IV

Option Pricing by Esscher Transforms

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Abstract

The Esscher transform is a time-honored tool in actuarial science. This paper shows that the Esscher transform is also an efficient technique for valuing derivative securities if the logarithms of the prices of the primitive securities are governed by certain stochastic processes with stationary and independent increments. This family of processes includes the Wiener process, the Poisson process, the gamma process, and the inverse Gaussian process. An Esscher transform of such a stock-price process induces an equivalent probability measure on the process. The Esscher parameter or parameter vector is determined so that the discounted price of each primitive security is a martingale under the new probability measure. The price of any derivative security is simply calculated as the expectation, with respect to the equivalent martingale measure, of the discounted payoffs. Straightforward consequences of the method of Esscher transforms include, among others, the celebrated Black-Scholes option-pricing formula, the binomial option-pricing formula, and formulas for pricing options on the maximum and minimum of multiple risky assets. Tables of numerical values for the prices of certain European call options (calculated according to four different models for stock-price movements) are also provided.

1. Introduction

The Esscher transform [35] is a time-honored tool in actuarial science. Members of the Society of Actuaries were introduced to it by Kahn's survey paper [51] and Wooddy's Study Note [79]. In this paper we show that the Esscher transform is also an efficient technique for valuing derivative securities if the logarithms of the prices of the primitive securities are governed by certain stochastic processes with stationary and independent increments. This family of processes includes the Wiener process, the Poisson process, the gamma process, and the inverse Gaussian process. Our modeling of stock-price movements by means of the gamma process and the inverse Gaussian process seems to be new. Straightforward consequences of the proposed method include, among others, the celebrated Black-Scholes option-pricing formula, the binomial option-pricing formula, and formulas for pricing options on the maximum and minimum of multiple risky assets.

For a probability density function $f(x)$, let h be a real number such that

$$M(h) = \int_{-\infty}^{\infty} e^{hx} f(x)\,dx$$

exists. As a function in x,

$$f(x;h) = \frac{e^{hx} f(x)}{M(h)}$$

is a probability density, and it is called the Esscher transform (parameter h) of the original distribution. The Esscher transform was developed to approximate the aggregate claim amount distribution around a point of interest, x_0, by applying an analytic approximation (the Edgeworth series) to the transformed distribution with the parameter h chosen such that the new mean is equal to x_0. When the Esscher transform is used to calculate a stop-loss premium, the parameter h is usually determined by specifying the mean of the transformed distribution as the retention limit. Further discussions and details on the method of Esscher transforms can be found in risk theory books such as [6], [7], [27], [38], and [70]; see also Jensen's paper [49].

In this paper we show that the Esscher transform can be extended readily to a certain class of stochastic processes, which includes some of those commonly used to model stock-price movements. The parameter h is determined so that the modified probability measure is an equivalent martingale measure, with respect to which the prices of securities are expected discounted payouts.

Our first application of the method of Esscher transforms is formula (2.15), which is a general expression for the value of a European call option on a non-dividend-paying stock and includes the Black-Scholes option-pricing formula, the pure-jump option-pricing formula, and the binomial option-pricing formula as special cases. We also introduce two new models for stock-price movements; the first one is defined in terms of the gamma process and the second in terms of the inverse Gaussian process. Formulas for pricing European call options on stocks with such price movements are also given, and numerical tables (calculated according to four different models) are provided.

In the second half of this paper, we extend the method of Esscher transforms to price derivative securities of multiple risky assets or asset pools. The main result is as follows: Assume that the risk-free force of interest is constant and denote it by δ. For $t \geq 0$, let $S_1(t)$, $S_2(t)$, …, $S_n(t)$ denote the prices of n non-dividend-paying stocks or assets at time t. Assume that the vector

$$(\ln[S_1(t)/S_1(0)], \ln[S_2(t)/S_2(0)], \ldots, \ln[S_n(t)/S_n(0)])'$$

is governed by a stochastic process that has independent and stationary increments and that is continuous in probability. Let g be a real-valued measurable function of n variables. Then, for $\tau \geq 0$,

$$\begin{aligned} &\mathrm{E}^*[e^{-\delta\tau}S_j(\tau)g(S_1(\tau), S_2(\tau), \ldots, S_n(\tau))] \\ &\qquad = S_j(0)\mathrm{E}^{**}[g(S_1(\tau), S_2(\tau), \ldots, S_n(\tau))], \end{aligned}$$

where the expectation on the left-hand side is taken with respect to the risk-neutral Esscher transform and the expectation on the right-hand side is taken with respect to another specified Esscher transform. It is shown that many classical option-pricing formulas are straightforward consequences of this result.

A useful introduction to the subject of options and other derivative securities can be found in Boyle's book [15], which was published recently by the Society of Actuaries. Kolb's book [52] is a collection of 44 articles on derivative securities by various authors; most of these articles are descriptive and not mathematical. For an intellectual history of option-pricing theory, see Chapter 11 of Bernstein's book [9].

In this paper the risk-free interest rate is assumed to be constant. We also assume that the market is frictionless and trading is continuous. There are no taxes, no transaction costs, and no restriction on borrowing or short sales. All securities are perfectly divisible. It is now understood that, in such a securities market model, the absence of arbitrage is "essentially" equivalent to the existence of an equivalent martingale measure, with respect to which the price of a random payment is the expected discounted value. Some authors ([5], [34], [67]) call this result the "Fundamental Theorem of Asset Pricing." In a general setting, the equivalent martingale measure is not unique; the merit of the risk-neutral Esscher transform is that it provides a general, transparent and unambiguous solution.

In the next section we use some basic ideas from the theory of stochastic processes. Two standard references are Breiman's book [18] and Feller's book [36].

2. Risk-Neutral Esscher Transform

For $t \geq 0$, $S(t)$ denotes the price of a non-dividend-paying stock or security at time t. We assume that there is a stochastic process, $\{X(t)\}_{t \geq 0}$, with stationary and independent increments, $X(0)=0$, such that

$$S(t) = S(0)e^{X(t)}, \quad t \geq 0. \tag{2.1}$$

For each t, the random variable $X(t)$, which may be interpreted as the continuously compounded rate of return over the t periods, has an infinitely divisible distribution [18, Proposition 14.16]. Let

$$F(x, t) = \Pr[X(t) \leq x] \tag{2.2}$$

be its cumulative distribution function, and

$$M(z, t) = \mathrm{E}[e^{zX(t)}] \tag{2.3}$$

its moment-generating function. By assuming that $M(z, t)$ is continuous at $t=0$, it can be proved that

$$M(z, t) = [M(z, 1)]^t \tag{2.4}$$

([18, Section 14.4], [36, Section IX.5]). We assume that (2.4) holds.

For simplicity, let us assume that the random variable $X(t)$ has a density

$$f(x, t) = \frac{d}{dx}F(x, t), \; t > 0;$$

then

$$M(z, t) = \int_{-\infty}^{\infty} e^{zx} f(x, t)dx.$$

Let h be a real number for which $M(h, t)$ exists. (It follows from (2.4) that, if $M(h, t)$ exists for one positive number t, it exists for all positive t.) We now introduce the Esscher transform (parameter h) of the process $\{X(t)\}$. This is again a process with stationary and independent increments, whereby the new probability density function of $X(t)$, $t>0$, is

$$f(x, t;h) = \frac{e^{hx} f(x, t)}{\int_{-\infty}^{\infty} e^{hy} f(y, t) dy}$$

$$= \frac{e^{hx} f(x, t)}{M(h, t)}. \qquad (2.5)$$

That is, the modified distribution of $X(t)$ is the Esscher transform of the original distribution. The corresponding moment-generating function is

$$M(z, t;h) = \int_{-\infty}^{\infty} e^{zx} f(x, t;h) dx$$

$$= \frac{M(z+h, t)}{M(h, t)}. \qquad (2.6)$$

By (2.4),

$$M(z, t; h) = [M(z, 1; h)]^t. \qquad (2.7)$$

The Esscher transform of a single *random variable* is a well-established concept in the risk theory literature. Here, we consider the Esscher transform of a *stochastic process*. In other words, the probability measure of the process has been modified. Because the exponential function is positive, the modified probability measure is *equivalent* to the original probability measure; that is, both probability measures have the same null sets (sets of probability measure zero).

We want to ensure that the stock prices of the model are internally consistent. Thus we seek $h=h^*$, so that the discounted stock price process, $\{e^{-\delta t}S(t)\}_{t\geq 0}$, is a martingale with respect to the probability measure corresponding to h^*. In particular,

$$S(0) = \mathrm{E}^*[e^{-\delta t}S(t)]$$
$$= e^{-\delta t}\, \mathrm{E}^*[S(t)],$$

where δ denotes the constant risk-free force of interest. By (2.1), the parameter h^* is the solution of the equation

$$1 = e^{-\delta t}\, \mathrm{E}^*[e^{X(t)}],$$

or

$$e^{\delta t} = M(1, t; h^*). \qquad (2.8)$$

From (2.7) we see that the solution does not depend on t, and we may set $t=1$:

$$e^{\delta} = M(1, 1; h^*), \qquad (2.9)$$

or

$$\delta = \ln[M(1, 1; h^*)]. \qquad (2.10)$$

It can be shown that the parameter h^* is unique [40]. We call the Esscher transform of parameter h^* the *risk-neutral Esscher transform*, and the corresponding equivalent martingale measure the *risk-neutral Esscher measure*. Note that, although the risk-neutral Esscher measure is unique, there may be other equivalent martingale measures; see the paper by Delbaen and Haezendonck [30] for a study on equivalent martingale measures of compound Poisson processes.

