SOCIETY OF ACTUARIES
Quantitative Finance and Investment Core

Exam QFICORE

MORNING SESSION

Date: Wednesday, April 26, 2017
Time: 8:30 a.m. – 11:45 a.m.

INSTRUCTIONS TO CANDIDATES

General Instructions

1. This examination has a total of 100 points. It consists of a morning session (worth 60 points) and an afternoon session (worth 40 points).

   a) The morning session consists of 10 questions numbered 1 through 10.

   b) The afternoon session consists of 7 questions numbered 11 through 17.

   The points for each question are indicated at the beginning of the question.

2. Failure to stop writing after time is called will result in the disqualification of your answers or further disciplinary action.

3. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions on the exam booklet.

Written-Answer Instructions

1. Write your candidate number at the top of each sheet. Your name must not appear.

2. Write on only one side of a sheet. Start each question on a fresh sheet. On each sheet, write the number of the question that you are answering. Do not answer more than one question on a single sheet.

3. The answer should be confined to the question as set.

4. When you are asked to calculate, show all your work including any applicable formulas. When you are asked to recommend, provide proper justification supporting your recommendation.

5. When you finish, insert all your written-answer sheets into the Essay Answer Envelope. Be sure to hand in all your answer sheets because they cannot be accepted later. Seal the envelope and write your candidate number in the space provided on the outside of the envelope. Check the appropriate box to indicate morning or afternoon session for Exam QFICORE.

6. Be sure your written-answer envelope is signed because if it is not, your examination will not be graded.

Tournez le cahier d'examen pour la version française.

475 N. Martingale Road
Schaumburg, IL 60173-2226

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****BEGINNING OF EXAMINATION****
Morning Session

1. (*7 points*) A complete market contains one risk-free asset B_t and two correlated risky assets $S_{1,t}$ and $S_{2,t}$ which have the following diffusion processes:

$$B_0 = 1$$
$$dB_t = rB_t dt$$
$$\frac{dS_{1,t}}{S_{1,t}} = \mu_1 dt + \sigma_1 dW_{1,t}$$
$$\frac{dS_{2,t}}{S_{2,t}} = \mu_2 dt + \sigma_2 dW_{2,t}$$
$$dW_{1,t} dW_{2,t} = \rho dt$$

where $r, -1 \le \rho \le 1$, $\mu_1, \mu_2, \sigma_1 > 0$ and $\sigma_2 > 0$ are constants, and $W_{1,t}$ and $W_{2,t}$ are standard Wiener processes under the measure $\mathbb{P}$.

Let $Z_t = S_{1,t} S_{2,t}$. You are pricing a derivative on Z_t with a payoff of $(Z_T)^\tau$ at time T where τ is a constant power.

(a) (*2 points*) Express the diffusion process of Z_t as $\frac{dZ_t}{Z_t} = \mu dt + \sigma dW_t$ where W_t is a standard Wiener process under $\mathbb{P}$ by deriving μ, σ, and W_t in terms of $\rho, \mu_1, \mu_2, \sigma_1, \sigma_2, W_{1,t},$ and $W_{2,t}$.

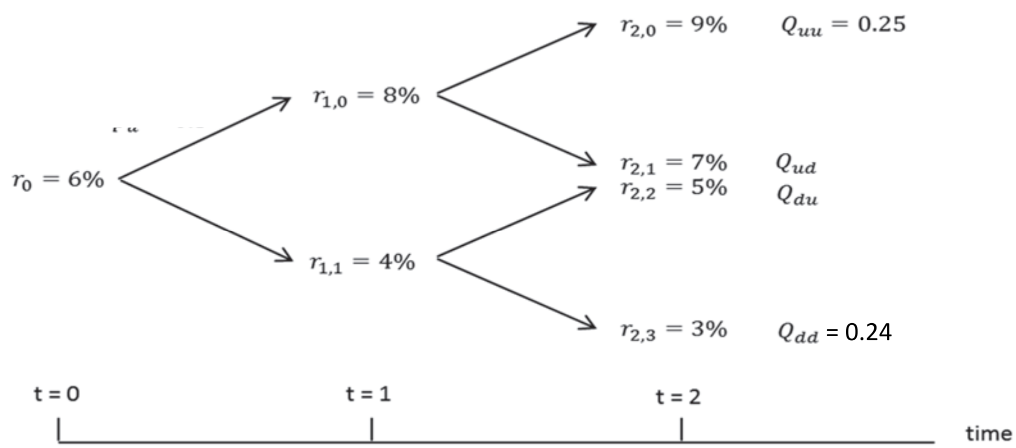
At time t, consider a self-financing portfolio valued at Π_t holding ϕ_t units of Z_t and ψ_t units of B_t such that $\Pi_T = Z_T^\tau$ almost surely.

