Merging Asset Allocation and Longevity Insurance: An Optimal Perspective on Payout Annuities

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1. Introduction

A number of recent articles in the *JFP* – for example Ameriks, Veres and Warshawsky (2001), Duff (2001), Bengen (2001) and Goodman (2002) – have focused financial planners attention on longevity risk and the probability of outliving wealth.

Indeed, the shift in retirement funding from professionally managed defined benefit plans to defined contribution personal savings vehicles also implies that investors need to make their own decisions on how to allocate retirement savings, as well as what product should be used to generate income in retirement. There are two important risk factors investors must consider when making these decisions: 1) Financial market risk, i.e., volatility in the capital markets which induces portfolio values to fluctuate up and down. If the market drops or corrections occur early during retirement, the portfolio may not be able to cushion the added stress of systematic withdrawals. This may make the portfolio unable to provide the necessary income for the desired lifestyle or it may simply run out of money too soon. 2) Longevity risk, i.e., the risk of living too long or outliving your portfolio. Life expectancies have been increasing, and retirees should be aware of the substantial chances for a long retirement, and plan accordingly. This risk is faced by every investor, especially those taking advantage of early retirement offers or those who have a family history of a longevity.

Traditionally, asset allocation is determined by constructing efficient portfolios for various risk levels based on modern portfolio theory (MPT)[3]. Then, based on the investor's risk tolerance, one of the efficient portfolios is chosen. MPT is widely accepted in the academic and finance industries as the primary tool for developing asset allocations. Its effectiveness is questionable, however, when dealing with asset allocations for individual investors in retirement, since longevity risk is not considered. The purpose of this article is to review the need for longevity insurance during retirement, and then

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[3] Markowitz (1952) and Merton (1971).

establish a framework to study the total asset allocation decision in retirement, which includes both conventional asset classes and immediate payout annuity products.

2. Why do my clients need longevity insurance?

Americans are living longer on average than ever before. The probability that an individual retiring at age 65 will reach age 80 is over 70% for females, and over 62% for males. When combined with the life expectancy of a spouse, the odds reach nearly 90% that at least one spouse will live to 80. And there's an over 80% chance at least one spouse will live to age 85. For a broader sense of the potential longevity risk, Table 1 illustrates how long a 65-year-old can expect to live.[4]

Table 1: The Conditional Probability of Survival at Age 65

To Age:	Single Female	Single Male	At Least One Member of a Couple
70	93.8%	92.0%	99.5%
75	84.4%	79.9%	96.9%
80	70.9%	62.7%	89.1%
85	52.8%	41.0%	72.2%
90	31.6%	19.6%	45.0%
95	13.4%	5.8%	18.4%

Source: Society of Actuaries RP-2000 Table

For example, the probability that at least one spouse will reach age 75 is computed as follows: 1 - (1-0.938)*(1-0.920) = 99.5%. As the reader can see from the table, ***longevity risk*** – the risk of outliving one's resources – is very substantial and is the main reason that we believe lifetime annuities (alternatively known as payout) will grow in popularity.

3. Payout annuity and its insurance against longevity risk

Longevity risk can be hedged away with insurance products, namely lifetime payout annuities. A lifetime payout annuity is an insurance product that exchanges an accumulated investment into payments that the insurance company pays out over a specified time or, in this case, over the lifetime of the investor. Payout annuities are the exact opposite of traditional life (or more aptly named premature death) insurance.

There are two basic types of payout annuities: fixed and variable. A fixed payout annuity pays a fixed dollar amount each period, perhaps with a COLA adjustment, in real or nominal terms. A variable annuity's payments fluctuate in value depending on the investments held and, therefore, disbursements will also fluctuate. The payment from a lifetime payout annuity is contingent upon the life of the investor. If the investor dies,

[4] We have chosen age 65 as the standard baseline for retirement, although similar numbers can be generated for any age.

he/she will no longer receive any payments, unless a special guarantee period or estate benefit was purchased at the same time, which is normally paid for by reducing the benefit stream.

There has been a substantial amount of recent literature on the topic of the costs and benefits of life annuities, and space constraints prevent us from giving providing a comprehensive review**.** Roughly speaking, the relevant literature can be partitioned into the following categories:

The first category consists of the theoretical insurance economics literature that investigates the equilibrium supply and demand of life annuities in the context of a complete market and utility-maximizing investors. This includes the classical work by Yaari (1965), as well as Richard (1975), Brugiavini (1993), Yagi and Nishigaki (1993) and Milevsky and Young (2002). Broadly speaking, their main conclusions are that life annuities should play a substantial role in a retiree's portfolio.