To evaluate a derivative security (whose future payments depend on the evolution of the stock price), we calculate the expected discounted value of the implied payments; the expectation is with respect to the risk-neutral Esscher measure. Let us consider a European call option on the stock with exercise price K and exercise date τ, $\tau>0$. The value of this option (at time 0) is

$$\mathrm{E}^*[e^{-\delta\tau}(S(\tau)-K)_+], \qquad (2.11)$$

where $x_+ = x$ if $x>0$, and $x_+ = 0$ if $x\leq 0$. With the definition

$$\kappa = \ln[K/S(0)], \qquad (2.12)$$

(2. 11) becomes

$$e^{-\delta\tau}\int_{\kappa}^{\infty}[S(0)e^x - K] f(x, \tau;h^*) dx$$

$$= e^{-\delta\tau}S(0)\int_{\kappa}^{\infty} e^x f(x, \tau;h^*) dx$$

$$- e^{-\delta\tau}K[1-F(\kappa, \tau;h^*)]. \qquad (2.13)$$

It follows from (2.5), (2.6) and (2.8) that

$$e^x f(x, \tau;h^*) = \frac{e^{(h^*+1)x} f(x, \tau)}{M(h^*, \tau)}$$

$$= \frac{M(h^*+1, \tau)}{M(h^*, \tau)} f(x, \tau;h^*+1)$$

$$= M(1, \tau;h^*) f(x, \tau;h^*+1)$$

$$= e^{\delta\tau} f(x, \tau;h^*+1). \qquad (2.14)$$

Thus the value of the European call option with exercise price K and exercise date τ is

$$S(0)[1-F(\kappa, \tau; h^*+1)] - e^{-\delta\tau}K[1-F(\kappa, \tau; h^*)]. \qquad (2.15)$$

In Sections 3 and 4, this general formula is applied repeatedly. It is shown that (2.15) contains, among others, the celebrated Black-Scholes option-pricing formula as a special case.

2.1 Remarks

In the general case in which the distribution function $F(x, t)$ is not necessarily differentiable, we can define the Esscher transform in terms of Stieltjes integrals. That is, we replace (2.5) by

$$dF(x,t;h) = \frac{e^{hx}dF(x,t)}{\int_{-\infty}^{\infty} e^{hy}dF(y,t)}$$

$$= \frac{e^{hx}dF(x,t)}{M(h,t)}. \qquad (2.1.1)$$

(In his paper [35] Esscher did not assume that the individual claim amount distribution function is differentiable.) Formula (2.15) remains valid.

That the condition of no arbitrage is intimately related to the existence of an equivalent martingale measure was first pointed out by Harrison and Kreps [42] and by Harrison and Pliska [43]. Their results are rooted in the idea of risk-neutral valuation of Cox and Ross [24]. For an insightful introduction to the subject, see Duffie's recent book [32]. In a finite discrete-time model, the absence of arbitrage opportunities is equivalent to the existence of an equivalent martingale measure ([28], [67]). In a more general setting, the characterization is more delicate, and we have to replace the term "equivalent to" by "essentially equivalent to." Discussion of the details is beyond the scope of this paper; some recent papers are [4], [5], [23], [29], [44], [53], [59], [68], and [69].

The idea of changing the probability measure to obtain a consistent positive linear pricing rule had appeared in the actuarial literature in the context of equilibrium reinsurance markets ([12], [13], [19], [20], [39], and [73]); see also [77], [2], and [78].

Observe that the option-pricing formula (2.15) can be written as

$$S(0)\Pr[S(\tau) > K; h^* + 1] - e^{-\delta\tau}K\Pr[S(\tau) > K; h^*],$$

where the first probability is evaluated with respect to the Esscher transform with parameter h^*+1, while the second probability is calculated with respect to the risk-neutral Esscher transform. Generalizations of this result are given in Section 6.

To construct a stochastic process $\{X(t)\}$ with stationary and independent increments, $X(0)=0$, and

$$M(z, t) = [M(z, 1)]^t,$$

we can apply the following theorem [18, Proposition 14.19]: Given the moment-generating function $\zeta(z)$ of an infinitely divisible distribution, there is a unique stochastic process $\{W(t)\}$ with stationary and independent increments, $W(0)=0$, such that

$$E[e^{zW(t)}] = [\zeta(z)]^t.$$

The normal distribution, the Poisson distribution, the gamma distribution, and the inverse Gaussian distribution are four examples of infinitely divisible distributions. In the following sections, we consider stock-price movements modeled with such processes.

3. Three Classical Option Formulas

In this section we apply the results of Section 2 to derive European call option formulas in three classical models for stock-price movements. These three formulas can be found in textbooks on options, such as those by Cox and Rubinstein [26], Gibson [41] and Hull [47]. Note that Hull's book [47] is a textbook for the Society of Actuaries Course F-480 examination.

3.1 Logarithm of Stock Price as a Wiener Process

Here we make the classical assumption that the stock prices are lognormally distributed. Let the stochastic process $\{X(t)\}$ be a Wiener process with mean per unit time μ and with variance per unit time σ^2. Let $N(x; \mu, \sigma^2)$ denote the normal distribution function with mean μ and variance σ^2. Then

$$F(x, t) = N(x; \mu t, \sigma^2 t)$$

and

$$M(z, t) = \exp[(\mu z + \tfrac{1}{2}\sigma^2 z^2)t].$$

It follows from (2.6) that

$$M(z, t; h) = \exp\{[(\mu + h\sigma^2)z + \tfrac{1}{2}\sigma^2 z^2]t\}.$$

Hence the Esscher transform (parameter h) of the Wiener process is again a Wiener process, with modified mean per unit time

$$\mu + h\sigma^2$$

and unchanged variance per unit time σ^2. Thus

$$F(x, t; h) = N(x; (\mu + h\sigma^2)t, \sigma^2 t).$$

From (2. 10) we obtain

$$\delta = (\mu + h^*\sigma^2) + \tfrac{1}{2}\sigma^2.$$

Consequently, the transformed process has mean per unit time

$$\begin{aligned} \mu^* &= \mu + h^*\sigma^2 \\ &= \delta - (\sigma^2/2). \end{aligned} \tag{3.1.1}$$

It now follows from (2.15) that the value of the European call option is

$$\begin{aligned} &S(0)[1 - N(\kappa; (\mu^* + \sigma^2)\tau, \sigma^2\tau)] - e^{-\delta\tau} K[1 - N(\kappa; \mu^*\tau, \sigma^2\tau)] \\ &\quad = S(0)[1 - N(\kappa; (\delta + \tfrac{1}{2}\sigma^2)\tau, \sigma^2\tau)] \\ &\qquad - e^{-\delta\tau} K[1 - N(\kappa; (\delta - \tfrac{1}{2}\sigma^2)\tau, \sigma^2\tau)]. \end{aligned} \tag{3.1.2}$$

In terms of the standard normal distribution function Φ, this result can be expressed as

$$S(0)\Phi\left(\frac{-\kappa + (\delta + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right) - e^{-\delta\tau} K\Phi\left(\frac{-\kappa + (\delta - \sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right), \tag{3.1.3}$$

which is the classical Black-Scholes option-pricing formula [11]. Note that μ does not appear in (3.1.3).

3.2 Logarithm of Stock Price as a Shifted Poisson Process

Next we consider the so-called pure jump model. The pricing of options on stocks with such stochastic movements was discussed by Cox and Ross [24]; however, they did not provide an option-pricing formula. The option-pricing formula for this model appeared several years later in the paper by Cox, Ross and Rubinstein [25, p. 255]; it was derived as a limiting case of the binomial option-pricing formula. (We deduce the binomial option-pricing formula by the Esscher transform method in Section 3.3.) A more thorough discussion of the derivation can be found in the paper by Page and Sanders [61].

Here the assumption is that

$$X(t) = kN(t) - ct, \tag{3.2.1}$$

where $\{N(t)\}$ is a Poisson process with parameter λ, and k and c are positive constants. Let

$$\Lambda(x; \theta) = \sum_{0 \le j \le x} \frac{e^{-\theta}\theta^j}{j!}$$

be the cumulative Poisson distribution function with parameter θ. Then the cumulative distribution function of $X(t)$ is

$$F(x, t) = \Lambda\left(\frac{x + ct}{k}; \lambda t\right). \tag{3.2.2}$$

Since

$$E[e^{zN(t)}] = \exp[\lambda t(e^z - 1)],$$

we have

$$\begin{aligned} M(z, t) &= E(e^{z[kN(t) - ct]}) \\ &= e^{[\lambda(e^{zk} - 1) - cz]t}, \end{aligned} \tag{3.2.3}$$

from which we obtain

$$M(z, t; h) = e^{[\lambda e^{hk}(e^{zk} - 1) - cz]t}. \tag{3.2.4}$$

Hence the Esscher transform (parameter h) of the shifted Poisson process is again a shifted Poisson process, with modified Poisson parameter λe^{hk}. Formula (2.10) is the condition that

$$\delta = \lambda e^{h^*k}(e^k - 1) - c. \tag{3.2.5}$$

Thus a derivative security is evaluated according to the modified Poisson parameter

$$\begin{aligned} \lambda^* &= \lambda e^{h^*k}. \\ &= (\delta + c)/(e^k - 1). \end{aligned} \tag{3.2.6}$$

For example, the price of a European call option is, according to (2.15) and (3.2.2),

$$\begin{aligned} &S(0)[1 - \Lambda((\kappa + c\tau)/k; \lambda^* e^k \tau)] \\ &\qquad - Ke^{-\delta\tau}[1 - \Lambda((\kappa + c\tau)/k; \lambda^*\tau)]. \end{aligned} \tag{3.2.7}$$

Formula (3.2.7) can be found in textbooks on options such as those by Cox and Rubinstein [26, p. 366], Gibson [41, p. 168] and Hull [47, p. 454]. Note that the Poisson parameter λ does not appear in (3.2.7).