(b) (*2 points*) Show, using Girsanov's theorem, that by changing the measure $\mathbb{P}$ to an equivalent risk-neutral measure $\mathbb{Q}$, $e^{-rt}\Pi_t$ is a $\mathbb{Q}$-martingale.

(c) (*1 point*) Find the diffusion process for Z_t under the measure $\mathbb{Q}$.

(d) (*2 points*) Calculate the value of the derivative at time t.

2. (*6 points*) The one-period spot rate r follows the interest rate tree given below.



The following instruments are traded:

- A default-free savings account
- A default-free bond maturing at time 3 with value 1 (price at time 0 = 0.84)
- A default-free bond maturing at time 2 with value 1 (price at time 0 = 0.89)

(a) (*2.5 points*) Calculate the remaining two risk-neutral probabilities Q_{ud} and Q_{du}.

(b) (*2 points*) Determine the time-0 price of a caplet on r_2 with notional of 1000, cap $K = 6\%$ and expiry at time 3 using:

(i) Risk-neutral measure

(ii) Forward measure

2. **Continued**

Now assume the forward rate follows the stochastic differential equation (SDE) below under the real-world measure:

$$dF_t = \mu F_t dt + \sigma F_t dW_t$$

where W_t is a standard Wiener process.

Let C_t be the price of a caplet on LIBOR rate L_t with tenor $\delta = 1$, cap rate *K*, notional amount $N = 1$, and maturity at time *T*.

Your colleague calculated C_t using the following formula:

$$C_t = F_t N(d_1) - e^{-r(T-t)} K N(d_2)$$

where r is the risk-free rate, $d_1 = \dfrac{\ln\left(\dfrac{F_t}{K}\right) + \left(r + \dfrac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$ and $d_2 = d_1 - \sigma\sqrt{T-t}$.

(c) (*1.5 points*) Identify errors of this approach for calculating C_t .

3. (*8 points*) Let (Ω, F, P) be a probability space and $\{W_t : t \geq 0\}$ be a standard Wiener process. Let $\{X_t : t \geq 0\}$ and $\{Y_t : t \geq 0\}$ be the processes defined by:

$$X_t = \int_0^t W_s ds$$
$$Y_t = \int_0^t W_s^2 ds$$

(a) (*1 point*) Prove that $E(W_t^4) = 3t^2$.

(b) (*7 points*) Derive an expression in terms of t for each of the following:

(i) $Var(X_t)$

(ii) $Var(Y_t)$

(iii) $Cov(X_t, Y_t)$

4. (*5 points*) Stock price S_t follows geometric mean reverting process with a standard Wiener process $\{W_t : t > 0\}$

$$dS_t = \alpha(\theta - \log S_t)S_t dt + \sigma S_t dW_t$$

where α, θ, and σ are positive constants.

Let $Y_t = \log S_t$.

(a) (*2 points*) Prove by using Ito's lemma that for $t \le T$.

$$Y_T = e^{-\alpha(T-t)}Y_t + \left(\theta - \frac{\sigma^2}{2\alpha}\right)\left(1 - e^{-\alpha(T-t)}\right) + \int_t^T \sigma e^{-\alpha(T-s)} dW_s.$$

(b) (*1 point*) Derive the mean and the variance of Y_T at time $t \le T$.

(c) (*2 points*) Derive $E(S_T)$ at time $t \le T$.

5. (*7 points*) You are given the following information of a European call option on a stock as of September 30, 2016:

- Market price of the call option = \$6.92
- Call option strike price = \$100
- Call option's remaining time to expiration = 1 year
- Stock price = \$100
- Stock dividend yield = 0%
- Continuously compound risk-free interest rate = 2% for all maturities
- All assumptions underlying the Black-Scholes option pricing model hold

You firmly believe that the stock's actual volatility σ will be 20%.

(a) (*1 point*) Calculate the call option price that is consistent with your volatility view.

You can trade the call option on 9/30/2016 only, but you can trade the stock on any day when the market is open. All your trades are to be conducted at the market price.

(b) (*1 point*) Propose a trading strategy on 9/30/2016 involving the call option and/or its underlying stock that guarantees a profit if your volatility view proves correct.

(c) (*2.5 points*) Provide a mathematical derivation to support your proposed strategy in part (b).