The empirical annuity literature examines the actual pricing of these products, and whether consumers are getting their money's worth. These include a sequence of papers by Brown, Warshawsky, Mitchell and Poterba (1999, 2000, 2001) in various combinations.

A third and final strand attempts to create normative models that help investors decide how much to annuitize, when to annuitize and the appropriate asset mix within annuities.These include the work by Milevsky (2000, 2001), Kapur and Orszag (1999), and Blake, Cairns and Dowd (2000).

3.1 Fixed payout annuity

Chart 1 illustrates the payment stream from a fixed immediate (a.k.a. payout, or lifetime) annuity. With an initial premium or purchase amount of $100,000, the annual income payments for a 65 year-old male in today's environment would be $706.14 per month, or $8,474 per year.[5] The straight line represents the annual payments before inflation. People who enjoy the security of a steady and predictable stream of income may find a fixed annuity appealing. The drawback of a fixed annuity becomes evident over time. Since the payments are the same year after year, purchasing power is eroded as the annuitant gets older. The second curved line in the image represents the same payment stream after a hypothetical 3.2% inflation rate is factored in.[6] While the annuitant still receives the same amount, it is no longer able to purchase as much as it used to.

Despite the benefits of longevity insurance and fixed payout amounts, there are disadvantages with a portfolio that consists solely of fixed annuities. First, because the nominal value of the payment will remain fixed for the rest of the annuitant's life, the

[5] This is the average quote obtained by the authors in mid-July, 2002, assuming a 65-year-old male and a $100,000 premium. The payments from different companies can differ quite substantially from week to week and from the best to the worst insurance company quotes.
[6] The average inflation rate from 1926 to 2001 is 3.2% in the U.S.

value of the payments in real terms (after inflation) will decline over time. Chart 2 displays the inflation rate during the last 30 years, as measured by changes in the level of the Consumer Price Index (CPI). Notice that although the inflation rate in the U.S. is currently under 2%, this number is at the low end of the historical record. In fact, as recently as the early 1990s, the inflation rate was over 6%, and in the early 1980s, it went as high as 13%. The (arithmetic) average during the last 30 years was approximately 5% per annum.

Secondly, the investor cannot trade-out of the fixed payout annuity once it is purchased.[7] In other words, the lack of liquidity (and reversibility) within a fixed annuity impedes the optimal asset allocation process and makes the fixed annuity less desirable, all else being equal. See Browne, Milevsky and Salisbury (2003) for details on how to quantify this drawback.

Finally, it seems that the current payout rates from fixed payout annuities are at a historical low, which is consistent with the current interest rate environment. A 65 year-old female might have received as much as $1,150 per month in the early 1980s, in exchange for the same $100,000 initial premium. Today the $100,000 buys closer to $700 per month. In fact, we are currently at historical lows on the interest rate cycle, and this may be one of the worst times to lock in an interest for the rest of one's life. Recall that once the individual has purchased a life annuity they can no longer cash-in or sell the insurance contract. While we obviously want to refrain from speculating – and encouraging others to speculate -- on the long-term direction of interest rates, we want to remind the reader that locking-in a fixed annuity is implicitly a market timing play. This is why we believe that variable payout annuities will continue to grow in popularity.

3.2 Variable payout annuities

A variable payout annuity is an insurance product that exchanges an accumulated investment value into annuity <u>units</u> that the insurance company pays out over the lifetime of the investor. The annuity payments fluctuate in value depending on the investments held and, therefore, disbursements will also fluctuate. Thus, instead of getting fixed annuity payments, the annuitant receives the equivalent of a fixed number of fund units. The insurance company converts these fund units into dollars at the going net asset value. Therefore, the cash-flow from the variable payout annuity fluctuates with the underlying investments.

Chart 3 illustrates the annuity payment stream in real terms from a 50% stock/50% bond portfolio using a life only payment option in an immediate variable annuity. We generated a Monte Carlo simulation to illustrate the various payment scenarios. The

[7] There are payout annuities available that allow the investor to withdraw money from them, but the investor typically has to pay a surrender or market value adjustment charge. Furthermore, this would only apply during the certain period of the annuity where payments are guaranteed regardless of life status. In this paper, we will focus our discussion on the basic type of payout annuities, which does not offer early withdrawal or death benefits.

simulation is generated using historical return statistics of stocks, bonds, and inflation from 1926–2001, a $100,000 initial portfolio, and a 3% Assumed Investment Return (AIR). While the actuarial mechanics behind the AIR are beyond the scope of this paper, one can think of it as a method of front-loading or back-loading annuity payments. The initial payment at age 65 is estimated to be $6,615.[8] The three lines show the 10th, 25th, and the 50th percentile. In other words, there is a 10% chance that annual inflation-adjusted annuity payments would have fallen below $5,000, a 25% chance that they would have fallen below $7,027 or higher, and a 50% chance that they would have grown to over $10,182.