3.3 Logarithm of Stock Price as a Random Walk

A very popular model for pricing options is the binomial model, which is a discrete-time model. Although this paper focuses on continuous-time models, we think that it is worthwhile to digress and derive the binomial option-pricing formula by the Esscher transform method, because of its importance in the literature. Indeed, the two papers in *TSA*, by Clancy [22] and Pedersen, Shiu, and Thorlacius [63], on the pricing of options on bonds, are based on models of the binomial type.

The binomial option-pricing formula was given in the papers by Cox, Ross and Rubinstein [25] and by Rendleman and Bartter [65]. In their paper [25], Cox, Ross and Rubinstein acknowledged their debt to Nobel laureate W.F. Sharpe for suggesting the idea.

Here, we assume that the stock price,

$$S(t) = S(0)e^{X(t)},\ t = 0, 1, 2, \ldots,$$

is a discrete-time stochastic process. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables. Define $X(0)=0$ and, for $t = 1, 2, 3, \ldots, \tau$,

$$X(t) = X_1 + X_2 + \ldots + X_t. \tag{3.3.1}$$

Let Ω denote the set of points on which X_j has positive probability. Assume that Ω is finite and consists of more than one point; let a be its smallest element and b its largest. To avoid arbitrages, we suppose that

$$a < \delta < b.$$

Let us assume that $\{S(t)\}$ is a multiplicative binomial process; that is, Ω consists of exactly two points:

$$\Omega = \{a, b\}.$$

Suppose that

$$\Pr(X_j = b) = p$$

and

$$\Pr(X_j = a) = 1 - p.$$

Let

$$B(x;n,\theta) = \sum_{0 \le j \le x} \binom{n}{j} \theta^j (1-\theta)^{n-j}$$

denote the cumulative binomial distribution function with parameters n and θ. Then the cumulative distribution function of $X(t)$ is

$$F(x,t) = \Pr\left(\sum_{j=1}^{t} X_j \le x\right) = B\left(\frac{x-at}{b-a};t,p\right).$$

Since

$$M(z,t) = \mathrm{E}[e^{zX(t)}] = [(1-p)e^{az} + pe^{bz}]^t, \tag{3.3.2}$$

we have

$$M(z,t;h) = M(z+h,t)/M(h,t) = \{[1-\pi(h)]e^{az} + \pi(h)e^{bz}\}^t, \tag{3.3.3}$$

where

$$\pi(h) = \frac{pe^{bh}}{(1-p)e^{ah} + pe^{bh}}. \tag{3.3.4}$$

Formula (2.9) is the condition that

$$e^{\delta} = [1-\pi(h^*)]e^a + \pi(h^*)]e^b, \tag{3.3.5}$$

from which it follows that

$$\pi(h^*) = \frac{e^{\delta} - e^a}{e^b - e^a}. \tag{3.3.6}$$

According to (2.15), the value of the European call option with exercise price K and exercise date τ is

$$S(0)\left[1 - B\left(\frac{\kappa - a\tau}{b-a};\tau, \pi(h^*+1)\right)\right]$$
$$-Ke^{-\delta\tau}\left[1 - B\left(\frac{\kappa - a\tau}{b-a};\tau, \pi(h^*)\right)\right], \tag{3.3.7}$$

where

$$\pi(h^*+1) = \frac{\pi(h^*)e^b}{[1-\pi(h^*)]e^a + \pi(h^*)e^b} = \pi(h^*)e^{b-\delta}.$$

Note that it is not necessary to know the probability p to price the option, since it is replaced by $\pi(h^*)$.

4. Two New Models

In this section we present two continuous-time models for stock-price movements. Similar to the pure jump model in Section 3.2, we assume here that

$$S(t) = S(0)e^{X(t)} = S(0)e^{Y(t)-ct},$$

where c is a constant. The stochastic process $\{Y(t)\}$ in the first model is a gamma process and in the second model an inverse Gaussian process. These two stochastic processes have been used to model aggregate insurance claims [33]. Recall that, in the pure jump model, all jumps are of the same size. However, this is not the case in these two models.

4.1 Logarithm of Stock Price as a Shifted Gamma Process

We assume that

$$X(t) = Y(t) - ct, \tag{4.1.1}$$

where $\{Y(t)\}$ is a gamma process with parameters α and β, and the positive constant c is a third parameter. Let $G(x;\ \alpha,\ \beta)$ denote the gamma distribution with shape parameter α and scale parameter β,

$$G(x;\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^x y^{\alpha-1}e^{-\beta y}dy, \quad x \geq 0.$$

Then

$$F(x, t) = G(x + ct;\ \alpha t,\ \beta) \tag{4.1.2}$$

and

$$M(z, t) = \left(\frac{\beta}{\beta - z}\right)^{\alpha t} e^{-ctz}, \quad z < \beta. \tag{4.1.3}$$

Hence

$$M(z, t;h) = \left(\frac{\beta - h}{\beta - h - z}\right)^{\alpha t} e^{-ctz}, \quad z < \beta - h, \tag{4.1.4}$$

which shows that the transformed process is of the same type, with β replaced by $\beta - h$. Formula (2.9) means that

$$e^\delta = \left(\frac{\beta - h^*}{\beta - h^* - 1}\right)^\alpha e^{-c}. \tag{4.1.5}$$

Define

$$\beta^* = \beta - h^*.$$

It follows from (4.1.5) that

$$\beta^* = \frac{1}{1 - e^{-(c+\delta)/\alpha}}. \tag{4.1.6}$$

According to (2.15) and (4.1.2), the value of the European call option is

$$S(0)[1 - G(\kappa + c\tau;\ \alpha\tau,\ \beta^* - 1)] - Ke^{-\delta\tau}[1 - G(\kappa + c\tau;\ \alpha\tau,\ \beta^*)]. \tag{4.1.7}$$

Note that the scale parameter β does not appear in (4.1.6) and (4.1.7).

4.2 Logarithm of Stock Price as a Shifted Inverse Gaussian Process

Here, we also assume that

$$X(t) = Y(t) - ct,$$

but $\{Y(t)\}$ is now an inverse Gaussian process with parameters a and b. Let $J(x;\ a,\ b)$ denote the inverse Gaussian distribution function,

$$J(x;a,b) = \Phi\left(\frac{-a}{\sqrt{2x}} + \sqrt{2bx}\right) + e^{2a\sqrt{b}}\Phi\left(\frac{-a}{\sqrt{2x}} - \sqrt{2bx}\right), \quad x > 0, \tag{4.2.1}$$

where Φ is the standard normal distribution function. (Panjer and Willmot's book [62], which was published recently by the Society of Actuaries, has an extensive discussion on the inverse Gaussian distribution.) Then

$$F(x, t) = J(x + ct;\ at,\ b). \tag{4.2.2}$$

Since the moment-generating function of the inverse Gaussian distribution is

$$e^{a(\sqrt{b} - \sqrt{b-z})}, \quad z < b,$$

we have

$$M(z, t) = e^{at(\sqrt{b} - \sqrt{b-z}) - ctz}, \quad z < b. \tag{4.2.3}$$

Consequently,

$$M(z, t;h) = e^{at(\sqrt{b-h} - \sqrt{b-h-z}) - ctz}, \quad z < b-h, \tag{4.2.4}$$

which shows that the transformed process is of the same type, with b replaced by b–h. Formula (2.10) leads to the condition

$$\delta = a(\sqrt{b - h^*} - \sqrt{b - h^* - 1}) - c. \tag{4.2.5}$$

Writing $b^* = b - h^*$, we have

$$\sqrt{b^*} - \sqrt{b^* - 1} = \frac{c + \delta}{a}, \tag{4.2.6}$$

which is an implicit equation for b^*. It follows from (2.15) that the value of the European call option with exercise price K and exercise date τ is

$$S(0)[1 - J(\kappa + c\tau; a\tau,\ b^* - 1)] - Ke^{-\delta\tau}[1 - J(\kappa + c\tau; a\tau,\ b^*)]. \tag{4.2.7}$$

Note that the parameter b does not appear in (4.2.6) and (4.2.7).

5. Numerical Examples

In this section we present numerical values for various European call options for the four continuous-time models. These values illustrate quantitatively some of the verbal statements in Table 17.1 of Hull's book [47, p. 438]. We thank François Dufresne for his computer expertise.

If we assume that $\{X(t)\}$ is a Wiener process, only one parameter (σ^2, the variance per unit time) has to be estimated for applying Formula (3.1.3). This is a main reason for the popularity of the Black-Scholes formula. Suppose that, for a certain stock, σ=0.2 and $S(0)$=100. Consider a European call option with exercise price K=90 six months from now (τ=0.5). With a constant risk-free force of interest δ=0.1, the value of the European call option according to (3.1.3) is

$$100\Phi(1.1693)-90e^{-0.05}\Phi(1.0279) = 15.29.$$

Table 1 gives the European call option values for various exercise prices K and times to maturity τ. For option values corresponding to different values of σ, see Table 14.1 of Ingersoll's book [48, p. 314].

TABLE 1
BLACK-SCHOLES OPTION PRICES
[$S(0)$=100, δ=0.1, σ=0.2]

Exercise Price (K)	Time to Maturity			
	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$	$\tau = 1$
80	21.99	24.03	26.04	27.99
85	17.21	19.52	21.74	23.86
90	12.65	15.29	17.72	19.99
95	8.58	11.50	14.07	16.44
100	5.30	8.28	10.88	13.27
105	2.95	5.69	8.18	10.52
110	1.47	3.74	5.99	8.18
115	0.66	2.35	4.28	6.26
120	0.27	1.42	2.98	4.71

If the logarithm of the stock price does not follow a symmetric distribution, the assumption of a Wiener process is not appropriate. Suppose that the process $\{X(t)\}$ has mean per unit time μ, variance per unit time σ^2, and third central moment per unit time θ^3. Let $\gamma = \theta^3/\sigma^3$ denote the coefficient of skewness of $X(1)$. Then

$$\begin{aligned}\ln\{E[e^{zX(t)}]\} &= \ln[M(z, t)] \\ &= t\ln[M(z, 1)] \\ &= t[\mu z + \sigma^2 z^2/2 + \theta^3 z^3/3! + \ldots] \\ &= t[\mu z + \sigma^2 z^2/2 + \gamma\sigma^3 z^3/3! + \ldots]. \quad (5.1)\end{aligned}$$

In the following we assume, as in the Wiener process example, σ=0.2, $S(0)$=100 and δ=0.1. Furthermore, we assume μ= 0.1 and γ=1.

