III

Why Did ALM Become Important?

The central premise of ALM from its early days in the Redington's (1952) paper has been the integrated treatment of assets and liabilities. Yet, only in the last two and a half decades has this issue acquired greater significance in the management of financial intermediaries. Early insurance companies, especially Friendly Societies, were troubled by unpredictability of their benefit disbursements. As the principles of actuarial sciences developed, this unpredictability gave in to a better understanding of cash flows related to product pricing. Thus, the C-2 risk, which was initially the main concern for the management, has gradually moved to the background. One can, of course, argue that the features of the insurance products that facilitate disintermediation are a part of the C-2 risk but, undoubtedly, these features are very closely related to the C-3 risk.

The Golden Age of U.S. insurers, the 1950s and 1960s, was characterized by nearly complete knowledge of claim-related cash flows because of actuarial knowledge, and by predictability of other cash flows (i.e., lapses, surrenders, new business, investment returns) because of an economic environment providing stability to those factors (Black and Skipper 1994). One could say that Golden Age was the "quiet before the storm." Subsequent developments have been noted by Sametz (1987), including:

- Unprecedented levels of inflation, and unpredictability of the inflation rate.
- Unprecedented levels of volatility of financial markets, especially interest rates.
- Unprecedented deregulation, consumerism, and competition.

All of these led to greater efficiency in consumer behavior, disintermediation, and change in the insurance industry position, versus other financial institutions. These, in turn, resulted in the insurance industry experiencing the common denominator in those three factors—"the unprecedented"—which first and foremost meant unpredictability of cash flows, or even a complete makeover of the nature of those cash flows.

For example, annuities, which historically have been a relatively unimportant part of the life insurance industry used primarily to provide an income stream after retirement, acquire new significance as savings vehicles through the use of single and flexible premium-deferred annuities, and the recent extraordinary growth of variable annuities (Tullis and Polkinghorn 1992). In 1982, total annuity reserves of U.S. life companies exceeded life insurance reserves for the first time, and by the 1990s they reached twice the level of life reserves. The popularity of annuities and other investment-related products in the United States has been aided by the provisions of the Tax Reform Act of 1986 (Babbel and Stricker 1987). According to Asay, Bouyoucos, and Marciano (1993), three major milestones in the recent history of the life insurance industry occurred:

- In the early 1980s, the short-term interest rates were at record highs, causing massive disintermediation as policyholders fled to higher yield.
- In the mid 1980s, a record decline in the level of nominal interest rates resulted in refinancing and prepayments of a large portion of insurers' portfolios.
- At the end of the 1980s, insurers pursuing higher yields often were caught taking too much credit risk in their investment portfolios.

The market nature of insurance products has changed as well. Ostaszewski (1998) points out that the historical *Paul v. Virginia* Supreme Court decision of 1867, which led to the present system of state regulation of insurance, appeared to have been based on the perception of insurance as a private contract between two local parties (therefore, no interstate commerce in insurance, and . . . no federal regulation).

Even though there are no traded markets in insurance products, there has been a decisive move towards competitive pricing of insurance, with mortality protection becoming nearly a commodity, and catastrophe futures markets under development.

These changes have resulted in a relative decline of importance of the C-2 risk, with the sole exceptions possibly being the catastrophe risks insured by property-casualty companies, and economically complex claim processes faced by the health insurers (recall that the product features facilitating disintermediation are very closely tied to the C-3 risk). At the same time, the relative significance of the C-1 and C-3 risks has increased. We believe that these two risks are indeed becoming integrated, as pointed out in the Chapter 2 discussion of credit risk on bonds derived from option values. This is reinforced by the RBC formula for life insurers discussed in that chapter as well.

The immunization approach should help in dealing with the new situation of insurance firms, but this is not necessarily the case. The very nature of financial intermediation may indeed pose an obstacle to successful implementation of immunizing strategies. The classical immunization of Redington (1952; see also the discussion in Kellison 1991), presented in Chapter 2, began with the approximation:

$$f(x + \Delta x) \approx f(x) + f'(x)\Delta x.$$

This has led to the strategy:

$$S'(i) = 0, \text{ i.e., } A'(i) = L'(i) \tag{3.1}$$

to protect the surplus value, or

$$\frac{d \ln\left(\frac{A(i)}{L(i)}\right)}{di} = 0, \text{ i.e., } \frac{d \ln(A(i))}{di} = \frac{d \ln(L(i))}{di} \tag{3.2}$$

to protect surplus ratio.

It is well known from calculus that the condition (3.1) gives the *critical point* of the surplus as a function of the interest rate. A continuous function attains a local minimum at a critical point if its second derivative is positive, and a local maximum if its second derivative is negative. Therefore, if condition (1) is satisfied and additionally

$$S''(i) > 0, \tag{3.3}$$

then the surplus function will have a local minimum at the current level of *i,* and any change in *i* will benefit the intermediary. This is equivalent to

$$\frac{d^2A}{di^2} > \frac{d^2L}{di^2}. \tag{3.4}$$

We will term this second derivative with respect to *i,* the *dollar convexity* of a financial instrument. Similarly, if Equation (3.2) is satisfied and additionally

$$\frac{d^2 \ln\left(\frac{A(i)}{L(i)}\right)}{di^2} > 0, \tag{3.5}$$

then the ratio of assets to liabilities will have a local minimum at the current level of *i,* similarly benefiting the financial intermediary in case of any changes in interest rates. This, in turn, is equivalent to

