathematical development of the risk pricing models. It is hoped that this paper, which is entirely nonmathematical, will stimulate discussion of why, rather than how, insurance risks should be traded.

Insurance Risk

A first step in the examination of trading insurance risk is to examine the concept of an *insurance risk*. Specifically, the terms "risk" and "insurance" can have broad interpretations and, apart from strict legal constraints, the concepts of *sharing risk* and *providing insurance* need not be confined to the insurance markets alone.

Broadly, a risk can be defined simply as an event whose occurrence triggers a loss. In most jurisdictions, the type of loss (and, therefore, the type of risk) that can be covered by insurance companies, and cannot be provided by others, is defined by law. A loss exposure is the set of circumstances that presents the possibility of a loss, whether or not the loss actually takes place. The event triggering the loss can be a one-time event, such as a death resulting in a claim on a life insurance policy. Alternatively, there can be several events of varying degrees of loss occurring within a preset time period, as in automobile insurance, for example.

Just as broadly, insurance can be defined simply as the act of sharing the risk of loss. More precisely, all or part of the *financial consequence* of the loss is shared. The financial consequence of a loss provides a means of quantifying the risk. There are certain risks

where the "value" of the loss cannot be defined. An example is the value of a life. However, by concentrating on the financial consequence (however broadly defined) of the loss of that life, an insurance contract can be formed.

Another definition of insurance* is: "Insurance is a device for the reduction of uncertainty of one party, called the insured, through the transfer of particular risks to another party, called the insurer, who offers a restoration, at least in part, of economic losses suffered by the insured."

Based on these broad definitions of risk and insurance, "insurance risk" can be defined as those loss exposures that are covered (or insured) by insurance companies. This is a retrospective view since it considers only those types of risks that have been covered by insurance companies at any time in the past. The definition can include a prospective view as well if it includes risks that are coverable (or insurable) by insurance companies.

Without allowing for any new developments in the insurance and capital markets, how does an insurance company decide whether a risk is insurable? There are several criteria which are discussed extensively in the actuarial and insurance literature. One can presume rather simplistically that all risks that are insurable are currently being insured by insurance companies and that those risks that are not insurable are not being insured. But that cannot be the whole truth.

There are several cases of risks that one might consider insurable based on objective criteria, where coverage is not available in the insurance market. For example, certain types of catastrophic risks may not be covered to the extent desired by the insured in some locations. There are two fundamental reasons for this:

1. the price (premium) to cover the risks, and
2. the sources of capital available to cover the risks.

One aspect of risk transfer that has not been discussed so far is the price or premium at which the risk will be ceded by the insured and accepted by the insurer. The premium makes *all* risks insurable. This is not as absurd as it may sound. All risks are insurable in the context that the premium sets a threshold below which the risk is self-insured and above which it is insured by another party. This is demonstrated clearly in the captive insurance market and in the reinsurance market. In the captive insurance market, for example, an industrial company sets up a captive which will cover the risks of its sponsor. It may decide to retain all the risk or reinsure part of it.

The price of risk transfer does not exist in isolation. It depends on the sources of capital available to share the risk. If the capital sources are small in relation to the risks, then the demand for capital is greater than supply and some risks will remain uncovered. Expanding the capital available to cover insurance risks should, therefore, provide the capacity to bring premiums to levels where a transfer of risk becomes economically feasible.

Allowing, now, for new developments in the insurance and capital markets, how does an insurance company decide whether a risk is insurable? It can stay with its conventional criteria for defining insurability, some of which may be artifacts of idiosyncratic rules and regulations, or it can expand its criteria. One way to expand the definition of insurability is to consider whether there are entities, insurance companies or otherwise, who will be willing to participate in the risk. This does have a price dimension to it as well, but the distinction here is that the search for capital may lead to other risk takers who are willing to participate in risks not conventionally covered by the insurance industry.

As an example of risk taking, consider a bank that sells an interest rate cap. In exchange for a premium, the bank takes the risk in exchange for a premium that it will have to make payments at several points in time over the life of the cap if interest rates rise above the strike level. Presumably the buyer of the cap is hedging its exposure to rising interest rates. This sounds like insurance, except that the principle of insurable interest is not necessarily present. A key difference in the way the bank and an insurer manage this risk is that the bank will make a conscious decision about the value of this risk that it wishes to retain. It could keep none of the risk simply by entering into offsetting transactions in the cash and futures markets, or it could keep all of the risk. Therefore, the bank is not necessarily a risk taker; it is a risk manager.