5.1 Shifted Poisson Process Model

By (5. 1) and (3.2.3), equating the first three central moments in the shifted Poisson process model yields the equations

$$\lambda k - c = \mu,$$
$$\lambda k^2 = \sigma^2$$

and

$$\lambda k^3 = \gamma\sigma^3,$$

from which we obtain

$$k = \gamma\sigma = 0.2,$$
$$\lambda = \gamma^{-2} = 1$$

and

$$c = (\sigma/\gamma) - \mu$$
$$= 0.1. \quad (5.1.1)$$

The resulting value for λ is not needed, since the calculations are done for λ^* in accordance with (3.2.6). Table 2 gives the European call option values computed with Formula (3.2.7) for various exercise prices K and times to maturity τ.

TABLE 2
POISSON PROCESS MODEL OPTION PRICES
[$S(0)$=100, δ=0.1, μ=0.1, σ=0.2, γ=1]

Exercise Price (K)	Time to Maturity			
	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$	$\tau = 1$
80	21.98	23.90	25.78	27.61
85	17.10	19.15	21.14	23.09
90	12.22	14.39	16.50	18.56
95	7.35	9.63	12.91	15.70
100	4.39	7.83	10.63	13.01
105	3.40	6.10	8.35	10.31
110	2.42	4.37	6.06	7.62
115	1.43	2.64	4.32	6.42
120	0.60	1.96	3.63	5.38

5.2 Shifted Gamma Process Model

By (5.1) and (4.1.3), matching the first three central moments in the shifted gamma process model yields the equations

$$(\alpha/\beta) - c = \mu,$$
$$\alpha/\beta^2 = \sigma^2$$

and

$$2\alpha/\beta^3 = \theta^3 = \gamma\sigma^3,$$

from which it follows that

$$\alpha = 4/\gamma^2 = 4,$$
$$\beta = 2/(\sigma\gamma) = 10$$

and

$$c = (2\sigma/\gamma) - \mu$$
$$= 0.3. \qquad (5.2.1)$$

The resulting value for β is not needed, since the calculations are done for β^* in accordance with (4.1.6). Table 3 gives the European call option values computed with Formula (4.1.7) for various exercise prices K and times to maturity τ.

TABLE 3
GAMMA PROCESS MODEL OPTION PRICES
[$S(0)$=100, δ=0.1, μ=0.1, σ=0.2, γ=1]

Exercise Price (K)	Time to Maturity			
	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$	$\tau = 1$
80	21.98	23.90	25.78	27.62
85	17.10	19.15	21.18	23.24
90	12.22	14.50	16.89	19.17
95	7.60	10.59	13.20	15.59
100	4.66	7.61	10.18	12.55
105	2.93	5.45	7.80	10.03
110	1.88	3.91	5.96	7.99
115	1.23	2.82	4.55	6.35
120	0.82	2.05	3.48	5.05

5.3 Shifted Inverse Gaussian Process Model

By (5.1) and (4.2.3), matching the first three central moments in the shifted inverse Gaussian process model yields the equations

$$ab^{-1/2}/2 - c = \mu,$$
$$ab^{-3/2}/4 = \sigma^2$$

and

$$3ab^{-5/2}/8 = \theta^3 = \gamma\sigma^3,$$

from which it follows that

$$a = 3(6\sigma/\gamma^3)^{1/2} = 3(1.2)^{1/2},$$
$$b = 3/(2\sigma\gamma) = 7.5$$

and

$$c = (3\sigma/\gamma) - \mu$$
$$= 0.5.$$

The resulting value for b is not needed, since the calculations are done for b^* in accordance with (4.2.6):

$$\sqrt{b^*} - \sqrt{b^* - 1} = \frac{c + \delta}{a}$$
$$= \frac{0.2}{\sqrt{1.2}},$$

from which we obtain

$$b^* = 8\tfrac{1}{120}.$$

(That b^* is a rational number is atypical.) Table 4 gives the European call option values computed with Formula (4.2.7) for various exercise prices K and times to maturity τ.

TABLE 4
INVERSE GAUSSIAN PROCESS MODEL OPTION PRICES
[$S(0)$=100, δ=0.1, μ=0.1, σ=0.2, γ=1]

Exercise Price (K)	Time to Maturity			
	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$	$\tau = 1$
80	21.98	23.90	25.78	27.64
85	17.10	19.15	21.22	23.27
90	12.22	14.56	16.95	19.21
95	7.70	10.63	13.23	15.61
100	4.67	7.61	10.18	12.54
105	2.88	5.41	7.77	10.01
110	1.83	3.86	5.91	7.95
115	1.20	2.77	4.50	6.31
120	0.80	2.01	3.44	5.01

5.4 Remarks

The four continuous-time models have in common that, in each case, all but one parameter can be read off from the sample path of the process. The parameters that are not inherent in the sample paths are μ, λ, β, and b.

In each case the probability measure is transformed by altering the respective parameter.

It can be shown that the limit for $\gamma \to 0$ of each of the models of Sections 3.2, 4.1 and 4.2 is the classical lognormal model of Section 3.1. In this sense these three models, in particular, Formulas (3.2.7), (4.1.7) and (4.2.7), are generalizations of the classical lognormal model and the Black-Scholes formula.

Stock-price models in the form of

$$S(t) = S(0)e^{ct-Y(t)},$$

as opposed to

$$S(t) = S(0)e^{Y(t)-ct},$$

are equally tractable. However, they are less realistic, since they imply a negative third central moment of the logarithm of stock prices.

Let us write down Equations (5.1.1), (5.2.1) and (5.3.1) in one place:

$$c = (\sigma/\gamma) - \mu, \qquad (5.1.1)$$

$$c = (2\sigma/\gamma) - \mu, \qquad (5.2.1)$$

$$c = (3\sigma/\gamma) - \mu. \qquad (5.3.1)$$

It is interesting to observe how these three formulas for the downward drift coefficient c differ. It turns out that these processes are special cases of a general family, which has been studied by Dufresne, Gerber and Shiu [33] in the context of collective risk theory. For further elaboration, see Sections 5 and 6 of our paper [40].

Eight months after this paper was submitted for publication, Heston's paper [45] appeared. Heston [45] has also introduced the gamma process for modeling stock-price movements. His Formula (10a) can be shown to be the same as our Formula (4.1.7).

6. Options on Several Risky Assets

In this section we generalize the method of Esscher transforms to price derivative securities of multiple risky assets or asset pools. Some of the related papers in the finance literature are [16], [17], [21], [37], [50], [56], [57], [58], [66], [75], and [76]. An obvious application of such results is portfolio insurance, or devising hedging strategies to protect portfolios of assets against losses ([3], [54], [55]). Other applications, such as the valuation of bonds involving one or more foreign currencies and pricing the quality option in Treasury bond futures, can be found in the cited references. In the actuarial literature, there are papers such as [3], [8], [14], [71] and [72]. The papers by Bell and Sherris [8] and by Sherris [72] study pension funds with benefit designs offering resignation, death and/or retirement benefits that are the greater of two alternative benefits. The two alternatives are typically a multiple of final (average) salary and the accumulation of contributions. Such a benefit design provides the plan participants an option on the maximum of two random benefit amounts.

For $t \geq 0$, let $S_1(t), S_2(t), \ldots, S_n(t)$ denote the prices of n non-dividend-paying stocks or assets at time t. Write

$$X_j(t) = \ln[S_j(t)/S_j(0)], \quad j = 1, 2, \ldots, n, \qquad (6.1)$$

and

$$\mathbf{X}(t) = (X_1(t), X_2(t), \ldots, X_n(t))'.$$

Let R^n denote the set of column vectors with n real entries. Let

$$F(\mathbf{x}, t) = \Pr[\mathbf{X}(t) \leq \mathbf{x}], \qquad \mathbf{x} \in R^n,$$

be the cumulative distribution function of the random vector $\mathbf{X}(t)$, and

$$M(\mathbf{z}, t) = \mathrm{E}[e^{\mathbf{z}'\mathbf{X}(t)}], \qquad \mathbf{z} \in R^n,$$

its moment-generating function. In the rest of this paper we assume that $\{\mathbf{X}(t)\}_{t\geq 0}$ is a stochastic process with independent and stationary increments and that

$$M(\mathbf{z}, t) = M(\mathbf{z}, 1)^t, \qquad t \geq 0. \qquad (6.2)$$

For simplicity, we also assume that the random vector $\mathbf{X}(t)$ has density

$$f(\mathbf{x}, t) = \frac{\partial^n}{\partial x_1 \partial x_2 \ldots \partial x_n} F(\mathbf{x}, t), \qquad t > 0.$$

Then the modified density of $\mathbf{X}(t)$ under the Esscher transform with parameter vector $\mathbf{h}$ is

$$f(\mathbf{x}, t; \mathbf{h}) = \frac{e^{\mathbf{h}'\mathbf{x}} f(\mathbf{x}, t)}{M(\mathbf{h}, t)}$$

and the corresponding moment-generating function is

$$M(\mathbf{z}, t; \mathbf{h}) = M(\mathbf{z} + \mathbf{h}, t)/M(\mathbf{h}, t).$$

The Esscher transform (parameter vector $\mathbf{h}$) of the process $\{\mathbf{X}(t)\}$ is again a process with stationary and independent increments, and

$$M(\mathbf{z}, t; \mathbf{h}) = [M(\mathbf{z}, 1; \mathbf{h})]^t. \qquad (6.3)$$

In the general case where the density function $f(\mathbf{x}, t)$ may not exist, we define the Esscher transform in terms of Stieltjes integrals, as we did in (2.1.1).