$$\frac{d^2 \ln(A(i))}{di^2} > \frac{d^2 \ln(L(i))}{di^2}. \tag{3.6}$$

This second derivative of the logarithm of the price of a financial instrument with respect to *i* is convenient to use in the problem as defined here. We will call it M^2 (*M-squared*, or *measure of dispersion*) or *logarithmic convexity*. The second derivative of the logarithm of the price with respect to the force of interest (under flat yield curve assumption), will be called M_M^2 (*Macaulay measure of dispersion,* or *Macaulary logarithmic convexity*). The standard measure of convexity (Boyle 1992) of $P(i)$ is:

$$C = \frac{1}{P}\frac{d^2P}{di^2}. \tag{3.7}$$

The immunizing condition (3.6) can also be written in terms of convexity as defined in Equation (3.7), as Equation (3.6) is stated with the assumption that durations of assets and liabilities are set equal to each other. However, convexity, as defined by Equation (3.7), increases with coupon for bonds of the same duration, while logarithmic convexity does not. Logarithmic convexity generally increases as dispersion of a security's cash flows increases (a notion we will return to later).

This naturally leads to the full classical immunization model, which can be summarized as follows.

To protect the absolute surplus level, set:

(i) $A'(i) = L'(i)$, that is, dollar duration of assets equal to the dollar duration of liabilities, and
(ii) $d^2A/di^2 > d^2L/di^2$, that is, choose assets with more dollar convexity than the liabilities.

To protect the surplus ratio level, set:

(i) $d \ln(A(i))/di = d \ln(L(i))/di$, that is, duration of assets equal to the duration of liabilities, and

(ii) $d^2 \ln(A(i))/di^2 > d^2 \ln(L(i))/di^2$, that is, choose assets with more logarithmic convexity than the liabilities.

These ideas seem to imply that the life of an insurer, or a banker, can be made very simple indeed. Why, then, such an unprecedented level of anxiety about the interest rate risk, why the insolvencies in the early 1990s, and why all the renewed interest in the interest rate risk? One might be immediately tempted to ask: Why is this strategy of *minimization* of surplus pursued at all? Isn't it unnatural to seek the minimum point of one's wealth? As it turns out, this is not so—immunization in practice means pursuit of the point of maximum wealth, and, to a degree, maximum interest rate risk. Thus, the claim of a simple life through immunization has been greatly exaggerated. Additionally, immunization, as specified above, rests on conditions under which riskless arbitrage opportunities exist, making the approach quite unrealistic.

Before proceeding with a more formal examination, let us try to understand better what convexity means. It is defined as the opposite of the rate of change of duration with respect to the interest rate i (or the second derivative of the logarithm of the price). As noted in Chapter 2, if the security analyzed has a price $P(i)$, and it produces certain cash flows CF_t at times t in the future, we have:

$$\frac{dP}{di} = \sum_{t \geq 0} CF_t(-t)(1 + i)^{-t-1} \tag{3.8}$$

and

$$\frac{d(\ln P)}{di} = \frac{\frac{dP}{di}}{P} = -\frac{\sum_{t\geq 0} tCF_t(1 + i)^{-t-1}}{\sum_{t\geq 0} CF_t(1 + i)^{-t}}. \tag{3.9}$$

The dollar convexity of this security equals:

$$\frac{d^2P}{di^2} = \sum_{t\geq 0} t(t + 1)CF_t(1 + i)^{-t-2}. \tag{3.10}$$

We can immediately see that for a security with deterministic cash flows its dollar convexity must be positive. What is less obvious from formula (3.10) is the relationship between dollar convexity and the actual cash flows. When is convexity large, and when is it small? This is easiest to see first for the $P = P(\delta)$ functional relationship of the price of a financial instrument to the force of interest (instantaneous forward rate, assuming flat yield curve). Then we will analyze it for the price $P = P(i)$ as the function of the interest rate. If the cash flows are deterministic CF_t at times t in the future, then

$$P(\delta) = \sum_{t\geq 0} CF_t e^{-\delta t}. \tag{3.11}$$

If

$$D_M = -\frac{\frac{dP}{d\delta}}{P} = \frac{\sum_{t\geq 0} tCF_t e^{-\delta t}}{\sum_{t\geq 0} CF_t e^{-\delta t}} \tag{3.12}$$

is the duration with respect to force of interest (Macaulay duration), and

$$C_M = \frac{1}{P}\frac{d^2P}{d\delta^2} \tag{3.13}$$

is the convexity measure with respect to force of interest (termed here *Macaulay convexity*), and we define:

$$w_t = \frac{CF_t e^{-\delta t}}{\sum_{t\geq 0} CF_t e^{-\delta t}} \tag{3.14}$$

then

$$\begin{aligned}\frac{d^2(\ln P)}{d\delta^2} &= M_M^2 = C_M - D_M^2 \\ &= \sum_{t\geq 0} t^2 w_t - D_M^2 = \sum_{t\geq 0} (t - D_M)^2 w_t,\end{aligned} \tag{3.15}$$

providing an analogue of the concept of variance of probability of distribution (as the weights w_t sum up to one), and illustrating that the logarithmic convexity increases with dispersion of cash flows. In fact, Equation (3.15) demonstrates that, in a manner similar to probability distributions, it is the dispersion around the duration value that determines the size of convexity, or, more precisely, it determines the sensitivity of the duration measure to changes in interest rates. The "probability weights" are provided by the relative weights of present values of cash flows in relation to the overall present value of the security. Note also that Macaulay convexity equals

$$C_M = M_M^2 + D_M^2$$

and it is the second moment of the said probability distribution, thus increasing with both dispersion and the square of Macaulay duration.

