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Valuing American Options in a Path Simulation Model

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Abstract

The goal of this paper is to dispel the prevailing belief that American-style options cannot be valued efficiently in a simulation model, and thus remove what has been considered a major impediment to the use of simulation models for valuing financial instruments. We present a general algorithm for estimating the value of American options on an underlying instrument or index for which the arbitrage-free probability distribution of paths through time can be simulated. The general algorithm is tested by an example for which the exact option premium can be determined.

1. Introduction

Mathematicians seem to resort to simulation models to analyze a problem only when all other methods fail to yield a solution. In financial economics, evidence of this tendency to avoid simulation models is found in the proliferation of published binomial and multinomial lattice solutions (or their equivalent) to the problem of valuing instruments with cash flows or payoffs contingent on interest rates or stock prices [1], [2], [4], [5], [8], [9], [10], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], and [23]. The standard approach to valuing an American option is to utilize a one-factor (continuous) model of the stochastic price behavior of the option's underlying asset, then create a binomial or multinomial (discrete) connected lattice representation of that stochastic process, and finally solve the valuation problem by backward induction on the lattice. Market-makers who deal in today's complicated financial instruments and investors who buy and sell them are beginning to sense a need for more realistic multifactor models of the stochastic dynamics of interest rates, foreign exchange rates, stock prices, and commodity prices. These more complex models demand analysis by simulation, because constructing approximate solutions (whether by means of lattices or otherwise) to the nonlinear differential and integral equations to which they give rise is extremely difficult.

In general, the use of simulation models for valuing financial instruments has been restricted to assets that have path-dependent cash flows or payoffs, for example, mortgage-backed securities, including collateralized mortgage obligations (CMOs), and esoteric derivative instruments, such as "look-back" options [7], [11]. (An exception, at least in the academic literature if not in practice, is the paper by Boyle [3], which examines how Monte Carlo simulation can be used to value European-style options.) Indeed, it has been thought that simulation models could not be used to value American-style options efficiently, if at all ([7], [8], and Pedersen's discussion of [24]). Ideally, a broker-dealer would like to be able to use a *single* method to value its entire book, and a financial intermediary would like to be able to use a *single* method to analyze its entire asset-liability condition. I believe that simulation models offer that possibility.

Simulation models consume large amounts of computer processing time, and some problems have heretofore required too much execution time to be handled practically by simulation. However, the arrival of powerful workstations, servers, and parallel-processors has rendered simulation feasible in many situations where it previously was not, a condition that can only improve with time as the pace of major technological advances accelerates. In many situations, a *single sample* of paths can be generated and then *used repeatedly* to value many different instruments, for example, a dealer's entire book of interest rate swaps, caps, floors, and swaptions; a dealer's entire book of stock index derivatives or of currency swaps and options; or a financial intermediary's entire portfolio of fixed-income securities. Simulation may not be the best method when each financial instrument must be valued on the basis of its own random sample of paths, but this situation can often be avoided by designing the simulation properly.

The algorithm for valuing American options is described in Section 2 and tested by an example in Section 3. The issue of bias in the estimator of the option premium is examined in Section 4, after which the example is revisited in Section 5. Finally, Section 6 summarizes the paper.

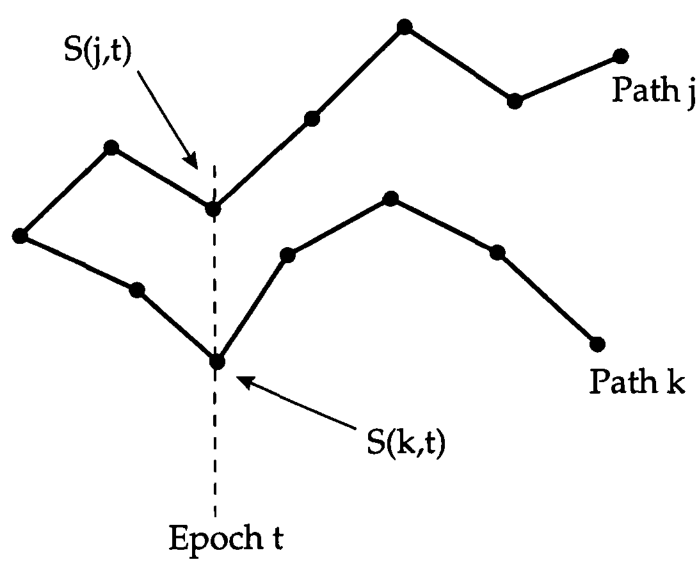
2. The Valuation Algorithm

A textbook by Cox and Rubinstein [6] provides a comprehensive treatment of the subject of options. We assume that the reader is familiar with the general subject area, including various models for pricing options. For convenience, the option's underlying asset is referred to as a "stock," but the entire development in this section applies to any type of asset or index for which the arbitrage-free probability distribution of paths through time can be simulated. In my earlier paper [24] we discuss what is meant by "arbitrage-free" and show how arbitrage-free paths of interest rates can be sampled stochastically. The example in Section 3 of this paper utilizes paths of stock prices that are sampled from a probability distribution that is arbitrage-free because its mean has been adjusted appropriately.