(d) (*0.5 points*) Determine the present value of expected profit of your strategy on 9/30/2016.

(e) (*1 point*) Determine the net cash position of your strategy on 9/30/2016.

(f) (*1 point*) Outline pros and cons of your strategy.

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6. (*7 points*) Suppose that the price S_t of a stock at time t follows the following SDE

$$dS_t = rS_t dt + \sigma S_t dW_t$$

where W_t is a standard Wiener process under the risk-neutral measure $\mathbb{Q}$ with filtration $\{F_t\}, r$ is the constant risk-free rate, and $\sigma > 0$ is constant.

(a) (*1 point*) Show that

$$S_t = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma W_t}$$

A European chooser option is an option where, at a specified future time $u,$ the holder can choose whether the option is a call or a put, both of which mature at time $T > u$ and have the same strike price $K.$

The value of the chooser option at time u equals the larger of the value of the call and the value of the put.

Denote by

- $C(S_t, t, K, T)$ the value at time t of the European call with strike price K and maturity T
- $P(S_t, t, K, T)$ the value at time t of the European put with strike price K and maturity T
- $H(S_t, t, K, u, T)$ the value at time t of the European chooser option with strike price K, choice to be made at time u, and maturity $T > u$

(b) (*1 point*) Show that the value of the European chooser option

$$H(S_u, u, K, u, T) = \max\left(C(S_u, u, K, T), C(S_u, u, K, T) + e^{-r(T-u)}K - S_u\right)$$

(c) (*1.5 points*) Show that

$$H(S_0, 0, K, u, T) = C(S_0, 0, K, T) + e^{-rT}E\left[\max\left(0, K - S_0 e^{rT} e^{\sigma W_u - \frac{u\sigma^2}{2}}\right)\right]$$

where $E[\ \ ]$ is the expectation under the risk-neutral measure $\mathbb{Q}$.

6. **Continued**

(d) (*2.5 points*) Show that

$$H\left(S_0,0,K,u,T\right)=S_0\left(N\left(d_1\right)-N\left(-d_1^*\right)\right)+Ke^{-rT}\left(N\left(-d_2^*\right)-N\left(d_2\right)\right)$$

$$where\ d_1=\frac{\ln\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2=d_1-\sigma\sqrt{T}$$

$$d_1^*=\frac{\ln\left(\frac{S_0}{K}\right)+\left(rT+\frac{\sigma^2}{2}u\right)}{\sigma\sqrt{u}}$$

$$d_2^*=d_1^*-\sigma\sqrt{u}$$

(e) (*0.5 points*) Derive the delta of the chooser option.

(f) (*0.5 points*) Calculate the limits of deltas of European chooser options:

(i) When the underlying stock price S_0 approaches to zero.

(ii) When the underlying stock price S_0 approaches to infinity.

7. (*4 points*) For an option on a stock, "Gamma" and "Speed" are the second- and third-order sensitivity of the option value to the underlying stock price movement, respectively. Assuming that the option value follows the Black-Scholes' pricing formula, you are to derive Gamma and Speed for a European call option on a non-dividend paying stock.

Notations:

K = strike price of the option

S = stock price

T = remaining maturity of the option

r = risk-free interest rate

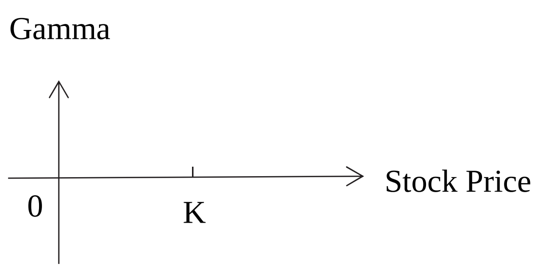
σ = volatility of the stock price

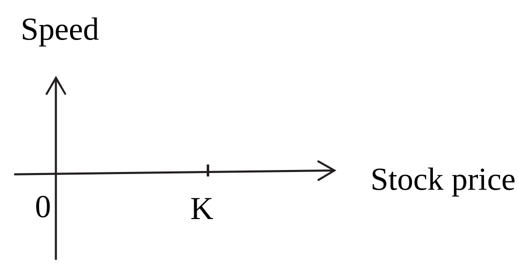
(a) (*1.5 points*)

(i) Derive Gamma in terms of the notations given above.

(ii) Derive Speed in terms of Gamma and the notations given above.

(b) (*1.5 points*) Sketch the following two graphs that illustrate how Gamma and Speed change with the underlying stock price, respectively, based on your derivations in parts (a)(i) and (a)(ii) above. Explain the shape of each of your graphs.