Let us now turn our attention to the convexity measure with respect to the interest rate. We can calculate directly, that

$$\frac{d(\ln P)}{di} = \frac{\sum_{t\geq 0} (-t)CF_t(1+i)^{-t-1}}{\sum_{t\geq 0} CF_t(1+i)^{-t}},$$

and

$$\frac{d^2(\ln P)}{di^2}$$

$$= \frac{\left(\sum_{t\geq 0} t(t+1)CF_t(1+i)^{-t-2}\right)\left(\sum_{t\geq 0} CF_t(1+i)^{-t}\right) - \left(\sum_{t\geq 0} tCF_t(1+i)^{-t-1}\right)\left(\sum_{t\geq 0} tCF_t(1+i)^{-t-1}\right)}{\left(\sum_{t\geq 0} CF_t(1+i)^{-t}\right)^2}$$

$$= \frac{1}{(1+i)^2}\frac{\sum_{t\geq 0} t(t+1)CF_t(1+i)^{-t}}{\sum_{t\geq 0} CF_t(1+i)^{-t}}$$

$$- \frac{1}{(1+i)^2}\left(\frac{\sum_{t\geq 0} tCF_t(1+i)^{-t}}{\sum_{t\geq 0} CF_t(1+i)^{-t}}\right)^2$$

$$= \frac{1}{(1+i)^2} C_M + \frac{1}{(1+i)^2} D_M - \frac{1}{(1+i)^2} D_M^2$$

On the other hand, from (3.10) we see that convexity is:

$$C = \frac{\frac{d^2P}{di^2}}{P}$$

$$= \frac{1}{(1+i)^2}\frac{\sum_{t\geq 0} t(t+1)CF_t(1+i)^{-t}}{\sum_{t\geq 0} CF_t(1+i)^{-t}} \quad (3.16)$$

$$= \frac{1}{(1+i)^2} C_M + \frac{1}{(1+i)^2} D_M.$$

Therefore

$$\frac{d^2(\ln P)}{di^2} = M^2 = C - D^2, \quad (3.17)$$

providing a perfectly analogous, to that for the force of interest, interpretation of M^2, as the measure of dispersion, and convexity increasing with both dispersion of cash flows and their duration. For practical purposes, we therefore need to remember that more dispersed cash flows tend to be more convex, and longer duration cash flows tend to be more convex. In particular, of the following two portfolios

(1) Bullet (a single cash flow at duration D), and

(2) Barbell (two cash flows, one before duration D, one after it, with the duration of the combined portfolio equal to D);

the barbell portfolio offers more convexity. Thus, the traditional convexity measure increases with the increase in the coupon of a bond with the same maturity, but this is greatly influenced by the fact that C is equal to the sum of logarithmic convexity and the square of duration, with the square of duration falling with an increase in the coupon (if maturity remains unchanged). Overall, of two patterns of payments with the same present value and the same duration, one can expect greater convexity from the pattern with a greater dispersion of cash flows. This simple observation bears some significance to the insurance business, especially life insurance.

Before we proceed to the analysis specific to the life insurance industry, let us summarize the key relationships between various measures of interest rate sensitivity:

$$M_M^2 = \frac{d^2 \ln P}{d\delta^2} = C_M - D_M^2,$$

$$D = \frac{1}{1+i} D_M, \; C = \frac{1}{(1+i)^2}(C_M + D_M),$$

$$M^2 = \frac{d^2 \ln P}{di^2} = \frac{1}{(1+i)^2}(C_M + D_M - D_M^2)$$

$$= C - D^2.$$

It is also quite interesting to observe that there is one more interpretation of convexity, which illustrates the effect of dispersion of cash flows on convexity. Let us write $A_t = CF_t(1+i)^{-t}$. Note that:

$$\frac{d^2(\ln P)}{di^2} = \frac{1}{(1+i)^2}$$

$$\frac{\left(\sum_{t\geq 0} t^2A_t\right)\left(\sum_{t\geq 0} A_t\right) - \left(\sum_{t\geq 0} tA_t\right)\left(\sum_{t\geq 0} tA_t\right) + \left(\sum_{t\geq 0} tA_t\right)\left(\sum_{t\geq 0} A_t\right)}{\left(\sum_{t\geq 0} A_t\right)^2} = \frac{1}{(1+i)^2}$$

$$\frac{\left(\sum_{t\geq 0} A_s\right)\left(\sum_{t\geq 0} t^2A_t\right) - \left(\sum_{t\geq 0} sA_s\right)\left(\sum_{t\geq 0} tA_t\right)}{\left(\sum_{t\geq 0} A_t\right)^2}$$

$$+ \frac{1}{(1+i)^2} D_M = \frac{1}{(1+i)^2}$$

$$\frac{\sum_t \sum_{s \neq t} t^2 A_s A_t - \sum_t \sum_{s \neq t} st A_s A_t}{\left(\sum_{t \geq 0} A_t\right)^2}$$

$$+ \frac{1}{(1+i)^2} D_M = \frac{1}{(1+i)^2}$$

$$\frac{\sum_t \sum_{s<t} (t-s)^2 A_s A_t}{\left(\sum_{t \geq 0} A_t\right)^2}$$

$$+ \frac{1}{(1+i)^2} D_M.$$

This formula, because of the expression $(t - s)^2$, again reinforces the fact that the dispersion of the cash flows is, in addition to duration, a key consideration for convexity. Note that the above also shows that:

$$M_M^2 = \frac{d^2 \ln P}{d\delta^2} = \frac{\sum_t \sum_{s<t} (t-s)^2 CF_s (1+i)^{-s} CF_t (1+i)^{-t}}{\left(\sum_{t \geq 0} CF_t (1+i)^{-t}\right)^2},$$

providing another illustration of the dispersion concept (see also Ostaszewski and Zwiesler, 2002). The above formula is effectively an interpretation of (3.15), because, for two independent identically distributed random variables S and T.