There are two questions that arise from this discussion.

1. How can more capital (capacity) be attracted to share in insurance risks?
2. How can the risks covered by insurance companies be expanded?

The rest of this paper will consider the first of these questions. In doing so, some aspects concerning the second question may arise. This discussion has brought

*Irving Pfeffer, *Insurance and Economic Theory* (Homewood, Ill.: Richard D. Irwin, 1956), p. 53

us to the point where it is relevant to consider the meaning of tradable insurance risk.

Tradable Insurance Risk

When an insured buys an insurance policy from an insurer, insurance risk is being traded. When an insurer cedes some risk to a reinsurer, insurance risk is being traded. When a reinsurer retrocedes some risk to a retrocessionaire, insurance risk is being traded. If all this trading is taking place, why is there a need to discuss the concept of tradable insurance risk? The reason is that the trading is taking place within the insurance market only.

There may be entities in the broader capital markets willing to share in insurance risks. This statement raises several issues which are addressed in this section.

1. Are these entities currently providing capital to cover insurance risks?
2. If they are not, what are the obstacles?
3. How would they prefer to provide capital, i.e., what is the form of risk sharing?
4. What is the nature of the risks they are willing to share?

In answer to the first question, some entities do provide capital to cover insurance risks, but there are many that do not. The insurance market does not exist in isolation. It is part of a broader capital market. Insurance companies and reinsurers have capital that allow them to provide the capacity to cover insurable risks. This capital is primarily in the form of equity in the insurance operating companies and, where permitted by regulation, debt capital (although debt is often at a holding company with the proceeds downstreamed to the operating entity).

The answer to the second question lies in the form of investment that an entity can make in an insurance enterprise. Some investors have restrictions dictated by investment policy or regulation, on the types of investments that they can make. With some exceptions, the majority of the capital provided to the insurance industry is in the form of direct equity, either at the insurance operating company or at a holding company. Opportunities for fixed-income investors in this sector are limited. In addition, for those investors who want to leverage their investment, there is no real derivatives market in insurance risk apart from the Catastrophic Futures (CAT) market at the Chicago Board of Trade (CBOT). In order to increase the capacity to cover insurance risks, the form of risk sharing has to be expanded to better serve the needs of these types of investors.

When capital is provided to an insurance company, it participates in more than just the pure insurance risk; it also shares in the fortunes of the insurance company as a business enterprise. Is it possible, or even desirable, to separate the two? It may be if the objective is to attract new participants to provide capital. Unless the form of risk sharing is modified, only the same investment vehicles (equity and some debt) and the same participants will be present.

In order to increase the fixed income and derivative vehicles for investing in insurance risk, it is necessary to consider the general nature of the risk in which investors will be willing to participate. Broadly, it should be the opposite of the risk in an equity investment. The investor should be able to calculate a fair value of the investment reasonably efficiently. The risk should be finite and subject the investor to the pure insurance risk only; otherwise, it is an equity investment. The risk definition should be such that the trading of insurance risk can take place in the secondary capital markets, in contrast to the secondary insurance markets of reinsurance and retrocession.

It is probably difficult to isolate the pure insurance risk without also including the skill of the insurance company's personnel who underwrite and manage the risk. The price (premium) to cover the risk includes the cost of managing the business. This cost may be an average over several lines of business and over several time periods. The actual allocation of cost when assessing profitability of a policy may be different from that assumed in the pricing. At best, therefore, the isolation is a matter of degree.

In addition, one of the fundamental principles of insurance is the pooling of risks to minimize the variance of operating results over time. Isolating the pure insurance risk and taking it out of the insurance company's intertemporal setting may *increase* the capital necessary to cover the risk. This defeats the purpose.

The last two paragraphs point to some of the difficulties of isolating pure insurance risk. However, it may still be necessary to make the separation in order to develop a market for trading insurance risk outside the traditional group of investors. One possible framework to consider when making the separation of pure insurance risk from enterprise risk is that of a *hedgeable risk.* If a risk is "hedgeable," then there is a basis for price discovery. That, in turn, forms the basis for active trading. This is considered in the next section.