An intern states that we should reduce the absolute level of Speed, which would result in a more stable Gamma hedge.

(c) (*1 point*) Critique the intern's statement.

8. (*5 points*) The Heath-Jarrow Morton (HJM) model's starting point is to define zero-coupon bond prices $P(t,T)$ maturing at time T in the risk-neutral world, which at time t must yield $r(t)$ the risk-free short rate.

$P(t,T)$ follows the process:

$$dP(t,T) = r(t)P(t,T)dt + v(t,T)P(t,T)dW_t$$

where $v(t,T)$ is the time-dependent volatility of $P(t,T)$ and W_t is a standard Brownian motion.

The instantaneous forward rate $F(t,T)$ for time T observed at t under HJM follows:

$$dF(t,T) = m(t,T)dt + s(t,T)dW_t$$

where $m(t,T)$ is the time-dependent drift and $s(t,T)$ is the volatility.

(a) (*0.5 points*) Describe the pros and cons of using the HJM model.

(b) (*1.5 points*) Show that $m(t,T) = s(t,T)\int_t^T s(t,u)du$

(Hint: Express $F(t,T)$ in terms of $P(t,T)$.)

Suppose that $s(t,T) = \sigma e^{-a(T-t)}$ for some positive constants α and σ.

(c) (*1.5 points*) Show that in this situation the diffusion coefficient of the bond price SDE is in the same form as the diffusion coefficient of the bond price SDE under the Hull-White model.

An investment analyst would like to model a number of swaptions. He is considering the following one-factor models:

- One-factor HJM
- One-factor BGM

(d) (*0.5 points*) Outline the main features of the two models under the risk-neutral measure.

(e) (*1 point*) Describe the advantages and disadvantages of using these models to price and hedge exotic interest rate derivatives.

9. (*5 points*) Let $(\Omega, \mathcal{F}, \mathrm{P})$, be a probability space.

(a) (*1 point*) State the three conditions for a stochastic process to be a martingale.

Let B_t be a standard Brownian motion in $(\Omega, \mathcal{F}, \mathrm{P})$.

(b) (*2 points*) Demonstrate that the process

$$X_t = B_t^2 - t$$

is a martingale by showing that each of the three conditions in part (a) holds.

(c) (*1.5 points*) Derive the stochastic differential equation for the process

$$Y_t = B_t^4 - 6tB_t^2 + ct^2$$

where c is a given constant by using Ito's lemma.

(d) (*0.5 points*) Determine the value(s) of the constant c, if any, for the process Y_t in part (c) to be a martingale.

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10. (*6 points*) You have been assigned to a team in an ALM group to provide support on portfolio immunization. During the kick-off meeting, your colleague made the following statements:

I. The immunization target rate of return is less than the yield to maturity.
II. Immunization is a set-and-forget strategy.
III. The liquidity of securities used to construct an immunized portfolio is relevant.
IV. If a portfolio is immunized against a change in the market yield at a given horizon by matching portfolio duration to the horizon, the portfolio faces no risk except for default risk.
V. In order to have a clear picture of the economic surplus of the portfolio, one merely needs to focus on the duration of the company's assets.
VI. To assure multiple liability immunization against parallel rate shifts, the distribution of durations of the liabilities must have a wider range than the distribution of durations of individual portfolio assets.

(a) (*2 points*) Critique each of your colleague's statements.

The company has decided to construct a portfolio consisting of three bonds in equal par amounts of $1 million each. The initial values and durations are:

	Market Value (including accrued interest)	Duration
Bond 1	1,010,697	4.660
Bond 2	998,619	1.879
Bond 3	1,097,032	8.598

After one year, the portfolio values including a shift in the yield curve are shown in the table below.

	Market Value (including accrued interest)	Duration
Bond 1	1,008,983	3.753
Bond 2	1,000,054	0.901
Bond 3	1,088,490	7.791

(b) (*1.5 points*) Calculate the amount of cash required to rebalance the portfolio in order to maintain the dollar duration at the initial level, assuming the proportion by par value is unchanged.

10. **Continued**

Alternatively, assume that you want to maintain the dollar duration at the initial level and also keep the total portfolio par value at $3 million. You will achieve these objectives by reducing the par value of one bond and simultaneously increasing the par values of the other two bonds by equal amounts.

(c) (*2.5 points*) Calculate the new weights by par value for all three bonds.

****END OF EXAMINATION****
Morning Session

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