As pointed out at the beginning of this chapter, early insurance companies were troubled by the unpredictability of their claim flows, but as the industry matured, liabilities cash flows have matured, stabilized, and, as a consequence, become more dispersed. In view of that evolutionary pattern, the insurance industry's increased concern about the interest rate risk is, to a great degree, caused by the maturity of the industry and greater dispersion of its combined portfolio liabilities cash flows. The insurance industry can be viewed as a net "provider of convexity" in the national economy.

Griffin (1995) points out that, in view of this situation, it may be wise for life insurers to pursue strategies of buying convexity, i.e., purchasing securities whose sensitivity of duration to interest rates is positive and high. He lists securities that can be purchased by an insurance firm seeking to increase the convexity of its portfolio. They include:

- Puttable bonds, or bonds that give holders the right to redeem the bond at par at some point in time. These bonds are relatively rare, and a notable (yet unavailable to insurers) large issue of them are the special issue U.S. Treasury Bonds held by the Social Security System trust funds.
- Bond warrants, which give the holder the right to purchase at par a fixed-coupon corporate bond during a specified period.
- Adjustable rate preferred stocks.
- Interest rate caps, which give the right to receive payments when a selected interest rate index is above a specified level, and floors, which give the right to receive payments if the index is below a certain level.
- Options on Treasury, agency, and corporate bonds, as well as futures contracts on Treasury bonds, and on interest rate swaps.

The purchase of such securities is costly, again indicating that the common goal of the two classical immunization approaches of acquiring assets with greater convexity than liabilities makes this methodology somewhat inappropriate for insurance enterprises. If, however, one pursues immunization while being "short convexity," the very same models indicate that the strategy of duration matching maximizes interest rate risk, at least locally (i.e., with respect to small changes in interest rates). Shiu (1990) points out that, under certain conditions, this local maximization of risk may turn out to be global; that is, an immunized company may lose part of its surplus under *any* parallel yield curve shift. Of course, the assumption of a flat yield curve—identical annual interest rates regardless of maturity of cash flows—is unrealistic. In fact, as pointed out by Boyle (1978) and Milgrom (1985), this assumption would result in arbitrage opportunities. But classical immunization rests precisely on the pursuit of arbitrage; it is a strategy in which the asset purchase is entirely funded from liability and, once executed, brings a riskless profit to the insurance firm under any (parallel) shift in the yield curve.

Shiu allows the force of forward interest rates $\delta = \delta(t)$ to vary with maturity, and analyzes the surplus S of an enterprise under conditions of varying interest rates, with the change in interest rates $\varepsilon = \varepsilon(t)$ also being a function of time. Let N_t be the net cash flow of the enterprise at time t, and we have:

$$S(\delta) = \sum_{t \geq 0} N_t e^{-\int_0^t \delta(s) ds}, \text{ and} \tag{3.18}$$

$$S(\delta + \varepsilon) = \sum_{t \geq 0} N_t e^{-\int_0^t (\delta(s) + \varepsilon(s)) ds}. \tag{3.19}$$

Define

$$n_t = N_t e^{-\int_0^t \delta(s)ds} \tag{3.20}$$

and

$$f(t) = e^{-\int_0^t \delta(s)ds}. \tag{3.21}$$

Then

$$S(\delta + \varepsilon) - S(\delta) = \sum_{t\geq 0} n_t(f(t) - 1). \tag{3.22}$$

Using Taylor's formula with integral remainder, we have:

$$\begin{aligned} f(t) &= f(0) + tf'(0) + \int_0^t (t - w)f''(w)dw \\ &= 1 - t\varepsilon(0) + \int_0^t (t - w)f''(w)dw. \end{aligned} \tag{3.23}$$

Therefore

$$S(\delta + \varepsilon) - S(\delta) = -\varepsilon(0) \sum_{t\geq 0} tn_t + \sum_{t\geq 0} n_t \int_0^t (t - w)f''(w)dw. \tag{3.24}$$

Define $x_+ = \max(x,0)$. Applying the Fubini Theorem to interchange the order of integration and summation, we obtain:

$$\begin{aligned} &\sum_{t\geq 0} n_t \int_0^t (t - w)f''(w)dw \\ &= \sum_{t\geq 0} n_t \int_0^\infty (t - w)_+ f''(w)dw \\ &= \int_0^\infty \left(\sum_{t\geq 0} n_t(t - w)_+\right) f''(w)dw. \end{aligned} \tag{3.25}$$

If the cash flows satisfy one of the following conditions for positive w:

$$\sum_{t\geq 0} n_t(t - w)_+ \geq 0 \tag{3.26}$$

or

$$\sum_{t\geq 0} n_t(t - w)_+ \leq 0, \tag{3.27}$$

then by the Mean Value Theorem for integrals, there is a number $x > 0$ such that

$$\int_0^\infty \left(\sum_{t\geq 0} n_t(t - w)_+\right) f''(w)dw = f''(\xi) \int_0^\infty \left(\sum_{t\geq 0} n_t(t - w)_+\right) dw. \tag{3.28}$$