We consider how to evaluate put and call options on a stock. The options are exercisable only at specified epochs t_1, t_2, ..., t_N, which are indexed 1, 2, ..., N for convenience. The origin of time is $t = 0$, which is indexed as epoch 0. The options can be considered to be first exercisable at epoch 0 or at epoch 1, as appropriate. A path of stock prices is a sequence $S(0)$, $S(1)$, $S(2)$, ..., $S(N)$, in which the arguments of S refer to the epoch indexes at which the stock prices occur. *All* paths of stock prices emanate from the initial stock price $S(0)$. The simulation procedure involves the random generation of a finite sample of R such paths and the estimation of option prices from that sample. The k-th path in the sample is represented by the sequence $S(0)$, $S(k,1)$, $S(k,2)$, ..., $S(k,N)$, in which the first index refers to the path and the second index refers to the epoch. Two paths of stock prices are represented in Figure 1. Let $d(k,t)$ be the present value at epoch t on path k of a \$1 payment occurring at epoch $t + 1$ on path k. Let $D(k,t)$ be the present value at epoch 0 of a \$1 payment occurring at epoch t on path k, computed as the product of the discount factors $d(k,s)$ from $s=0$ to $s=t-1$.

FIGURE 1

TWO ILLUSTRATIVE PATHS OF STOCK PRICES SAMPLED FROM A DISCRETE-TIME CONTINUOUS-STATE MODEL OF STOCK PRICE MOVEMENTS



Assume that the option has strike prices that can depend on the date of exercise but not on the stock price at the time of exercise. Let $X(1)$, $X(2)$, ..., $X(N)$ denote the sequence of exercise prices at epochs 1, 2, ..., N, respectively. Typical stock options have a constant strike price X independent of date of exercise, but typical call options in private placement bonds do not. The intrinsic value $I(k,t)$ of the option on path k at epoch t can now be defined as:

$$I(k,t) = \begin{cases} \text{maximum } [0, S(k,t) - X(t)] & \text{for a call option} \\ \text{maximum } [0, X(t) - S(k,t)] & \text{for a put option.} \end{cases}$$

Finally, let $z(k,t)$ be the "exercise-or-hold" indicator variable, which takes the value 0 if the option is not exercised at epoch t on path k and which takes the value 1 if the option is exercised at epoch t on path k. Clearly, either $z(k,t)=0$ at all epochs t along path k or $z(k,t_*)= 1$ at one and only one epoch t_* along path k. If such a t_* exists, it is the date at which the option is exercised on path k.

The price of any asset is known at epoch 0 if its cash flows are known at all epochs along all possible paths. That price is calculated in two steps: first, compute for each path k the present value at epoch 0 of the asset's cash flows along that path using the path-specific discount factors $D(k,t)$, and second, average across all paths the present values computed in the first step. The paths must be drawn from the appropriate arbitrage-free distribution. More details on this general valuation procedure can be found in my paper [24]. On a given stock price path, the "cash flow" for an option is 0 at every epoch other than the one at which the option is exercised. At exercise, the option's "cash flow" is equal to its payoff, which is its intrinsic value. Assuming the usual situation that all randomly sampled stock price paths are equally likely with probability weight R^{-1}, we can express the option premium estimator by the following equation:

$$\text{Premium Estimator} = R^{-1} \sum_{\substack{\text{all} \\ \text{paths} \\ k}} \sum_{\substack{\text{all} \\ \text{epochs} \\ t}} z(k,t)D(k,t)I(k,t).$$

Thus, to estimate the price of the option, we need to estimate the exercise-or-hold indicator function $z(k,t)$, given a finite sample of R paths drawn from an arbitrage-free distribution of paths. The algorithm presented in this section for estimating $z(k,t)$ mimics the standard backward induction algorithm implemented on a connected lattice for estimating the value of an American option. A discussion of this standard technique can be found in the textbook by Cox and Rubinstein [6].

The backward induction is begun at the latest epoch at which the option can be exercised, that is, at its expiration date. On that date, represented by epoch N, the option, if it is still "alive" on path k (that is, not previously exercised), will be exercised if and only if $I(k,N)>0$. The general step of the backward induction performed at an arbitrary epoch t involves determining whether it is optimal to hold the option for possible exercise beyond epoch t or to exercise the option immediately at epoch t. This decision is made by comparing the option's "holding value" to its "exercise value." The option's exercise value is equal to its intrinsic value and can be directly calculated for each path, because the price of the underlying stock is known at each epoch on each path. The option's holding value on any path is calculated as the present value of the expected one-period-ahead option value.

Many believe that utilizing the path structure illustrated in Figure 1 precludes estimation of an option's holding values, because the only point from which *many* paths emerge is epoch 0. On any particular path, at any epoch $t>0$, only a *single* path is simulated. One might think that many paths would need to be simulated from each such point to estimate closely the mathematical expectation of the one-period-ahead option value. Unfortunately, such an approach would lead to a multinomial "tree" in which the number of paths grows exponentially with the number of epochs—a computational infeasibility. Instead, computational feasibility can be achieved by utilizing the path structure illustrated in Figure 1 and then estimating the option's holding value by means of a distinct partitioning at each epoch of the R paths into Q bundles of P paths each. The hope is that the paths within a given bundle are sufficiently alike that they can be considered to have the same expected one-period-ahead option value; in other words, Q must not be too small. The mathematical expectation of the one-period-ahead option value is estimated as an average over all the paths in the bundle. Thus, the estimate of the option holding value will be good only if there are sufficiently many paths in the bundle; in other words, P must not be too small.