The parameter vector $\mathbf{h}=\mathbf{h}^*$ is determined so that, for $j = 1, 2, \ldots, n$,

$$\{e^{-\delta t}S_j(t)\}_{t\geq 0}$$

is a martingale with respect to the modified probability measure. In particular,

$$S_j(0) = \mathrm{E}[e^{-\delta t}S_j(t);\ \mathbf{h}^*],\quad t\geq 0,\quad j = 1, 2, \ldots, n. \tag{6.4}$$

(Note that these conditions are independent of t.) The value of a derivative security is calculated as the expectation, with respect to the modified probability measure, of the discounted value of its payoffs.

Define

$$\mathbf{1}_j = (0, 0, \ldots, 0, 1, 0, \ldots, 0)' \in R^n,$$

where the 1 in the column vector $\mathbf{1}_j$ is in the j-th position. Formulas (6.4) become

$$\begin{aligned} e^{\delta t} &= M(\mathbf{1}_j, t;\ \mathbf{h}^*) \\ &= [M(\mathbf{1}_j, 1;\ \mathbf{h}^*]^t,\quad t\geq 0,\quad j = 1, 2, \ldots, n, \end{aligned} \tag{6.5}$$

by (6.3). The following is the main result in this section.

Theorem

Let g be a real-valued measurable function of n variables. Then, for each positive t,

$$\begin{aligned} &\mathrm{E}[e^{-\delta t}S_j(t)g(S_1(t), S_2(t), \ldots, S_n(t));\ \mathbf{h}^*] \\ &= S_j(0)\mathrm{E}[g(S_1(t), S_2(t), \ldots, S_n(t));\ \mathbf{h}^* + \mathbf{1}_j]. \end{aligned} \tag{6.6}$$

Proof

The proof follows the same line of argument that we used in deriving the European call option formula (2.15). The expectation on the left-hand side of (6.6) is obtained by integrating

$$e^{-\delta t}S_j(0)e^{x_j}g(S_1(0)e^{x_1}, \ldots, S_n(0)e^{x_n})f(\mathbf{x}, t;\ \mathbf{h}^*)$$

with respect to $\mathbf{x} = (x_1, \ldots, x_n)'$ over R^n. Since

$$\begin{aligned} e^{x_j}f(\mathbf{x}, t;\ \mathbf{h}^*) &= \frac{e^{(\mathbf{h}^*+\mathbf{1}_j)'\mathbf{x}}f(\mathbf{x}, t)}{\mathrm{M}(\mathbf{h}^*, t)} \\ &= \frac{\mathrm{M}(\mathbf{h}^*+\mathbf{1}_j, t)}{\mathrm{M}(\mathbf{h}^*, t)}f(\mathbf{x}, t;\mathbf{h}^*+\mathbf{1}_j) \\ &= M(\mathbf{1}_j, t;\mathbf{h}^*)f(\mathbf{x}, t;\mathbf{h}^*+\mathbf{1}_j) \\ &= e^{\delta t}f(\mathbf{x}, t;\mathbf{h}^*+\mathbf{1}_j), \end{aligned}$$

the result follows. □

There is another way to derive the theorem. For $\mathbf{k} = (k_1, \ldots, k_n)'$, write

$$\mathbf{S}(t)^{\mathbf{k}} = S_1(t)^{k_1} \ldots S_n(t)^{k_n}.$$

Then

$$\begin{aligned} \mathrm{E}[\mathbf{S}(t)^{\mathbf{k}}g(\mathbf{S}(t));\ \mathbf{h}] &= \frac{\mathrm{E}[\mathbf{S}(t)^{\mathbf{k}}g(\mathbf{S}(t))e^{\mathbf{h}'\mathbf{X}(t)}]}{\mathrm{E}[e^{\mathbf{h}'\mathbf{X}(t)}]} \\ &= \frac{\mathrm{E}[\mathbf{S}(t)^{\mathbf{k}}g(\mathbf{S}(t))\mathbf{S}(t)^{\mathbf{h}}]}{\mathrm{E}[\mathbf{S}(t)^{\mathbf{h}}]} \\ &= \frac{\mathrm{E}[\mathbf{S}(t)^{\mathbf{k}+\mathbf{h}}]}{\mathrm{E}[\mathbf{S}(t)^{\mathbf{h}}]}\,\frac{\mathrm{E}[g(\mathbf{S}(t))\mathbf{S}(t)^{\mathbf{k}+\mathbf{h}}]}{\mathrm{E}[\mathbf{S}(t)^{\mathbf{k}+\mathbf{h}}]} \\ &= \mathrm{E}[\mathbf{S}(t)^{\mathbf{k}};\mathbf{h}]\mathrm{E}[g(\mathbf{S}(t));\mathbf{k}+\mathbf{h}]. \end{aligned}$$

Now the theorem follows from this factorization formula (with $\mathbf{h}=\mathbf{h}^*$ and $\mathbf{k}=\mathbf{1}_j$) and (6.4).

One of the first papers generalizing the Black-Scholes formula to pricing derivative securities of more than one risky asset is by Margrabe [57]. Assuming that the asset prices are geometric Brownian motions, Margrabe [57] derived a closed-form formula for the value of an option to exchange one risky asset for another at the end of a stated period. In other words, he determined the value at time 0 of a contract whose only payoff is at time τ, the value of which is

$$[S_1(\tau) - S_2(\tau)]_+.$$

Corollary 1

The value at time 0 of an option to exchange $S_2(\tau)$ for $S_1(\tau)$ at time τ is

$$S_1(0)\Pr[S_1(\tau) > S_2(\tau);\ \mathbf{h}^* + \mathbf{1}_1] - S_2(0)\Pr[S_1(\tau) > S_2(\tau);\ \mathbf{h}^* + \mathbf{1}_2].$$

Proof

The option value at time 0 is

$$\mathrm{E}(e^{-\delta\tau}[S_1(\tau) - S_2(\tau)]_+;\ \mathbf{h}^*).$$

Let $I(A)$ denote the indicator random variable of an event A. Then

$$\begin{aligned} [S_1(\tau) - S_2(\tau)]_+ &= [S_1(\tau) - S_2(\tau)]I[S_1(\tau) > S_2(\tau)] \\ &= S_1(\tau)I[S_1(\tau) > S_2(\tau)] - S_2(\tau)]I[S_1(\tau) > S_2(\tau))]. \end{aligned}$$

Thus

$$\begin{aligned} &\mathrm{E}(e^{-\delta\tau}[S_1(\tau) - S_2(\tau)]_+;\ \mathbf{h}^*) \\ &= \mathrm{E}(e^{-\delta\tau}[S_1(\tau)I[S_1(\tau) > S_2(\tau)];\ \mathbf{h}^*) \\ &\qquad - \mathrm{E}(e^{-\delta\tau}S_2(\tau)I[S_1(\tau) > S_2(\tau)];\ \mathbf{h}^*) \end{aligned}$$

$= S_1(0)\mathrm{E}(I[S_1(\tau) > S_2(\tau)]; \mathbf{h^*} + \mathbf{1}_1)$
$\quad - S_2(0)\mathrm{E}(I[S_1(\tau) > S_2(\tau)]; \mathbf{h^*} + \mathbf{1}_2),$

by the theorem. Since $\mathrm{E}[I(A)] = \Pr(A)$, the result follows. □

In Section 7 we discuss the geometric Brownian motion assumption and show that Margrabe's formula is an immediate consequence of Corollary 1. Now we give another derivation for the European call option formula (2.15).

Corollary 2

Formula (2.15) holds.

Proof

Consider n=2 with $S_1(t)$=$S(t)$ and $S_2(t)$=$Ke^{\delta(t-\tau)}$. Then

$$\mathbf{X}(t) = (X_1(t), X_2(t))' = (X(t), \delta t)',$$

$$M(\mathbf{z}, t) = M(z_1, t)e^{z_2\delta t},$$

and

$$M(\mathbf{z}, t; \mathbf{h}) = M(z_1, t; h_1)e^{z_2\delta t}. \qquad (6.7)$$

Since the parameter h_2 does not appear in the right-hand side of (6.7), the parameter h_2^* is arbitrary, and $h_1^* = h^*$. Thus the value of the European call option is

$$\begin{aligned}
&\mathrm{E}^*(e^{-\delta\tau}[S(\tau) - K]_+)\\
&= \mathrm{E}(e^{-\delta\tau}[S_1(\tau) - S_2(\tau)]_+; \mathbf{h^*})\\
&= S_1(0)\Pr[S_1(\tau) > S_2(\tau); \mathbf{h^*} + \mathbf{1}_1]\\
&\qquad - S_2(0)\Pr[S_1(\tau) > S_2(\tau); \mathbf{h^*} + \mathbf{1}_2]\\
&= S(0)\Pr[S(\tau) > K; h^* + 1] - e^{-\delta\tau} K\Pr[S(\tau) > K; h^*]\\
&= S(0)\{1 - \Pr[S(\tau) \le K; h^* + 1]\}\\
&\qquad - e^{-\delta\tau} K\{1 - \Pr[S(\tau) \le K; h^*]\},
\end{aligned}$$

which is formula (2.15). □

Margrabe's work [57] was extended by Stulz [75], who also assumed that the asset prices are geometric Brownian motions. By laborious calculation, Stulz derived formulas for valuing options on the maximum and the minimum of two risky assets; that is, he found the value at time 0 of a contract with payoff at time τ

$$(\mathrm{Max}[S_1(\tau), S_2(\tau)] - K)_+$$

and the value at time 0 of a contract with payoff at time τ

$$(\mathrm{Min}[S_1(\tau), S_2(\tau)] - K)_+.$$

These two option formulas of Stulz were generalized to the case of n risky assets by Johnson [50]. Indeed, one may further ask the following questions: How much should one pay at time 0 to obtain (the value of) the second-highest value asset at time τ? The third-highest value asset? The k-th highest value asset? More generally, what is the value of the European call option on the k-th highest value asset at time τ with exercise price K? Note again that, in the papers quoted in this paragraph, the asset prices are assumed to be geometric Brownian motions.