The integral on the right-hand side of Equation (3.28) can be simplified further by yet another application of the Fubini Theorem:

$$\begin{aligned} &\int_0^\infty \left(\sum_{t\geq 0} n_t(t - w)_+\right) dw \\ &= \sum_{t\geq 0} n_t \int_0^\infty (t - w)_+ dw \\ &= \sum_{t\geq 0} n_t \int_0^t (t - w)dw = \sum_{t\geq 0} n_t \frac{t^2}{2}. \end{aligned} \tag{3.29}$$

Therefore

$$S(\delta + \varepsilon) - S(\delta) = -\varepsilon(0) \sum_{t\geq 0} tn_t + \frac{1}{2} f''(\xi) \sum_{t\geq 0} t^2 n_t. \tag{3.30}$$

If an immunizing condition equivalent to dollar duration matching

$$\sum_{t\geq 0} tn_t = 0 \tag{3.31}$$

is imposed, then

$$S(\delta + \varepsilon) - S(\delta) = \frac{1}{2} f''(\xi) \sum_{t\geq 0} t^2 n_t. \tag{3.32}$$

By Equation (3.29), $\sum_{t\geq 0} t^2 n_t$ is positive if Equation (3.26) holds, and negative if Equation (3.27) holds. Also,

$$f''(s) = f(s)((\varepsilon(s))^2 - \varepsilon'(s)), \tag{3.33}$$

so that the sign of $f''(\xi)$ is the same as the sign of $(\varepsilon(s))^2 - \varepsilon'(s)$. Therefore Shiu's generalization of Redington's immunization is as follows:

To preserve the absolute surplus level, one should structure the cash flows in such a way that

- $\sum_{t\geq 0} tn_t = 0$ (i.e., the dollar duration of assets equals the dollar duration of liabilities), and
- the product $f''(\xi) \sum_{t\geq 0} t^2 n_t$ is as large as possible.

In particular, if the function $\varepsilon(s)$, representing the yield curve change, is constant, corresponding to a parallel yield curve shift, $f''(\xi)$ is positive. This means that, for such a change in yield curve, one should immunize by setting asset and liabilities dollar durations equal, while satisfying condition (3.26). On the other hand, if condition (3.27) is satisfied, the effect of dollar duration matching will be exactly the opposite of that normally desired in immunization—any parallel shift in yield curve will result in the deterioration of surplus.

Shiu (1990) demonstrates one important case when such deterioration is indeed assured. If $\Sigma_{t\geq 0} n_t = 0$ (i.e., there is no net investment), and $\Sigma_{t\geq 0} tn_t = 0$ (i.e., dollar durations of assets and liabilities are matched), then:

$$S(\delta + \varepsilon) - S(\delta) = S(\delta + \varepsilon) = \frac{1}{2} f''(\xi) \sum_{t \geq 0} t^2 n_t. \tag{3.34}$$

The immunizing conditions for the sequence $\{n_t\}$ imply that, unless it consists of zeros only, it must have at least two sign changes. If there are exactly two sign changes forming the pattern $-$, $+$, $-$, then for each convex function f, $\Sigma_{t \geq 0} n_t \phi(t) \leq 0$ (Goovaerts, De Vylder, and Haezendock 1984, p. 202, lemma 4) and consequently the inequality (3.27) is satisfied. One case when this pattern of signs is exhibited is when an insurance company issues single-premium immediate-annuity policies and invests all premium in a noncallable and default-free bond (assuming that the bond is of shorter maturity than that of the eventual annuity cash flow payout—and, of course, such assumption is nearly always true, as people live longer than bonds). Such a company will lose surplus under any parallel shift of the yield curve if it pursues the strategy of matching dollar duration of assets and liabilities.

As we can see, the result of duration matching as an ALM strategy can turn out to be exactly the opposite of the desired effect. Instead of being immunized, the company may end up being fully exposed to interest rate risk. For a moment, let us consider these liabilities cash flows as independent of interest rates. When choosing its investment baseline, the enterprise faces a harsh reality of modern capital markets which says that a typical fixed-income security available for purchase does not have the amount of convexity exhibited by its liabilities. Let us examine this by graphing logarithms of the prices of units of a 30-year annuity deferred by five years and a 15-year noncallable default-free 8% bond, both issued at par at the force of interest of 8%. This is presented in Figure 3.