In general, there is at least one bundle in which the decision for some paths is to hold the option, while the decision for the rest of the paths in the bundle is to exercise the option. Such a bundle generally has a "transition zone" in stock price from a decision to hold the option to a decision to exercise the option. Specifically, for a call option, there exist stock prices $S_L(t)$ and $S_U(t)$ at epoch t, with $S_L(t)<S_U(t)$, such that the decision is to hold for $S \le S_L(t)$ and to exercise for $S \ge S_U(t)$. However, for $S_L(t)<S<S_U(t)$, the decision is "inconsistent"; that is, there exist stock prices $S_l(t)$ and $S_u(t)$ such that $S_L(t)<S_l(t)<S_u(t)<S_U(t)$, yet the decision is to exercise at $S=S_l(t)$ and to hold at $S=S_u(t)$! The transition zone from an unambiguous hold decision to an unambiguous exercise decision often extends across several consecutive bundles. The algorithm can be refined to eliminate the transition zone by creating a "sharp boundary" at $S=S_*(t)$, such that the decision is to hold for $S<S_*(t)$ and to exercise for $S \ge S_*(t)$.

The general step that is performed at epoch t in the backward induction algorithm includes eight substeps, as follows:

1. Reorder the stock price paths by stock price, from lowest price to highest price for a call option or from highest price to lowest price for a put option. Reindex the paths from 1 to R according to the reordering.
2. For each path k, compute the intrinsic value $I(k,t)$ of the option.
3. Partition the set of R ordered paths into Q distinct bundles of P paths each. Assign the first P paths to the first bundle, the second P paths to the second bundle, and so on, and finally the last P paths to the Q-th bundle. We assume that P and Q are integer factors of R.
4. For each path k, the option's holding value $H(k,t)$ is computed as the following mathematical expectation taken over all paths in the bundle containing the path k:
$$H(k,t) = d(k,t)P^{-1} \sum_{\substack{\text{all } j \\ \text{in bundle} \\ \text{containing } k}} V(j, t+1).$$
The variable $V(k,t)$ is fully defined in substep 8 below. At epoch N, $V(k,N) = I(k,N)$ for all k.
5. For each path k, compare the holding value $H(k,t)$ to the intrinsic value $I(k,t)$, and decide "tentatively" whether to exercise the option or to hold it. Define an indicator variable $x(k,t)$ as follows:
$$x(k,t) = \begin{cases} 1 & \text{if} \quad I(k,t) > H(k,t) \quad \text{Exercise} \\ 0 & \text{if} \quad H(k,t) \ge I(k,t) \quad \text{Hold.} \end{cases}$$
6. Examine the sequence of 0's and 1's $\{x(k,t); k=1, 2, ..., R\}$. Determine a sharp boundary between the hold decision and the exercise decision as the start of the first string of 1's, the length of which exceeds the length of every subsequent string of 0's. Let $k_*(t)$ denote the path index (in the sample as *ordered* in substep 1 above) of the leading 1 in such a string. The transition zone between *hold* and *exercise* is defined as the sequence of 0's and 1's that begins with the first 1 and ends with the last 0. An example is given below:

$$\begin{array}{c} \text{Boundary} \\ \downarrow \\ 0\,0 \ldots 0\,1\,1\,0\,0\,0\,1\,1\,1\,1\,1\,0\,0\,1 \ldots 1\,1 \end{array}$$

7. Define a new exercise-or-hold indicator variable $y(k,t)$ that incorporates the sharp boundary as follows:
$$y(k,t) = \begin{cases} 1 & \text{for} \quad k \ge k_*(t) \\ 0 & \text{for} \quad k < k_*(t). \end{cases}$$
8. For each path k, define the current value $V(k,t)$ of the option as follows:
$$V(k,t) = \begin{cases} I(k,t) & \text{if} \quad y(k,t) = 1 \\ H(k,t) & \text{if} \quad y(k,t) = 0. \end{cases}$$

After the algorithm has been processed backward from epoch N to epoch 1 (or epoch 0 if immediate exercise is permitted), the indicator variable $z(k,t)$ for $t \le N$ is estimated as follows:
$$z(k,t) = \begin{cases} 1 & \text{if } y(k,t) = 1 \text{ and } y(k,s) = 0 \text{ for all } s < t \\ 0 & \text{otherwise.} \end{cases}$$

This completes the description of the algorithm for valuing an American option.

The partition of R paths into Q bundles of P paths each can be characterized by defining a "bundling parameter" α by means of the equation $Q=R^{\alpha}$, and therefore, $P=R^{1-\alpha}$. It is clear that $0 \le \alpha \le 1$. The value $\alpha=0$ corresponds to the partition into a single bundle of R paths, and the value $\alpha=1$ corresponds to the partition into R bundles of one path each. A particular American option valuation algorithm can now be fully described by the sample size R, the technique used to sample paths, and the bundling parameter α. If α is restricted to rational numbers, we can fix α and take sensible limits as $R \to \infty$ to investigate the convergence properties of the option premium estimator. For example, with $\alpha=2/5$, we can examine sample sizes equal to 2^5, 3^5, 4^5, ... paths for which we can study the estimators associated with the partitions $Q=2^2$, 3^2, 4^2, ... bundles and $P=2^3$, 3^3, 4^3, ... paths per bundle, respectively.