Transformation of Insurance Risk to Tradable Form

In order to convert an insurance risk into a tradable form, it is useful to consider the replication argument that forms the basis for developing the fair value of other investment vehicles. Essentially, if a risk can be hedged exactly (i.e., it is a hedgeable risk) and the price of the hedge vehicles can be observed, then the price of that risk is determined and is equal to the price of the hedge vehicles.

There are two risk transformation models that motivate the development of hedging insurance risk. The first is the derivatives model, and the second is the asset-backed securitization (ABS) model. The rest of this paper will focus on the derivatives model.

The derivatives model can be used where there is an *explicit index* available that is highly correlated with the insured risk. The insured risk can be characterized by the type of index used for hedging it. There are two types of insured risk:

1. *Insurance-specific*: Index generally *not* used by other noninsurance entities.
2. *Noninsurance-specific*: Index generally used by other entities for purposes other than managing insurance risk.

At present, there are many indices, but only one that is insurance-specific, i.e., the ISODATA index which forms the basis of the CAT futures contract at the CBOT. More indices are expected to be developed as interest in this subject increases.

Insurance risks are also correlated with some of the noninsurance-specific indices. In particular, life insurance policies have significant exposure to interest rate risk directly through the investment component of the insurance benefit and indirectly through the lapse behavior of policyholders. Hedging this risk using interest rate derivatives is already being done actively.

There are two risk transfer vehicles that can be used under this model.

1. *Naked derivatives*: These are futures, options, swaps, caps, floors, and other such instruments that are based on the performance of a measure underlying an index. Generally, counterparty risk of varying degrees exists with these instruments.
2. *Embedded derivatives*: These are structured notes or bonds that have the index-based naked derivatives embedded in the bond structure, not unlike the call feature of callable bonds or the prepayment feature of mortgage-backed securities. These bonds are generally of a quality acceptable to an investor.

The ABS Model can be used where no explicit index is available. Instead, an *implicit index* results from separating a pool of liabilities and paying investors a return based on the performance of the pool. To the extent that the investments in the pool are fixed income in nature, they can be called liability-backed securities (LBS). They can also be classified as structured notes or bonds. The specific elements of LBS will not be considered in this discussion.

Development of an Explicit Insurance-Specific Index

As mentioned above, the ISODATA index used for trading the CAT futures contract at the CBOT is the only insurance-specific index available for trading at present. Other bodies do report loss estimates, e.g., Property Claims Services (PCS) and Swiss Re's SIGMA, and these can be used to create customized reinsurance contracts.

Although the CBOT contract is a start in the right direction, it is by no means the final form. Both the index and the contract design have met with some criticism, which is only to be expected as the various market participants make their preferences known. Focusing on the index design alone, it is worthwhile asking whether an index based on industry loss experience (the ISODATA index) is necessarily the only way to proceed.

From an insurance industry perspective, the closer the index is to the loss experience, the better the ability to hedge the loss exposure of insurers. On the other hand, it does place a noninsurance industry expert, i.e., an investor, at a disadvantage. The investor is handicapped when translating weather information into loss experience. Over time, as both the investor and insurer understand the market better and as the contract and index evolve, this may not turn out to be such a big issue.

An alternative to a loss experience-based index would be to have an index based on some independently verifiable measure of a physical phenomenon that is highly correlated to loss experience. For example, in a flood prone area, the water level at any instant at a given location could form the basis for an index. Both the insurer and the investor have an equal chance of modeling the behavior of the water level since they

would both have access to scientists specializing in making such predictions. With such an index, only the insurer need worry about correlating the water level to a loss exposure and thereby developing a hedge strategy.

Other models combining these two extremes are also under consideration. As interest in this topic increases, it is likely that additional indices will be developed. At present, the London International Financial Futures Exchange (LIFFE) and several insurance intermediaries are racing to develop their own insurance-specific indices.

Participants in Traded Insurance Risk

Whether insurance risk is traded over-the-counter (OTC) or through an exchange, it is useful to consider the types of participants in the market. Simplistically, there are two types of participants: hedgers and risk takers. Hedgers are exposed to insurance risk and want to transfer some portion of it to the risk takers.