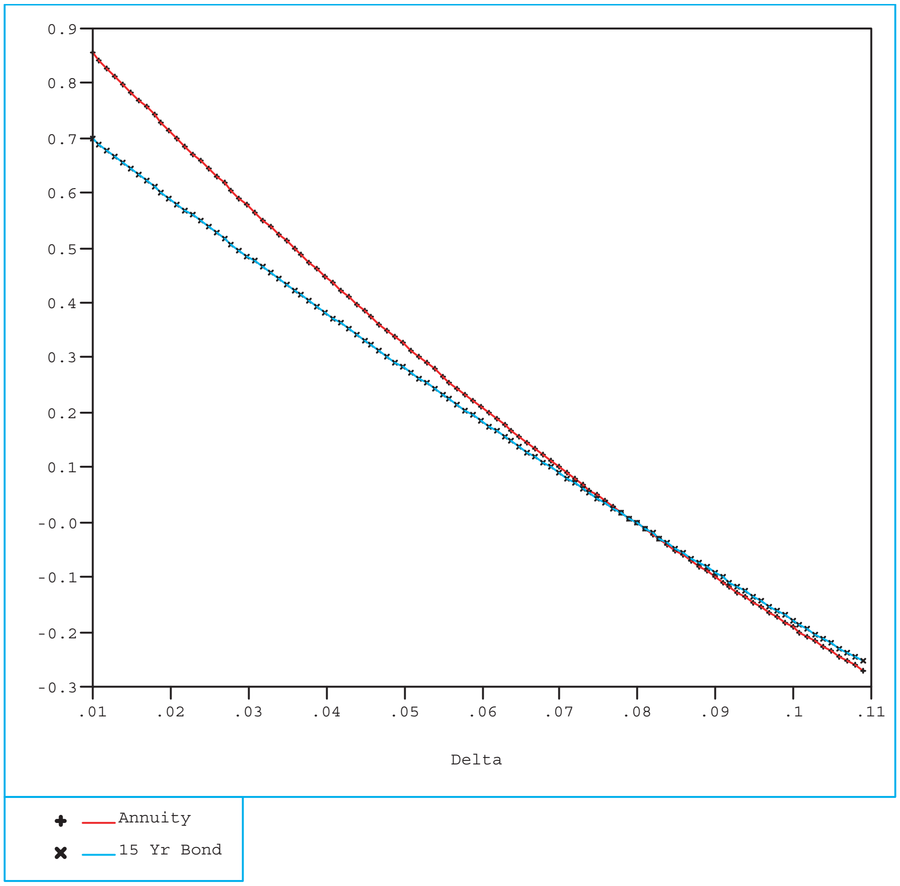
The annuity exhibits somewhat greater convexity. The asset-liability portfolio of corporate bonds (for a moment assumed noncallable and default free) and annuities, will, on a net basis, tend to have negative convexity. There is often a perception among some practitioners of the art and science of investing insurance company assets that negative convexity is always a result of interest rate options embedded in insurance company products and assets. However, it is important to stress that negative convexity of the asset-liability portfolio will manifest itself even in a simple portfolio, as suggested by Shiu (1990), that is, immediate annuities certain that is backed by noncallable default-free corporate bonds, which do not contain any embedded options.

But the practitioners are also correct. This negative convexity position will be reinforced if the insurer's assets have embedded option-like derivatives that tend to decrease the value of the asset and shorten its duration as interest rates fall, or have the opposite effect as interest rates rise. Similarly, if the liabilities' portfolio contains embedded option-like derivatives, then that tend to increase the market value of the liability, and lengthen its duration, as interest rates fall, The effect is opposite as interest rates rise, but the insurance company loses again: as rates rise, depreciation of assets value is greater than that of liabilities value.

What are these features that cause such unpleasant convexity consequences? On the liabilities side, the insurers are short (i.e., have sold) the following options:

- Life insurance and annuity policies must provide certain guarantees to policyholders, as required by the NAIC Standard Nonforfeiture Law. A policyholder surrendering a policy must receive certain portions of premiums already paid, accumulated with an interest rate which is usually bounded from below by the minimum interest rate guarantee. Such a guarantee is equivalent to the policyholder having the right to purchase a bond paying the minimum guaranteed interest rate, that is, a bond call. Policyholders also hold a bond put, as expressed by the right to surrender the policy in exchange for a cash value, or to exchange the policy for an annuity without tax consequences (this is particularly important in the case of tax-free 1035 exchanges of deferred annuities). Finally, policyholders often have the right to borrow funds from their life policies, and tend to utilize those rights quite efficiently as interest rates rise.
- Many deferred annuities offer interest rates that vary with a market index, or are adjusted in response to changes in the interest rates offered by competitors. Dividends paid by traditional life insurance policies also are subject to similar market pressures. Universal life policies were created solely for the purpose of being able to offer competitive market-related interest rates. All of these features of life and annuity products, combined with expectations and better information available to consumers, tend either to create options in products or increase the efficiency of exercising existing options.
- Many property-casualty (P&C) policies provide replacement cost protection or offer coverage linked to inflation in some other fashion. If, after a policy is issued the market value of the insured property or liability judgments increase, this is generally re-

FIGURE 3
Annuities have a lot of convexity



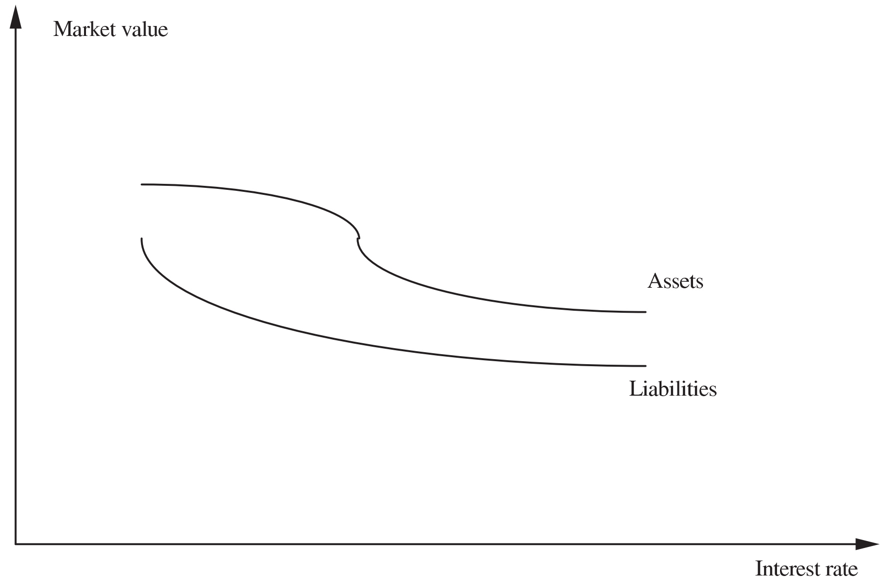
lated to inflation and the resulting higher nominal interest rate. Thus, despite the rise in interest rates, the insurer will not experience a decline in the market value of liabilities by effectively offering an "inflation option" in addition to the standard P&C coverage. However, if interest and inflation rates fall, the insurer will find itself offering excessive coverage, which would result in moral hazard—the insured having the right to put the item or event insured to the insurance company at above its market value.