For any exercise-hold decision algorithm with α fixed and $0<\alpha<1$, it can be proved that the option premium estimate must converge to the proper result as $R \to \infty$. This follows from the observation that the algorithm for determining the exercise-hold decision variable is based on the standard backward induction algorithm for valuing American options and that all sources of error arise from P, Q, and R being finite. For finite R, imprecision in the premium estimates arises because: (1) the continuous distribution of stock prices at each epoch is not sampled finely enough and (2) the mathematical expectation in substep 4 above is approximated by an average over a finite number of paths. Imprecision of the first type can be reduced by increasing Q, the number of bundles. Imprecision of the second type can be reduced by increasing P, the number of paths per bundle. For fixed R, increasing Q means decreasing P, and vice versa, implying a tradeoff between the first and second types of imprecision. However, if α is held constant at some value in the interval (0, 1), then both types of imprecision are eliminated simultaneously as $R \to \infty$, because then *both* $Q \to \infty$ and $P \to \infty$.

The distinction between the variables $y(k,t)$ and $x(k,t)$ disappears as $R \to \infty$ and α is held constant at a value other than 0 or 1. As $R \to \infty$, the boundary between a decision to exercise the option and a decision to hold the option becomes sharper and sharper; that is, at each epoch, the transition zone with alternating strings of 1's and 0's occurs over a smaller and smaller interval of stock prices. Defining a sharp boundary by means of substep 6 above generally improves the convergence of the algorithm considerably for any α in the interval (0,1) and also generally broadens considerably the interval of α over which the option premium estimates are good. In general, the option premium estimate based on a given sample size, sampling technique, and bundling parameter is more accurate when a sharp exercise-hold boundary is determined than when it is not. However, the *ultimate* convergence of the exercise-hold decision algorithm to the exact option premium does *not* depend at all on whether substeps 6 and 7 above are implemented. If substeps 6 and 7 were omitted from the algorithm, $x(k,t)$ would be used in lieu of $y(k,t)$, both in substep 8 and in the calculation of $z(k,t)$.

3. An Example

To test the algorithm presented in the preceding section, we consider the situation of a non-dividend-paying stock. Let $S(t)$ denote the price of the stock at time t. We assume that the random variable $\ln[S(t)/S(0)]$ is normally distributed with mean μt and variance $\sigma^2 t$. We further assume that the yield curve is flat and that interest rates are constant over time at an annual effective rate r. For the distribution of stock price movements to be arbitrage-free over time, it must be true that $\mu=\ln[1+r]-\sigma^2/2$. Refer to the textbook by Cox and Rubinstein [6] for a proof of this statement.

When a non-dividend-paying stock is the underlying asset, the price of an American call option must be exactly the same as the price of an otherwise identical European call option [6]. The price of an American put option must be no less than the price of an otherwise identical European put option, but the former will in general exceed the latter [6]. Therefore, we test the valuation algorithm on a *put option* that is first exercisable in one quarter of a year and is exercisable every quarter of a year thereafter until its expiration in three years. The stock price has logarithmic volatility σ equal to 30 percent. The initial price of the stock $S(0)$ is 40; the strike price X of the option is 45 at all epochs; and the annual effective interest rate r is 7 percent. Paths of stock price movements are generated randomly by stratified sampling of the standard normal density as described in my paper [24]. Random samples of size $7!=5{,}040$ are used so that many different partitions can be examined. Table 1 lists the values of the bundling parameter α that correspond to each of the 60 different partitions of 5,040 paths into equal bundles.

The exact price of the three-year American put option with quarterly exercise intervals was determined to be 7.941 by using a binomial lattice with 1,200 time periods constructed according to the procedure described in Cox and Rubinstein [6]. This is approximately 1.61 higher than the price of the corresponding three-year European put option. Using a *single* sample of 5,040 paths, the exercise-decision algorithm described in the preceding section was tested for all partitions having at least 12 bundles but no more than 420 bundles. The results are displayed in Figure 2.

In Figure 2, the solid line connecting "diamonds" corresponds to application of the algorithm without substeps 6 and 7—that is, with a *transition zone* from hold to exercise, not a sharp boundary between hold and exercise. The broken line connecting "squares" corresponds to application of the algorithm with substeps 6 and 7 included—that is, with a *sharp boundary* between hold and exercise. The horizontal line across the graph at a vertical axis value of 7.941 marks the exact option premium. Figure 2 clearly demonstrates the importance of including substeps 6 and 7 in the algorithm. When a sharp boundary is determined, the option premium estimates are essentially flat across an interval from α=0.29 to α=0.71 and cover a range of only 12 cents. However, when only a transition zone is utilized, the option premium estimates rise more or less steadily as the bundling parameter is increased and cover a range of approximately 63 cents, more than five times the range obtained when a sharp boundary is utilized!

TABLE 1

BUNDLING PARAMETER ALPHA FOR VARIOUS PARTITIONS OF 5,040 PATHS

Partition		Bundling Parameter Alpha	Partition		Bundling Parameter Alpha
Number of Bundles	Paths per Bundle		Number of Bundles	Paths per Bundle	
1	5040	0.00000	72	70	0.50165
2	2520	0.08131	80	63	0.51401
3	1680	0.12887	84	60	0.51973
4	1260	0.16261	90	56	0.52783
5	1008	0.18879	105	48	0.54591
6	840	0.21017	112	45	0.55348
7	720	0.22825	120	42	0.56157
8	630	0.24392	126	40	0.56730
9	560	0.25773	140	36	0.57965
10	504	0.27009	144	35	0.58296
12	420	0.29148	168	30	0.60104
14	360	0.30956	180	28	0.60913
15	336	0.31765	210	24	0.62721
16	315	0.32522	240	21	0.64288
18	280	0.33904	252	20	0.64860
20	252	0.35140	280	18	0.66096
21	240	0.35712	315	16	0.67478
24	210	0.37279	336	15	0.68235
28	180	0.39087	360	14	0.69044
30	168	0.39896	420	12	0.70052
35	144	0.41704	504	10	0.72991
36	140	0.42035	560	9	0.74227
40	126	0.43270	630	8	0.75608
42	120	0.43843	720	7	0.77175
45	112	0.44652	840	6	0.78983
48	105	0.45409	1008	5	0.81121
56	90	0.47217	1260	4	0.83739
60	84	0.48027	1680	3	0.87113
63	80	0.48599	2520	2	0.91869
70	72	0.49835	5040	1	1.00000