4. Optimal asset allocation mix with payout annuities

Smart asset allocation decisions that take advantage of the diversification benefits across different asset classes are an effective tool to manage and reduce market risk. Therefore, to help investors find the appropriate allocation of their savings in retirement, we must incorporate fixed and variable payout annuities into the traditional asset allocation models.

4.1 The Rationale

It makes little sense to offer a money market and bond fund in the savings portion of a personal pension plan, without offering an equity fund to complete the risk and return spectrum. So, too, it makes little sense to offer fixed payout annuities without offering variable payout annuities to balance out the risk. Clearly, the latter is the symmetric extension of the former. And, indeed, since there is a proper asset allocation involving savings (accumulation) products, the same applies to dissavings (consumption stage) products.

Classical asset allocation (savings) models used by the popular software vendors and advisor services input information on the investor's time horizon and risk aversion level in order to determine the appropriate asset mix. But, to incorporate payout annuities and retirement dynamics into asset allocation models, a proper model requires more information. This would include inputs such as the investor's subjective health estimate, the strength of bequest motives and pre-existing pension income.

We have developed a model for optimally allocating investment assets within and between two distinct categories. The two categories are annuitized assets and non-annuitized assets. The annuitized assets include fixed and variable immediate annuities. The non-annuitized assets include all types of investment instruments, such as mutual funds, stocks, bonds, and T-bills that do not contain a mortality-contingent income flow. In addition, our model incorporates the following decision factors:

- Investor's risk tolerance

[8] The initial payment is estimated by Ibbotson Associates.

- Investor's age
- Investor's subjective probability of survival
- Population objective (pricing) probability of survival
- Relative weights placed on consumption and bequest
- Investor's utility from "live" consumption and bequest
- Risk and return characteristics of risky and risk-free assets

The model is developed based on micro-economic models of consumer behavior. The appendix provides a more technical discussion about the model. Chart 4 provides a graphical illustration of the tradeoff between the desire for bequest and liquidity needs and existing pension income. The greater the desire for creating an estate, or bequest value, the lower the demand (or need for) payout annuities (PA). This is because life annuities trade-off longevity insurance against the creation of an estate.

4.2 Numerical Results

To understand the normative predictions of the model, let us look at several different cases so that we can see the effect of changing parameters on the optimal allocation. We will start with the capital market assumptions that will remain the same for all four cases. All cases will assume that the individual is a 60 year-old male who would like to allocate his portfolio across the two investment asset classes and the two mortality-contingent claim classes. Together, the four 'allocatable' products are: 1) risk-free asset; 2) risky asset; 3) immediate fixed annuity; and 4) immediate variable annuity. We assume that the return from the risk-free (T-bills) asset class is 5% per annum with no volatility. The return from the risky asset is log-normally distributed with a mean value of 10% and a standard deviation of 20%. (In other words the investment is expected to earn 10% per annum, but may actually earn as much as 30% or lose 10% in any given year.) This implies a risk premium of 5%, which is in line with forward-looking estimates for U.S. equity markets. As for the mortality parameters, we use a table provided by the U.S. based Society of Actuaries, called the Individual Annuity Mortality (IAM) 2000 basic table. These tables are the probabilities of survival for a healthy population of potential annuitants. Many people might feel they are less (or more) healthy than the numbers indicated by the IAM 2000; we will therefore allow the subjective probability of survival to be lower (or higher) than the objective probability of survival. The utility preferences will be taken from within the Constant Relative Risk Aversion (CRRA) family, with a CRRA coefficient of γ.

While space constrains us from providing a crash-course on micro-economic theory, the CRRA can be viewed as measuring a consumer's aversion to investing in risky assets. The greater the CRRA value, the lower is their appetite for risk. And, while we are fully cognizant that few if any investors can identify their personal CRRA value – and DNA testing has proven elusive so far – we strongly believe this normative framework can be used to guide a prudent asset mix and to educate the investor about the risks. Finally, we will employ the 20-year horizon as representing the one period. In other words, the individual intends to re-allocate (rebalance) assets after 20 years.[9] In practice, we would recommend investors rebalance their portfolio much *before* the 20 year horizon, which require a dynamic multi-period model. This additional dimension of 'when to rebalance' complexity is beyond the scope of this introductory paper, but is being addressed in a follow-up report by Peng and Milevsky (2003).