In order to better understand the market participants, it may be useful to discuss the stage at which *exact* hedging is desirable. (If a hedge is not exact, the hedger retains some basis risk.) There are three stages to consider: the insured, the insurer, and the reinsurers. The distinctions between these three levels are size of capital base and diversification of risk exposure.

Generally, the insured has the smallest capital base and the least diversification of risk exposure. Risk transfer to the next level is, therefore, desirable in the most complete form. This is accomplished through an insurance contract. The insurer has more capital than the insured and better risk diversification, but generally is not as well off as the reinsurers. With the exception of the very large, financially sound insurers, the risk transfer to the reinsurer should also be as complete as possible. The only way to do so is through a customized reinsurance contract.

The reinsurer, by virtue of its capital and the significant diversification of risks on its books, is in the strongest position to withstand the basis risk in any hedge that does not perfectly replicate the insurance risk. The term "hedgeable risk" can, therefore, be viewed in the context of a reinsurer wishing to retrocede part of its insurance risk to the capital markets; the risk is hedgeable if it can be transformed fully or partially from a pure insurance risk to a traded risk.

The risk takers can be categorized as investors and speculators. There is not necessarily a clear distinction between the two, but it is useful to consider them separately within the context of additional capital to be attracted to the insurance market. Both are necessary to develop an active market in traded insurance risks; the investors provide the additional capital that is the object of the exercise, and the speculators operate to enhance the efficiency of the market.

The investors run the range of participants in the fixed-income markets. They include pension funds, mutual funds, insurance companies, and other corporate investors. Until the traded insurance risks market develops sufficient liquidity, it is unlikely to attract retail investors and would, therefore, be a market for institutional investors. These institutional investors have significant capital to deploy. The attraction of this market to these investors is the potential of high-quality investments providing a spread over alternative investments, as well as risk diversification to the extent that the insured risk is not correlated with other investment risks.

The practical question then remains: how is capacity increased with these participants? The answer lies in the value of the risks that can be passed on to the investors. As reinsurers lay off their risk to investors, they free up some of their capital to cover other risks. Ultimately, if the reinsurers are able to pass all hedgeable risks to the investors, they will retain a portfolio of unhedgeable risks (or at least the portion of risk that they do not want to hedge.) The next section describes some possible applications.

Examples of Trading Insurance Risk

A. CAT risk cover hedged by CBOT CAT contract:

1. Reinsurance Need: A medium-sized insurer writes significant property insurance, with concentrations in Florida and New York. The reinsurer has a good relationship with this customer but is not willing to cover the CAT risk entirely.
2. Solution: Reinsurer can provide reinsurance as before, but lays off a portion of its risk via the CBOT CAT market. The effectiveness of the hedge will depend on the correlation of the reinsured CAT loss relative to the CBOT CAT loss. The reinsurer will be retaining the basis risk of the hedge.

B. Double-Trigger CAT Cover:

1. Reinsurance Need: A strong, well-capitalized insurer wishes to purchase CAT reinsurance at a cheaper cost than available through conventional programs. It believes that it can cover its CAT claims if bond markets remain stable or rise. The concern is that it may have to liquidate bonds when interest rates are high and a CAT claim has to be paid.
2. Solution: Provide a double trigger CAT cover which provides a sliding scale of cover up to some maximum limit. The premium for the coverage should be less than the cost of the underlying interest rate derivative (cap) alone and the cost of the CAT cover alone. Assuming that the CAT risk is not hedgeable using an explicit derivative, then only the interest rate risk would be hedged. The hedge would be based on assessing the probability of a CAT loss in the loss exposure period and combining that with an interest rate derivative. The hedge would be adequate if it were to cover the expected long-run experience on a large group of similar policies. With a small number of transactions and short-term covers, there may be a significant basis risk. The basis risk can be mitigated somewhat by building in a hedging strategy that increases in a nonlinear form as interest rates rise.

Conclusion

The market for trading insurance risks is in its early days. Many elements of this market are evolving. In particular, a thorough evaluation of risk transformation models may indicate that there are several alternatives to insurance futures that are worth considering. It may well be that these alternatives have to develop in tandem so that the traded risks market has greater breadth and depth. As these alternatives develop, so too will the market participants. And with them, the capital available to cover the risks that are insurable must grow.