On the asset side of the balance sheet, insurers tend also to be short options in the following ways:

- Life insurers are large purchasers of corporate bonds, which tend to contain calls or other refinancing provisions. As interest rates fall, bonds will be called, and the insurer will not experience the expected price appreciation of the bond, receiving only the stated exercise price. As interest rates rise, options will be out of the money and will not be exercised.
- P&C insurers tend to be large purchasers of municipal securities, which also often contain refinancing provisions.
- Insurers have large holdings of mortgages and mortgage-backed securities. When interest rates fall, borrowers tend to prepay or refinance their mortgages. However, when interest rates rise, borrowers tend to postpone prepayments and hold on to their mortgages, thus extending the duration of pass-through securities as well.

Therefore, the net position of an insurer typically resembles that of the *short straddle* (Babbel and Stricker 1987). A straddle (Hull 1993) is an option strategy of purchasing a call and a put at the same exercise price equal to the current price of a security. Insurers tend to write (i.e., issue) bond options, which are equivalent to interest rate options, both calls and puts. This creates a short position in both bond puts and calls, i.e., a short straddle. As a result, the relationship of assets and liabilities to interest rates has the shape resembling the one presented in Figure 4.

FIGURE 4
NEGATIVE CONVEXITY OF A SURPLUS



The graphs of natural logarithms of prices would also produce similar shapes.

Why do insurance companies put themselves in such a predicament? They are paid for the options they write, in two forms: (1) additional yield on their assets, when compared to noncallable assets, and (2) the ability to credit lower yield to their products than the yields earned on otherwise similar market instruments.

Are the payments received sufficient to justify the additional risks undertaken? There are two answers to this question. On the industrywide scale, over the long run, they must be, otherwise the industry would no longer exist. But in an individual company case, proper management of the enterprise requires developing a methodology for addressing this question. This is, in fact, the central issue of ALM. It ties into the key problem of modern finance concerning valuation of securities, especially contingent claims, because the short-straddle position is not only caused by the excess of convexity of liabilities over that of assets, but, overwhelmingly, by the options on both sides of the balance sheet. We will return to the relationship of that question to insurance company management in later chapters.

At this point, let us once again review immunization as a technique of ALM, but now let us be fully aware of the short straddle model. If we now position the two graphs in Figure 4 in such a way that that the slopes of their tangents are parallel (which implies equal dollar durations, or in the case of graphs of price logarithms, equal durations), then we will position ourselves exactly at the point of maximum of surplus (or surplus ratio). This must be a local maximum, because duration matching implies that dollar duration (or duration) of surplus is zero, and the first-order condition is satisfied, while negative convexity implies that the second derivative of surplus (or logarithm of the ratio A/L) is negative, so that the second-order condition for a maximum is met. Any small change in interest rates results in economic losses. Shiu's (1990) results further imply that, in certain circumstances, the maximum is global, and even a large change in the interest rates in the form of a parallel shift of the yield curve will lead to economic losses. At this point one can only ponder why anyone would pursue such a strategy.

Some additional details lurk beneath. Imagine an enterprise pursuing immunization as its basic strategy in the form of duration matching. This is, indeed, the popular approach to the problem, and many actuaries have heard the request from their investment portfolio managers to just give the investment department the duration of liabilities, and the investment people will handle it from there.

If Figure 4 is a proper (although necessarily simplified) illustration of the structure of the asset-liability portfolio of the enterprise, then let us examine the consequences of a sharp downturn in the overall level of interest rates on the portfolio. For the sake of simplicity, assume also that the firm does not have any risk of asset default. A sharp downturn in the level of interest rates can occur in an overall economic downturn in the country, for example, in the United States in the early 1990s. As a result of it, the firm suffers loss of surplus but, in addition to that, finds itself holding assets of significantly shorter duration than its liabilities. In the United States, this becomes especially pronounced if the original asset portfolio contained callable bonds and prepayable mortgages or mortgage-backed securities. The insurance company ends up holding much more cash than expected and fewer bonds and mortgages. Even if duration matching is not the investment baseline, the very existence of large amounts of cash will force the firm to extend duration through new securities purchases.

This is even more pronounced given the commitment to duration matching. The insurance firm will buy bonds and mortgages to satisfy new duration requirements. It will be just as unlikely to buy enough convexity to produce an overall positive convexity asset-liability portfolio. If the interest rates turn up again (as they do in an economic recovery, and did in the United States in 1994), the firm will *again* suffer economic loss, and cannot return to higher rates in exactly the same position as when the initial downturn in rates started. There is a common perception of actuaries as people who value stability. This may be, after all, correct.