To study the efficiency of estimation, the "70 bundles by 72 paths per bundle" partition was used on 1,000 independent samples of 5,040 paths each. Each sample gives rise to an estimate of the put option premium. The frequency histogram of these 1,000 estimates is plotted in Figure 3. The mean of the estimates is 7.971 and the standard deviation of the estimates is 0.053. The solid line graph superimposed on the frequency histogram is that of a normal density function with the same mean and standard deviation as the option premium estimator. We can see that the algorithm produces premium estimates that are normally distributed. What seems surprising is that the premium estimator is *biased*. The mean estimate of \$7.971 is 3 cents higher than the exact premium of \$7.941, which is about 17.9 times the standard deviation of $5.3/\sqrt{1000}$ cents. Despite the bias, the algorithm can estimate the option premium quite tightly.

FIGURE 2

PREMIUM ESTIMATES FOR 3-YEAR AMERICAN PUT OPTION

(5,040 PATHS PARTITIONED 40 WAYS INTO EXERCISE-DECISION BUNDLES)

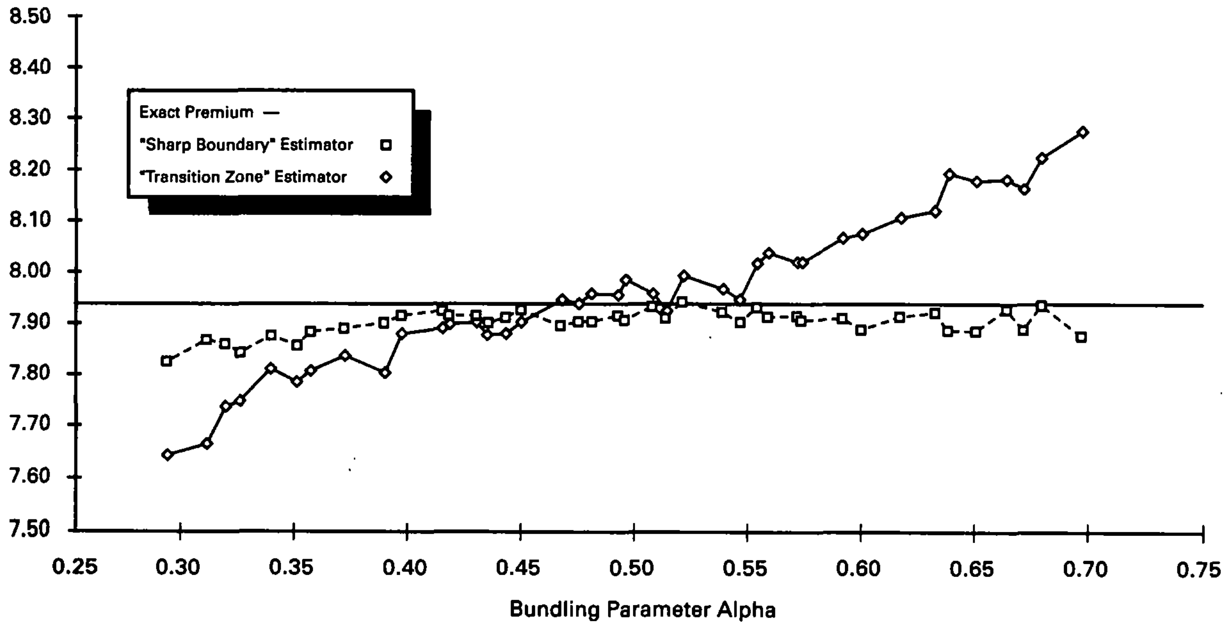
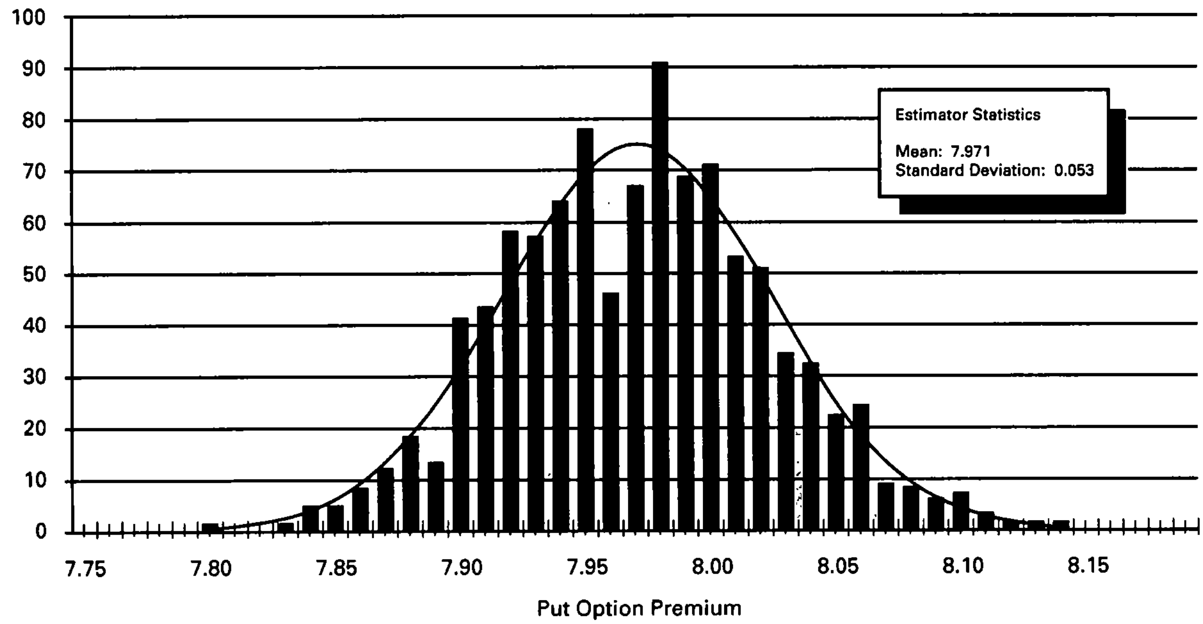