For a fixed time τ, τ>0, let $\mathbb{S}$ denote the set consisting of the random variables $\{S_j(\tau); j = 1, 2, \ldots, n\}$. Let $S_{[k]}$, denote the random variable defined by the k-th highest value of $\mathbb{S}$. Thus, $S_{[1]}$ and $S_{[n]}$ denote the maximum and minimum of $\mathbb{S}$, respectively.

Corollary 3

Assume that $\mathbf{X}(t)$ has a continuous distribution. Then the option to obtain the k-th highest value asset at time τ is worth

$$\sum_{j=1}^{n} S_j(0)\ \Pr(S_j(\tau) \text{ ranks } k\text{-th among } \mathbb{S}; \mathbf{h^*} + \mathbf{1}_j) \quad (6.8)$$

at time 0.

Proof

The option value at time 0 is

$$\mathrm{E}(e^{-\delta\tau}S_{[k]}; \mathbf{h^*}).$$

Since $\mathbf{X}(\tau)$ has a continuous distribution, we have the identity

$$S_{[k]} = \sum_{j=1}^{n} S_j(\tau)\,I[S_j(\tau) \text{ ranks } k\text{-th among } \mathbb{S}].$$

Formula (6.8) now follows from the theorem. □

Corollary 4

Assume that $\mathbf{X}(t)$ has a continuous distribution. Then the European call option on the k-th highest value asset at time τ with exercise price K is worth

$$\sum_{j=1}^{n} S_j(0)\Pr(S_j(\tau) > K \text{ and } S_j(\tau) \text{ ranks } k\text{-th among } \mathbb{S}; \mathbf{h^*} + \mathbf{1}_j)$$
$$- e^{-\delta\tau}K\Pr(S_{[k]} > K; \mathbf{h^*}) \qquad (6.9)$$

at time 0.

The proof for Corollary 4 is essentially a combination of the proofs for Corollary 2 and Corollary 3. Note that, when $K=0$, Corollary 4 becomes Corollary 3.

There are obviously many other applications of the theorem. For example, in a paper recently published in the *Journal of the Institute of Actuaries,* Sherris [71] analyzed the "capital gains tax option," whose payoff at time τ is

$$(S(\tau) - \text{Max}[C(\tau), K])_+,$$

where $S(\tau)$ denotes the price of a risky asset at time t and $C(t)$ denotes the value of an index at time t. Sherris's result follows from the formula

$$\begin{aligned}(S - \text{Max}[C, K])_+ &= S\,I(S > C \text{ and } S > K) \\ &\quad - [C\,I(S > C > K) + K\,I(S > K > C)].\end{aligned}$$

Let us end this section by showing that an American call option on the maximum of n non-dividend-paying stocks is never optimally exercised before its maturity date. Consequently, the value of the American option is given by Corollary 4 (with $k=1$). The proof is by two applications of Jensen's inequality:

$$\begin{aligned}&\text{E}[e^{-\delta t}(\text{Max}\{S_j(t)\} - K)_+; \mathbf{h}^*] \\ &\qquad \geq (\text{E}[e^{-\delta t}\text{Max}\{S_j(t)\}; \mathbf{h}^*] - e^{-\delta t}K)_+ \\ &\qquad \geq (\text{Max}\{\text{E}[e^{-\delta t}S_j(t); \mathbf{h}^*]\} - e^{-\delta t}K)_+ \\ &\qquad = (\text{Max}\{S_j(0)\} - e^{-\delta t}K)_+ \\ &\qquad \geq (\text{Max}\{S_j(0)\} - K)_+\end{aligned}$$

For $t>0$ and $\delta>0$, the last inequality is strict if the option is currently in the money, that is, if

$$\text{Max}\{S_j(0)\} > K.$$

7. Logarithms of Stock Prices as a Multidimensional Wiener Process

In the finance literature, the usual distribution assumption on the prices of the primitive securities is that they are geometric Brownian motions. In other words, $\{\mathbf{X}(t)\}$ is assumed to be an n-dimensional Wiener process. We now show that many results on options and derivative securities in the literature are relatively straightforward consequences of the theorem and its corollaries.

Following the notation in Chapter 12 of Hogg and Craig's textbook [46] for the Course 110 examination, we let $\boldsymbol{\mu}=(\mu_1, \mu_2, \ldots, \mu_n)'$ and $\mathbf{V}=(\sigma_{ij})$ denote the mean vector and the covariance matrix of $\mathbf{X}(1)$, respectively. It is assumed that $\mathbf{V}$ is nonsingular. For $t>0$, the density function of $\mathbf{X}(t)$ is

$$f(\mathbf{x}, t) = \frac{1}{(2\pi)^{n/2}|t\mathbf{V}|^{1/2}} e^{-(\mathbf{x} - t\boldsymbol{\mu})'(2t\mathbf{V})^{-1}(\mathbf{x} - t\boldsymbol{\mu})}, \quad \mathbf{x} \in R^n.$$

It can be shown [46, Section 12.1] that

$$M(\mathbf{z}, t) = \exp[t(\mathbf{z}'\boldsymbol{\mu} + \tfrac{1}{2}\,\mathbf{z}'\mathbf{V}\mathbf{z})], \quad \mathbf{z} \in R^n.$$

Thus, for $\mathbf{h} \in R^n$,

$$\begin{aligned}M(\mathbf{z}, t; \mathbf{h}) &= M(\mathbf{z} + \mathbf{h}, t)/M(\mathbf{h}, t) \\ &= \exp\{t[\mathbf{z}'(\boldsymbol{\mu} + \mathbf{V}\mathbf{h}) + \tfrac{1}{2}\,\mathbf{z}'\mathbf{V}\mathbf{z}]\}, \quad \mathbf{z} \in R^n,\end{aligned}$$

which shows that the Esscher transform (parameter vector $\mathbf{h}$) of the n-dimensional Wiener process is again an n-dimensional Wiener process, with modified mean vector per unit time

$$\boldsymbol{\mu} + \mathbf{V}\mathbf{h}$$

and unchanged covariance matrix per unit time $\mathbf{V}$. Equations (6.5) mean that, for $j = 1, 2, \ldots, n$,

$$\delta = \mathbf{1}_j'(\boldsymbol{\mu} + \mathbf{V}\mathbf{h}^*) + \tfrac{1}{2}\mathbf{1}_j'\mathbf{V}\mathbf{1}_j,$$

from which we obtain

$$\boldsymbol{\mu} + \mathbf{V}\mathbf{h}^* = (\delta - \tfrac{1}{2}\sigma_{11}, \delta - \tfrac{1}{2}\sigma_{22}, \ldots, \delta - \tfrac{1}{2}\sigma_{nn})'. \quad (7.1)$$

Consequently, the mean vector per unit time of the modified process with parameter vector $\mathbf{h}^*+\mathbf{1}_j$ is

$$\begin{aligned}\boldsymbol{\mu} + \mathbf{V}(\mathbf{h}^* + \mathbf{1}_j) = (&\delta + \sigma_{1j} - \tfrac{1}{2}\sigma_{11}, \delta + \sigma_{2j} - \tfrac{1}{2}\sigma_{22}, \ldots, \\ &\delta + \sigma_{nj} - \tfrac{1}{2}\sigma_{nn})'. \quad (7.2)\end{aligned}$$

Note that the right-hand sides of (7.1) and (7.2) do not contain any elements of $\boldsymbol{\mu}$.

To derive Margrabe's [57] main result, we evaluate the expectation

$$\text{E}(e^{-\delta\tau}[S_1(\tau) - S_2(\tau)]_+; \mathbf{h}^*),$$

which, by Corollary 1, is

$$\begin{aligned}&S_1(0)\Pr[S_1(\tau) > S_2(\tau); \mathbf{h}^*+\mathbf{1}_1] - S_2(0)\Pr[S_1(\tau) > S_2(\tau); \mathbf{h}^*+\mathbf{1}_2] \\ &\qquad = S_1(0)\Pr[Y < \xi; \mathbf{h}^*+\mathbf{1}_1] - S_2(0)\Pr[Y < \xi; \mathbf{h}^*+\mathbf{1}_2],\end{aligned}$$

where

$$Y = X_2(\tau) - X_1(\tau) \quad (7.3)$$

and

$$\xi = \ln[S_1(0)/S_2(0)]. \quad (7.4)$$

Now, Y is a normal random variable with respect to any Esscher transform,

$$E(Y; \mathbf{h^*} + \mathbf{1}_1) = [(\delta + \sigma_{21} - ½\sigma_{22}) - (\delta + \sigma_{11} - ½\sigma_{11})]\tau$$
$$= (-½\sigma_{11} + \sigma_{21} - ½\sigma_{22})\tau$$

and

$$E(Y; \mathbf{h^*} + \mathbf{1}_2) = [(\delta + \sigma_{22} - ½\sigma_{22}) - (\delta + \sigma_{12} - ½\sigma_{11})]\tau$$
$$= (-½\sigma_{11} + \sigma_{12} - ½\sigma_{22})\tau.$$

The variance of Y does not depend on the parameter vector; it is

$$(\sigma_{11} - 2\sigma_{12} + \sigma_{22})\tau.$$

With the definition

$$v^2 = \sigma_{11} - 2\sigma_{12} + \sigma_{22} \tag{7.5}$$

(the variance per unit time of the process $\{X_1(t)-X_2(t)\}$), we have

$$E(Y; \mathbf{h^*} + \mathbf{1}_1) = -v^2\tau/2,$$
$$E(Y; \mathbf{h^*} + \mathbf{1}_2) = v^2\tau/2$$

and

$$\mathrm{Var}(Y) = v^2\tau.$$

Thus the value (at time 0) of the option to exchange $S_2(\tau)$ for $S_1(\tau)$ at time τ is

$$S_1(0)\Phi\left(\frac{\xi + v^2\tau/2}{\upsilon\sqrt{\tau}}\right) - S_2(0)\Phi\left(\frac{\xi - v^2\tau/2}{\upsilon\sqrt{\tau}}\right), \tag{7.6}$$

which is the formula on p. 179 of Margrabe's paper [57].