One could argue that the issue in the above story lay in the company's inability to respond quickly enough to changing circumstances. As indicated in Chapter 2, immunization is a strategy that requires adjustments with the passage of time. It is formulated based on the parameters given at a current time. Is there a way to make it into a continuous ALM process? If the financial process describing capital asset prices is continuous and frictionless, and one or more state variables exist whose values specify all relevant information for investors, then Boyle (1978) proved that in the case of one state variable, continuous immunization can be achieved by continuously rebalancing the portfolio to maintain the asset duration equal to the liability duration. Duration is then defined as the logarithmic derivative with respect to the state variable. Boyle's model assumes that the asset and liability cash flows are deterministic, and it does not require consideration of second-order conditions. Similar continuous immunization was developed by Albrecht (1985) for a process described by several state variables. Nevertheless, as pointed out by Shiu (1991a, 1991b) as long as immunization assumes a riskless strategy with no net investment and certain profits, it rests on a riskless arbitrage and, thus, remains internally inconsistent.

To address some weaknesses of the classical immunization, Ho (1990) and Reitano (1991a, 1991b) developed a multivariate generalization of duration and convexity. They replaced the single interest rate parameter i by a yield curve vector $\vec{i} = (i_1, \ldots, i_n)$, where the coordinates of the yield curve vector correspond to certain set of "key" rates. Reitano (1991a) says: "For example, one might base a yield curve on observed market yields at maturities of 0.25, 0.5, 1, 2, 3, 4, 5, 7, 10, 20 and 30 years." The price function is then viewed as $P(i_1, \ldots, i_n)$, and instead of analyzing derivatives with respect to one interest rate variable, one could use multivariate calculus tools to study the price function. There is one, quite significant, objection that could be raised with respect to this approach immediately. When analyzing a function of two variables $f(x,y)$ we implicitly assume that the variables x and y are independent of each other, that is, that each of them has its derivative with respect to the other equal to zero. This is definitely *not* the case when various maturity interest rates are considered. Nevertheless, one can study such multivariate models for the purpose of better understanding their implications.

The negative partial logarithmic derivatives of $P(i_1, \ldots, i_n)$ are then termed *partial durations* (Reitano 1991a, 1991b), or *key-rate durations* (Ho 1990). The *total duration vector* is then defined as:

$$P'(i_1, \ldots, i_n) = -\frac{1}{P(i_1, \ldots, i_n)}\left(\frac{\partial P}{\partial i_1}, \ldots, \frac{\partial P}{\partial i_{in}}\right). \tag{3.35}$$

One can also introduce the standard notion of directional derivative of $P(i_1, \ldots, i_n)$ in the direction of a vector $\vec{\nu} = (\nu_1, \ldots, \nu_n)$:

$$P'_{\vec{\nu}}(i_1, \ldots, i_n) = \vec{\nu} \cdot \left(\frac{\partial P}{\partial i_1}, \ldots, \frac{\partial P}{\partial i_n}\right). \tag{3.36}$$

Note that the dot refers to the dot product of the vectors. The second derivative matrix can also be used to define the *total convexity*:

$$\frac{P''(i_1, \ldots, i_n)}{P(i_1, \ldots, i_n)} = \frac{1}{P(i_1, \ldots, i_n)}\left[\frac{\partial^2 P}{\partial i_k \partial i_l}\right]_{1\le k,l\le n}. \tag{3.37}$$

One can now view the surplus of an insurance firm as a function of the set of key interest rates chosen

$$S = S(i_1, \ldots, i_n) = S(\vec{i}), \tag{3.38}$$

and use multivariate calculus for two immunization algorithms, directly analogous to the one-dimensional case:

- To protect the absolute surplus level, set the first derivative (gradient) equal to zero

$$S'(\vec{i}) = 0, \tag{3.39}$$

and make the second derivative (Hessian) matrix positive definite.

- To protect the relative surplus level (i.e., surplus ratio), set:

$$-\frac{A'(\vec{i})}{A(\vec{i})} = -\frac{L'(\vec{i})}{L(\vec{i})}, \tag{3.40}$$

with the symbols *A, L* referring to assets and liabilities, respectively.

This approach, though creative and ingenious, does not eliminate the key problem of immunization strategies, namely that they do not eliminate interest rate risk, but rather tend to maximize it because the second-order conditions are unattainable in practice, resulting in strategies that only address the first-order conditions.

As pointed out by Milgrom (1972) and Shiu (1991a, 1991b), in practice, multivariate immunization will indeed lead to exact cash flow matching; as independence of the interest rate variables implies that cash flows at their respective maturities cannot be replicated by cash flows at other maturities. This can be easily shown by observing the following: Given a positive partial derivative of the liabilities with respect to a given key rate, if the assets have no cash flows of exactly that maturity, the partial derivative of their market value with respect to that key rate is zero, and immunization is impossible. If there are cash flows of that maturity from the asset, the first-order condition of immunization will force them to have exactly the same cash flows as the liabilities. Does this observation extend to the standard duration measure? Not exactly. The standard duration measure assumes, in a sense, that there is an underlying interest rate to which spot rates at various maturities respond. It does not require a contradictory assumption of independence of various spot rates.

Is there any value then in calculating key rate durations as presented above? They do represent sensitivities of financial instruments with respect to certain changes in the yield curve, and, as such, they do convey some information. For example, the gradient vector formed from partial durations of the surplus does indicate the direction of change in interest rates to which such surplus is most sensitive (this follows from a well-known property of the gradient vector shown in elementary calculus).

The message from these stories is that insurance enterprise management requires consistent economic valuation of the cash flows. Immunization does not work in theory, because it assumes violation of the principle of no arbitrage, and it does not work in practice, because conditions beyond simple comparison of durations of assets and liabilities generally cannot be met, or are not consistent with the nature of the insurance enterprise. The answer, instead, lies in valuation of the cash flows of the business. This should come as no surprise, as such is precisely the message of modern finance about the valuation of any firm (Chew 1993).