FIGURE 3

FREQUENCY HISTOGRAM FOR α=0.50 PREMIUM ESTIMATOR

(BASED ON 1,000 SAMPLES OF 5,040 PATHS EACH)

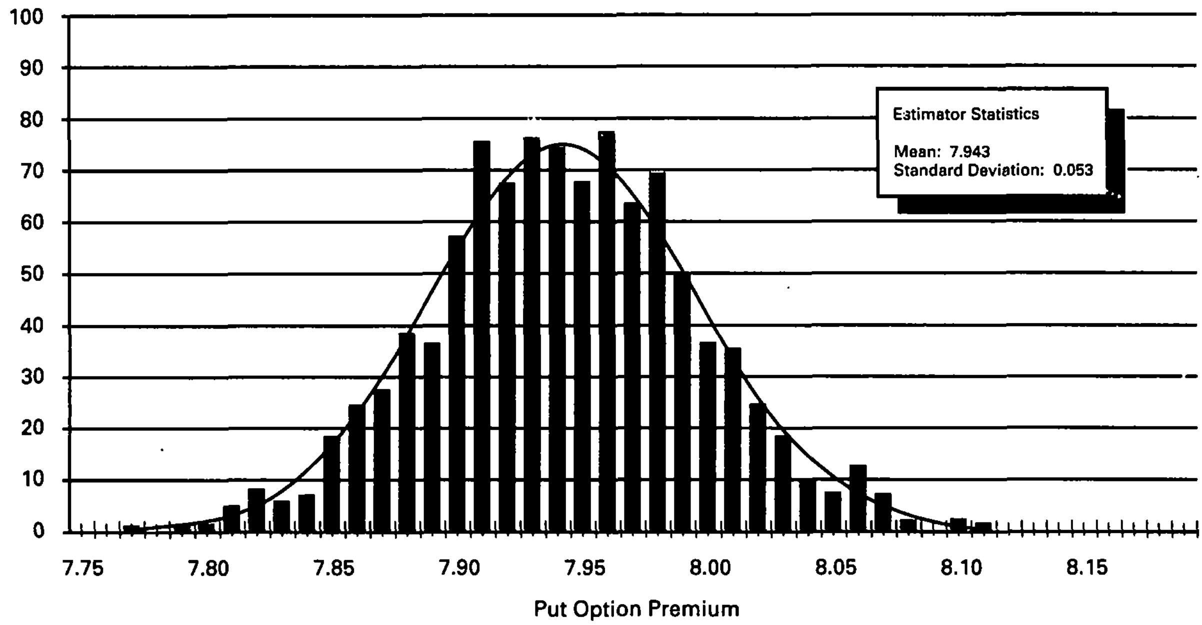


4. Estimator Bias

In this section, we investigate the source of the bias in the option premium estimates that was discovered by means of the example presented in the last section. It turns out that the bias arises because the "optimization" is done over a *finite* sample. The bias vanishes in the limit of infinite sample size. The description of the exercise-decision algorithm in Section 2 makes it evident that estimating the premium for an American option is equivalent to estimating the exercise-hold stock price boundary at each epoch at which the option can be exercised. Accordingly, we determined the "exact" boundary between holding and exercising the put option at each of the 12 exercise-decision epochs by using the Cox-Rubinstein binomial lattice that was described in the preceding section. With *full knowledge* of the exact exercise-hold boundaries, the American option premium was estimated again by simulation using the *same 1,000 samples* of 5,040 paths on which the results shown in Figure 3 were based. The resulting frequency histogram of the premium estimates is shown in Figure 4.

In Figure 4 the premium estimates are normally distributed. The standard deviation of the estimates is 5.3 cents, the same as in Figure 3. However, the mean of the estimates is $7.943, only 0.2 cents higher than the exact premium. This deviation is not statistically significant at a 5 percent level of confidence, since it is only about 1.2 times the standard deviation of $5.3/\sqrt{1000}$ cents. Thus, with full knowledge of the exact exercise-decision boundaries, the American option premium estimator is unbiased, even for finite samples of paths. We must conclude that the process of estimating the exercise-hold boundaries from a finite sample of paths introduces the bias. The following analysis demonstrates the truth of this assertion.