It is somewhat surprising that (7.6) does not depend on the risk-free force of interest, δ. Note also that, if $S_2(t)=Ke^{-\delta(\tau-t)}$, (7.6) becomes the Black-Scholes formula (3.1.3).

Next we calculate the value (at time 0) of the option to receive the greater of $S_1(\tau)$ and $S_2(\tau)$ at time τ. Because of the identity

$$\mathrm{Max}[S_1(\tau), S_2(\tau)] = S_2(\tau) + [S_1(\tau) - S_2(\tau)]_+,$$

the option value is

$$S_2(0) + e^{-\delta\tau}E([S_1(\tau) - S_2(\tau)]_+; \mathbf{h^*}),$$

which, by (7.6), is

$$S_1(0)\Phi\left(\frac{\xi + v^2\tau/2}{\upsilon\sqrt{\tau}}\right) + S_2(0)\left[1 - \Phi\left(\frac{\xi - v^2\tau/2}{\upsilon\sqrt{\tau}}\right)\right]$$
$$= S_1(0)\Phi\left(\frac{\xi + v^2\tau/2}{\upsilon\sqrt{\tau}}\right) + S_2(0)\Phi\left(\frac{-\xi + v^2\tau/2}{\upsilon\sqrt{\tau}}\right)$$
$$= S_1(0)\Phi\left(\frac{\ln[S_1(0)/S_2(0)] + v^2\tau/2}{\upsilon\sqrt{\tau}}\right)$$
$$+ S_2(0)\Phi\left(\frac{\ln[S_2(0)/S_1(0)] + v^2\tau/2}{\upsilon\sqrt{\tau}}\right). \tag{7.7}$$

This result can also be obtained by applying Corollary 3 (with n=2). Again, it is noteworthy that (7.7) does not depend on δ.

Let us also derive the results in Stulz's paper [75] and in Johnson's paper [50]. By Corollary 4 (with n=2),

$$E(e^{-\delta\tau}\{\mathrm{Max}[S_1(\tau), S_2(\tau)] - K\}_+; \mathbf{h^*})$$
$$= E[e^{-\delta\tau}(S_{[1]} - K)_+; \mathbf{h^*}],$$
$$= S_1(0)\Pr[S_1(\tau) > K \text{ and } S_1(\tau) > S_2(\tau); \mathbf{h^*} + \mathbf{1}_1]$$
$$+ S_2(0)\Pr[S_2(\tau) > K \text{ and } S_2(\tau) > S_1(\tau); \mathbf{h^*} + \mathbf{1}_2]$$
$$-Ke^{-\delta\tau}\Pr[S_1(\tau) > K \text{ or } S_2(\tau) > K; \mathbf{h^*}]. \tag{7.8}$$

First, we evaluate the last probability term,

$$\Pr[S_1(\tau) > K \text{ or } S_2(\tau) > K; \mathbf{h^*}] =$$
$$1 - \Pr[S_1(\tau) \le K \text{ and } S_2(\tau) \le K; \mathbf{h^*}].$$

Similar to (2.12), define

$$\kappa_1 = \ln[K/S_1(0)] \tag{7.9}$$

and

$$\kappa_2 = \ln[K/S_2(0)]. \tag{7.10}$$

Then

$$\Pr[S_1(\tau) \le K \text{ and } S_2(\tau) \le K; \mathbf{h^*}]$$
$$= \Pr[X_1(\tau) \le \kappa_1 \text{ and } X_2(\tau) \le \kappa_2; \mathbf{h^*}]. \tag{7.11}$$

By (7.1)

$$E[X_1(\tau); \mathbf{h^*}] = (\delta - ½\sigma_{11})\tau$$

and

$$E[X_2(\tau); \mathbf{h^*}] = (\delta - ½\sigma_{22})\tau.$$

Let $\Phi_2(a, b; \rho)$ denote the bivariate cumulative standard normal distribution with upper limits of integration a and b and coefficient of correlation ρ. (For various properties of Φ_2, see Section 26.3 in the book by Abramowitz and Stegun [1]). Write

$$\sigma_i = \sqrt{\sigma_{ii}} \tag{7.12}$$

and

$$\rho_{ij} = \sigma_{ij}/(\sigma_i\sigma_j). \tag{7.13}$$

Then the probability defined by (7. 11) is

$$\Phi_2\left(\frac{\kappa_1-(\delta-\sigma_1^2/2)\tau}{\sigma_1\sqrt{\tau}}, \frac{\kappa_2-(\delta-\sigma_2^2/2)\tau}{\sigma_2\sqrt{\tau}}, \rho_{12}\right). \quad (7.14)$$

To obtain approximate numerical values for (7.14), we can use formulas (26.3.11) and (26.3.20) together with Figures 26.2, 26.3 and 26.4 in the book by Abramowitz and Stegun [1]. An algorithm to calculate the bivariate cumulative standard normal distribution to four-decimal-place accuracy can be found in the paper by Drezner [31]. The Drezner algorithm (with a typo corrected) can be found in Appendix 10B in Hull's book [47, p. 245] and in Appendix 13.1 in the book by Stoll and Whaley [74, p. 338].

Next, we evaluate the first probability term in (7.8),

$$\begin{aligned}&\Pr[S_1(\tau)>K \text{ and } S_1(\tau)>S_2(\tau); \mathbf{h^*}+\mathbf{1}_1]\\ &= \Pr[-X_1(\tau)<-\kappa_1 \text{ and } X_2(\tau)-X_1(\tau)<\xi; \mathbf{h^*}+\mathbf{1}_1], \quad (7.15)\end{aligned}$$

where the constants κ_1 and ξ are defined by (7.9) and (7.4), respectively. Now,

$$\begin{aligned}E[-X_1(\tau); \mathbf{h^*}+\mathbf{1}_1] &= -(\delta+\sigma_{11}-½\sigma_{11})\tau\\ &= -(\delta+½\sigma_{11})\tau,\\ E[X_2(\tau)-X_1(\tau); \mathbf{h^*}+\mathbf{1}_1] &= (-½\sigma_{11}+\sigma_{12}-½\sigma_{22})\tau\\ &= -\nu^2\tau/2,\\ \mathrm{Var}[X_2(\tau)-X_1(\tau); \mathbf{h}] &= (\sigma_{11}-2\sigma_{12}+\sigma_{22})\tau\\ &= \nu^2\tau\end{aligned}$$

and

$$\begin{aligned}\mathrm{Cov}[-X_1(\tau), X_2(\tau)-X_1(\tau); \mathbf{h}) &= \mathrm{Cov}(X_1(\tau), X_1(\tau)-X_2(\tau); \mathbf{h})\\ &= (\sigma_{11}-\sigma_{12})\tau\\ &= [\sigma_1(\sigma_1-\rho_{12}\sigma_2)]\tau,\end{aligned}$$

where ν is defined by (7.5). Thus (7.15) can be expressed as

$$\Phi_2\left(\frac{-\kappa_1+(\delta+½\sigma_1^2)\tau}{\sigma_1\sqrt{\tau}}, \frac{\xi+(\nu^2\tau/2)}{\nu\sqrt{\tau}}; \frac{\sigma_1-\rho_{12}\sigma_2}{\nu}\right). \quad (7.16)$$

By symmetry, we can write down the expression, in terms of the distribution Φ_2, for the second probability term in (7.8). Hence the value at time 0 of the European call option on the maximum of two risky assets with exercise price K and exercise date τ is

$$\begin{aligned}&S_1(0)\Phi_2\left(\frac{-\kappa_1+(\delta+½\sigma_1^2)\tau}{\sigma_1\sqrt{\tau}},\right.\\ &\qquad\left.\frac{\ln[S_1(0)/S_2(0)]+\nu^2\tau/2}{\nu\sqrt{\tau}}; \frac{\sigma_1-\rho_{12}\sigma_2}{\nu}\right)\\ &+S_2(0)\Phi_2\left(\frac{-\kappa_2+(\delta+½\sigma_2^2)\tau}{\sigma_2\sqrt{\tau}},\right.\\ &\qquad\left.\frac{\ln[S_2(0)/S_1(0)]+\nu^2\tau/2}{\nu\sqrt{\tau}}; \frac{\sigma_2-\rho_{12}\sigma_1}{\nu}\right)\\ &-Ke^{-\delta\tau}\\ &\quad\left[1-\Phi_2\left(\frac{\kappa_1-(\delta-½\sigma_1^2)\tau}{\sigma_1\sqrt{\tau}}, \frac{\kappa_2-(\delta-½\sigma_2^2)\tau}{\sigma_2\sqrt{\tau}}; \rho_{12}\right)\right],\end{aligned} \quad (7.17)$$

which is the same as equation (6) in Johnson's paper [50, p. 281].