FIGURE 4
FREQUENCY HISTOGRAM FOR "BEST" PREMIUM ESTIMATOR
(BASED ON 1,000 SAMPLES OF 5,040 PATHS EACH)



The exact price of an American option is the value given by the premium estimator equation in Section 2 when the *infinite sample space* of stock price paths and the *exact exercise-hold boundaries* are used. Determining the exact price of the option is equivalent to finding the exercise-hold boundaries at all exercise-decision epochs that maximize the value given by the premium estimator equation when the infinite sample space of stock price paths is used. An approximation to the exact price is obtained by finding the exercise-hold boundaries at all exercise-decision epochs that maximize the value given by the premium estimator equation when a *finite sample* of R stock price paths is used. A different approximation to the exact price is obtained by implementing the backward induction algorithm with eight substeps at each epoch that was described in Section 2. This latter estimate of the exact option price is itself an approximation to the former estimate of the exact option price, by reason of the construction of the backward induction algorithm as an optimization.

Let E_i denote the option premium estimate obtained when the i-th sample of R paths is used together with some premium estimation method. The dependence of the estimate on the estimation method used is denoted by an appropriate superscript. The superscript *∞-optimal* is used to represent the estimation method that utilizes the exact boundaries determined from the infinite sample space of stock price paths. The superscript *R-optimal* is used to represent the estimation method that utilizes the boundaries that optimize the value given by the premium estimator equation when the finite sample of R paths is used. Finally, the superscript *R-algorithm* is used to represent the estimation method that utilizes the boundaries determined from the eight-substep backward induction algorithm applied to the finite sample of R paths. As a consequence of the definitions of the various estimates and the construction of the different estimation methods, the following inequalities hold for *any* sample i consisting of R paths:

$$E_i^{\infty\text{-}optimal} \le E_i^{R\text{-}optimal} \text{ and } E_i^{R\text{-}algorithm} \le E_i^{R\text{-}optimal}.$$

Thus, the means of the various estimators computed over any finite number of samples of R paths each also satisfy the same inequalities. In practice, the strict inequality will hold "almost surely." When the sample size is infinite, the inequalities become equalities. Because the *∞-optimal* estimator is unbiased, the first inequality demonstrates that the *R-optimal* estimator must always have positive bias. The bias tends to zero as $R \rightarrow \infty$. Furthermore, the second inequality demonstrates that the *R-optimal* estimator must be positively biased relative to the *R-algorithm* estimator.

The relative bias tends to zero as $R \rightarrow \infty$. It is indeterminable whether the *R-algorithm* estimator has positive or negative bias with respect to the ∞*-optimal* estimator. The two inequalities also show that we should not try too hard to "perfect" the *R-algorithm* estimator in the sense of making it better approximate the *R-optimal* estimator, because the latter always has positive bias relative to the unbiased ∞*-optimal* estimator.

5. Example Revisited

Now that we understand that the sign of the bias of the *R-algorithm* estimator is indeterminable, but is likely to be positive if the *R-algorithm* estimates the *R-optimal* exercise-hold decision boundaries closely, we should conduct further empirical studies of the bias. Table 2 presents results obtained by using the *R-algorithm* estimator of Section 2 on 100 independent samples of 5,040 paths each by using a partition of 70 bundles by 72 paths per bundle. Results are shown for 3-year American put options with strike prices ranging from 10 to 100 in multiples of 5. All other assumptions are the same as in the earlier example. The "exact" premiums were calculated as before, by using the Cox-Rubinstein binomial lattice with 1,200 time intervals. The estimator bias ranges from a low of −0.7 cents to a high of +3.4 cents. The standard deviations of the estimates peak at 6.3 cents for a put option somewhat in the money. The premium estimates must be considered very accurate.

TABLE 2

STATISTICS FOR α=0.50 ESTIMATORS OF PREMIUMS FOR AMERICAN PUT OPTIONS (STOCK PRICE VOLATILITY OF 30 PERCENT)

Stock Price: 40
Option Expiration: 3.00 Years — Stock Volatility: 30 Percent
Exercise Interval: 0.25 Years — Annual Interest Rate: 7 Percent

Strike Price	'Exact' Premium*	Estimator Mean†	Estimator Bias	Estimator Standard Deviation†
10	0.003	0.003	0.000	0.001
15	0.046	0.046	0.000	0.005
20	0.242	0.239	−0.003	0.012
25	0.744	0.744	0.000	0.020
30	1.689	1.694	0.005	0.027
35	3.172	3.185	0.013	0.038
40	5.247	5.268	0.021	0.044
45	7.941	7.968	0.027	0.055
50	11.255	11.289	0.034	0.063
55	15.136	15.161	0.025	0.059
60	19.469	19.485	0.016	0.054
65	24.100	24.109	0.009	0.044
70	28.894	28.899	0.005	0.034
75	33.764	33.763	−0.001	0.028
80	38.665	38.662	−0.003	0.024
85	43.576	43.574	−0.002	0.017
90	48.491	48.486	−0.005	0.015
95	53.407	53.400	−0.007	0.014
100	58.323	58.316	−0.007	0.012

*Calculated using the Cox-Rubinstein binomial model with 1,200 time intervals.
†Calculated using a simulation model with 100 samples of 5,040 paths and exercise boundary determined by first dominant string of 1's in the transition zone.

Table 3 presents results similar to those in Table 2, but for a partition of 504 bundles by 10 paths per bundle. In this case, substep 6 of the exercise-decision algorithm was refined to account not only for the first dominant string of 1's in the transition zone but also the last dominant string of 0's in the transition zone. As in substep 6, a boundary index is determined as the start of the first string of 1's, the length of which exceeds the length of every subsequent string of 0's. Another boundary index is determined as the end of the last string of 0's, the length of which exceeds the length of every previous string of 1's. In many cases, the two boundaries are identical, but if not, the *dominant 0-string* boundary must occur before the *dominant 1-string* boundary. The boundary index actually used in the revised algorithm is the arithmetic mean of the two boundary indexes, rounded appropriately. The estimator bias shown in Table 3 ranges from a low of −1.2 cents to a high of +0.8 cents. The standard deviations of the estimates are generally a little higher than their counterparts in Table 2.

Table 4 presents results similar to those in Table 3, except that the stock price volatility has been doubled to 60 percent. Again, the estimator biases are small, ranging from −0.8 cents to +2.4 cents. The standard deviations of the estimates are much larger, but are still very small when expressed as a percentage of the exact premiums.

TABLE 3

STATISTICS FOR α=0.73 ESTIMATORS OF PREMIUMS FOR AMERICAN PUT OPTIONS (STOCK PRICE VOLATILITY OF 30 PERCENT)

Stock Price: 40 Option Expiration: 3.00 Years Exercise Interval: 0.25 Years			Stock Volatility: 30 Percent Annual Interest Rate: 7 Percent	
Strike Price	'Exact' Premium*	Estimator Mean†	Estimator Bias	Estimator Standard Deviation†
10	0.003	0.003	0.000	0.001
15	0.046	0.048	0.002	0.005
20	0.242	0.246	0.004	0.012
25	0.744	0.750	0.006	0.018
30	1.689	1.697	0.008	0.028
35	3.172	3.178	0.006	0.039
40	5.247	5.255	0.008	0.049
45	7.941	7.943	0.002	0.052
50	11.255	11.260	0.005	0.066
55	15.136	15.139	0.003	0.059
60	19.469	19.468	−0.001	0.058
65	24.100	24.094	−0.006	0.049
70	28.894	28.889	−0.005	0.037
75	33.764	33.752	−0.012	0.034
80	38.665	38.654	−0.011	0.032
85	43.576	43.566	−0.010	0.021
90	48.491	48.480	−0.011	0.021
95	53.407	53.398	−0.009	0.020
100	58.323	58.315	−0.008	0.016

*Calculated using the Cox-Rubinstein binomial model with 1,200 time intervals.

†Calculated using a simulation model with 100 samples of 5,040 paths and exercise boundary determined by dominant strings of both 0's and 1's in the transition zone.

TABLE 4
STATISTICS FOR α=0.73 ESTIMATORS OF PREMIUMS FOR AMERICAN PUT OPTIONS
(STOCK PRICE VOLATILITY OF 60 PERCENT)

Stock Price: 40
Option Expiration: 3.00 Years
Exercise Interval: 0.25 Years

Stock Volatility: 60 Percent
Annual Interest Rate: 7 Percent

Strike Price	'Exact' Premium*	Estimator Mean†	Estimator Bias	Estimator Standard Deviation†
10	0.486	0.489	0.003	0.013
15	1.409	1.414	0.005	0.022
20	2.810	2.815	0.005	0.034
25	4.636	4.643	0.007	0.044
30	6.834	6.840	0.006	0.054
35	9.357	9.367	0.010	0.064
40	12.162	12.180	0.018	0.075
45	15.220	15.240	0.020	0.088
50	18.504	18.526	0.022	0.102
55	21.986	22.009	0.023	0.120
60	25.650	25.667	0.017	0.118
65	29.475	29.493	0.018	0.131
70	33.453	33.477	0.024	0.129
75	37.566	37.584	0.018	0.148
80	41.798	41.809	0.011	0.145
85	46.137	46.139	0.002	0.149
90	50.571	50.567	–0.004	0.149
95	55.086	55.078	–0.008	0.144
100	59.670	59.662	–0.008	0.144

*Calculated using the Cox-Rubinstein binomial model with 1,200 time intervals.
†Calculated using a simulation model with 100 samples of 5,040 paths and exercise boundary determined by dominant strings of both 0's and 1's in the transition zone.

6. Summary and Conclusions

This paper has presented an algorithm for valuing American options in a path simulation model and has demonstrated its accuracy by an example involving a put option on a non-dividend-paying stock for which the exact premium could be determined. The demonstration of the existence of a useful algorithm for valuing American options in a path simulation model should remove what has been perceived as a major impediment to the use of simulation models in valuing a broker-dealer's derivatives book and in analyzing the asset-liability condition of financial intermediaries.

In many situations involving the use of multifactor models to describe realistic market price behavior, simulation is the only method that can handle the American option valuation problem satisfactorily. Furthermore, it is usually straightforward to apply a simulation technique, whereas solving complicated partial differential equations numerically generally requires great care as well as sophistication in applied mathematical methods. This paper has not dealt with some of the complexities that arise in determining exercise-hold decision boundaries when multifactor stochastic models of asset price behavior are utilized. Empirical studies that I have conducted suggest that some modification to the algorithm presented in this paper is required to handle those situations adequately. For example, the bundling must often be carried out in at least two dimensions rather than the single dimension presented in this paper. Boundary points become boundary lines or surfaces.

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