Let us also consider the expectation

$$E[e^{-\delta\tau}[\mathrm{Min}[S_1(\tau), S_2(\tau)]-K)_+; \mathbf{h^*}],$$

which by Corollary 4 (with n=2) is

$$\begin{aligned}&S_1(0)\Pr[K<S_1(\tau)<S_2(\tau); \mathbf{h^*}+\mathbf{1}_1]\\ &+S_2(0)\Pr[K<S_2(\tau)<S_1(\tau); \mathbf{h^*}+\mathbf{1}_2]\\ &-Ke^{-\delta\tau}\Pr[K<S_1(\tau) \text{ and } K<S_2(\tau); \mathbf{h^*}]\\ =\ &S_1(0)\Pr[-X_1(\tau)<-\kappa_1 \text{ and } X_1(\tau)-X_2(\tau)\\ &\qquad<\ln[S_2(0)/S_1(0)]; \mathbf{h^*}+\mathbf{1}_1]\\ &+S_2(0)\Pr[-X_2(\tau)<-\kappa_2 \text{ and } X_2(\tau)-X_1(\tau)\\ &\qquad<\ln[S_1(0)/S_2(0); \mathbf{h^*}+\mathbf{1}_2]\\ &-Ke^{-\delta\tau}\Pr[-X_1(\tau)<-\kappa_1 \text{ and } -X_2(\tau)<-\kappa_2; \mathbf{h^*}].\end{aligned}$$

By a calculation similar to the above, we obtain the value at time 0 of the European call option on the minimum of two risky assets with exercise price K and exercise date τ:

$$\begin{aligned}&S_1(0)\Phi_2\left(\frac{-\kappa_1+(\delta+½\sigma_1^2)\tau}{\sigma_1\sqrt{\tau}},\right.\\ &\qquad\left.\frac{\ln[S_2(0)/S_1(0)]-\nu^2\tau/2}{\nu\sqrt{\tau}}; \frac{\rho_{12}\sigma_2-\sigma_1}{\nu}\right)\end{aligned}$$

$$+ S_2(0)\Phi_2\left(\frac{-\kappa_2 + (\delta + {}^1\!/_2\,\sigma_2^2)\tau}{\sigma_2\sqrt{\tau}}, \frac{\ln[S_1(0)/S_2(0)] - \nu^2\tau/2}{\nu\sqrt{\tau}}; \frac{\rho_{12}\sigma_1 - \sigma_2}{\nu}\right)$$

$$-Ke^{-\delta\tau}$$

$$\Phi_2\left(\frac{-\kappa_1 + (\delta - {}^1\!/_2\sigma_1^2)\tau}{\sigma_1\sqrt{\tau}}, \frac{-\kappa_2 + (\delta - {}^1\!/_2\sigma_2^2)\tau}{\sigma_2\sqrt{\tau}}; \rho_{12}\right). \tag{7.18}$$

This is the same as formula (11) in Stulz's paper [75, p. 165] (both $\sigma^2\sqrt{\tau}$ should be $\sigma_2\tau$) and formula (8) in Johnson's paper [50, p. 281].

Because of the identity

$$(\text{Max}[S_1(\tau), S_2(\tau)] - K)_+ + (\text{Min}[S_1(\tau), S_2(\tau)] - K)_+ = [S_1(\tau) - K]_+ + [S_2(\tau) - K]_+,$$

the sum of (7.17) and (7.18) should be

$$S_1(0)\Phi\left(\frac{-\kappa_1 + (\delta + {}^1\!/_2\,\sigma_1^2)\tau}{\sigma_1\sqrt{\tau}}\right)$$

$$-Ke^{-\delta\tau}\Phi\left(\frac{-\kappa_1 + (\delta - {}^1\!/_2\,\sigma_1^2)\tau}{\sigma_1\sqrt{\tau}}\right)$$

$$+S_2(0)\Phi\left(\frac{-\kappa_2 + (\delta + {}^1\!/_2\,\sigma_2^2)\tau}{\sigma_2\sqrt{\tau}}\right)$$

$$-Ke^{-\delta\tau}\Phi\left(\frac{-\kappa_2 + (\delta - {}^1\!/_2\,\sigma_2^2)\tau}{\sigma_2\sqrt{\tau}}\right).$$

We can verify this algebraically by applying the formulas

$$\Phi_2(a, b; \rho) + \Phi_2(a, -b; -\rho) = \Phi(a)$$

and

$$\Phi_2(a, b; \rho) - \Phi_2(-a, -b; -\rho) = \Phi(a) - \Phi(-b) = \Phi(a) + \Phi(b) - 1.$$

Johnson [50] also gave formulas for European options on the maximum and the minimum of n risky assets with exercise price K. These formulas are of course special cases of Corollary 4. Let us end this section by showing how to evaluate the first probability term in (6.9) (with $k=1$),

$$\Pr[S_1(\tau) > K \text{ and } S_1(\tau) \text{ ranks first among } \mathbb{S}; \mathbf{h^*} + \mathbf{1}_1]$$
$$= \Pr[S_1(\tau) > K, S_1(\tau) > S_2(\tau), \ldots, S_1(\tau) > S_n(\tau); \mathbf{h^*} + \mathbf{1}_1]. \tag{7.19}$$

Write

$$\mathbf{W} = (0, X_2(\tau), X_3(\tau), \ldots, X_n(\tau))',$$
$$\mathbf{1} = (1, 1, 1, \ldots, 1)'$$

and

$$\mathbf{s} = (\ln[S_1(0)/K], \ln[S_1(0)/S_2(0)], \ln[S_1(0)/S_3(0)], \ldots, \ln[S_1(0)/S_n(0)])'.$$

Let $N_n(\mathbf{x}; \boldsymbol{\mu}, \mathbf{V})$ denote the n-dimensional normal distribution function with mean vector $\boldsymbol{\mu}$ and covariance matrix $\mathbf{V}$. Then the probability expressed by (7.19) is the same as

$$\Pr[\mathbf{W} - X_1(\tau)\mathbf{1} < \mathbf{s}; \mathbf{h^*} + \mathbf{1}_1]$$
$$= N_n(\mathbf{s}; \text{E}[\mathbf{W} - X_1(\tau)\mathbf{1}; \mathbf{h^*} + \mathbf{1}_1], \tau\mathbf{Y}), \tag{7.20}$$

where $\tau\mathbf{Y} = (\tau y_{ij})$ denotes the covariance matrix of the random vector $\mathbf{W} - X_1(\tau)\mathbf{1}$. By (7.2),

$$\text{E}[\mathbf{W} - X_1(\tau)\mathbf{1}; \mathbf{h^*} + \mathbf{1}_1]$$
$$= \text{E}(\mathbf{W}; \mathbf{h^*} + \mathbf{1}_1] - \text{E}[X_1(\tau)\mathbf{1}; \mathbf{h^*} + \mathbf{1}_1]$$
$$= (0, \delta + \sigma_{21} - \tfrac{1}{2}\sigma_{22}, \ldots, \delta + \sigma_{n1} - \tfrac{1}{2}\sigma_{nn})' - (\delta + \sigma_{11} - \tfrac{1}{2}\sigma_{11})\mathbf{1}$$
$$= (-\delta, \sigma_{21} - \tfrac{1}{2}\sigma_{22}, \ldots, \sigma_{n1} - \tfrac{1}{2}\sigma_{nn})' - \tfrac{1}{2}\sigma_{11}\mathbf{1}.$$

To find the matrix $\mathbf{Y}$, which is independent of $\mathbf{h^*}$, observe that

$$[\mathbf{W} - X_1(\tau)\mathbf{1}][\mathbf{W} - X_1(\tau)\mathbf{1}]'$$
$$= \mathbf{WW}' - X_1(\tau)[\mathbf{W1}' + \mathbf{1W}'] + [X_1(\tau)]^2\mathbf{11}'.$$

Thus

$$y_{11} = \sigma_{11};$$

for $i>1$,

$$y_{i1} = y_{1i} = -\sigma_{i1} + \sigma_{11};$$

and, for $i\neq 1$ and $j\neq 1$,

$$y_{ij} = y_{ji} = \sigma_{ij} - (\sigma_{i1} + \sigma_{j1}) + \sigma_{11}.$$

As a test of understanding of the method presented in this paper, the interested reader is encouraged to work out all the probability terms for evaluating the options with payoffs $(S_{[1]}-K)_+$ and $(S_{[n]}-K)_+$. Answers can be checked against the published formulas in Johnson's paper [50].

8. Conclusion

The option-pricing theory of Black and Scholes [11] is perhaps the most important advance in the theory of financial economics in the past two decades. Their theory has been extended in many directions, usually by applying sophisticated mathematical tools such as stochastic calculus and partial differential equations. A fundamental insight in the development of the theory was provided by Cox and Ross [24] when they pointed out the concept of risk-neutral valuation. This idea was further elaborated on by Harrison and Kreps [42] and by Harrison and Pliska [43] under the terminology of equivalent martingale measure.

Under the assumption of a constant risk-free interest rate, this paper shows how such equivalent martingale measures can be determined for a large class of stochastic models of asset price movements. Any Esscher transform of the stochastic process $\{X(t)\}$ provides an equivalent probability measure for the process; the parameter vector $\mathbf{h}^*$ is chosen such that the equivalent probability measure is also a martingale measure for the discounted value of each primitive security. The price of a derivative security is calculated as the expectation, with respect to the equivalent martingale measure, of the discounted payoffs. In other words, after an appropriate change of probability measure, the price of each security is simply an actuarial present value.

We hope that this paper helps demystify the procedure for valuing European options and other derivative securities. If actuaries can project the cash flow of a derivative security, they can value it by using what they learned as actuarial students—by discounting and averaging. The one difference is that averaging is done with respect to the risk-neutral Esscher measure, which this paper shows how to determine.

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