Case #1: Total Altruism and Complete Bequest Motives

In this case we assume the investor's utility is derived entirely from bequests. In other words the weight of his utility of bequest is assumed to be one, and the weight on his utility of consumption is zero, that is, A=0 and D=1. The objective probability of survival is 65% (roughly equal to the survival probability of a 60-year-old male in the next 20 years) and the subjective probability is the same 65%. In other words, we are assuming that the investor does not have any private information about his or her mortality status that might lead them to believe they are healthier, or less healthy than average. Using these input parameters in the model described above, the optimal allocations to the assets across various relative risk aversion levels are presented in Table 2 and Chart 5.

A few things should be evident from the table. First, immediate annuities get no allocation, since the investor only cares about bequest. The intuition for this result can be traced back to a classical paper by M. Yaari (*Review of Economics Studies*, 1965). Namely, if consumers are 100% altruistic, they will not waste the asset by annuitizing. Second, the allocation to stocks gradually decreases as the investor's risk aversion increases. Thus, without any consumption motive, this becomes the traditional allocation problem between risk-free and risky assets. This case can be used as an illustration for extraordinarily wealthy individuals, where the size of their portfolio far exceeds their consumption needs. In this case, bequest becomes the dominant factor. Annuities do not get any allocation, as they do not leave any money for the heirs. For example, for investors with a relative risk aversion level of 2, the optimal allocation is 36% to the risk-free asset and 64% to equity.

Case #2: No Bequest Motives

This case maintains the same age (gender), survival probability and time horizon, but completely eliminates the strength of bequest by replacing A=1 with D=0. In other words, 100% of the utility weight is placed on "live" consumption. The optimal allocations to the assets across various risk aversion levels are presented in Table 3 and Chart 6.

Since the returns on annuities are always higher than the returns on traditional assets – conditional on the retiree being alive -- the immediate annuities get 100% of the allocation. The allocation to the immediate variable annuity gradually decreases, while

[9] These assumptions can be easily modified to accommodate other utility functions, asset return distributions, mortality probabilities, and horizons. Note that because we are using a utility function that has constant relative risk aversion the initial wealth level does not have any impact on the allocations for the one-period model.

the allocation to the immediate fixed annuity increases as the risk aversion of the investor increases. This case can be used as an illustration for investors who would like to maximize their lifetime consumption and have no interest in leaving any money behind. (They are alternatively known as the “die broke” crowd.) All the savings should be used to purchase annuities. Overall, the optimal allocation between risky and risk-free assets (in this case, they are an immediate fixed annuity and an immediate variable annuity) are almost identical to that of Case #1. For investors with a risk aversion level of 2, the optimal allocation is 36% to immediate fixed annuity and 64% to immediate variable annuity.

Case #3: 20% Bequest Motives and 80% Consumption Motives

This case maintains the same age (gender), survival probability and time horizon, but changes the strength of bequest from D=0 to a more realistic D=0.2.[10] In other words, 80% of the utility weight is placed on “live” consumption. The optimal allocations to the assets across various risk aversion levels are presented in Table 4 and Chart 7.

There are several interesting results in the allocation. First, unlike the previous two cases, all four of the asset classes are present in the optimal allocations. This is because immediate annuities are more suitable (relative to traditional assets) for consumption and traditional investments are more suited for bequest motives in this one-period framework. When the investor has a more balanced motive between bequest and consumption, both immediate annuities and traditional asset classes are selected. In general, the higher the bequest motives, the more the investor should allocate to traditional investments and the less to immediate annuities.

Second, the allocation between risky (both VIA and equity) and risk-free (cash and FIA) is almost identical to that in Case #1 and Case #2 at comparable risk aversion levels. This indicates that the changes in the investor’s bequest vs. consumption motive do not significantly impact the investor’s behavior regarding risk. The optimal allocation between risky and risk-free assets is determined by the investor’s risk tolerance.

Third, we find the allocation to annuities decreases as the investor’s risk aversion increases. In other words, more risk averse investors will avoid immediate life annuities. This makes intuitive sense, since the investor could get little or no utility from immediate annuity investments if he or she dies shortly after the purchase. With traditional investments, there will be some left for their heirs. It seems that higher aversion to risk increases the implicit weight on the utility of bequest. For an investor with a risk aversion level of 2, the optimal allocation is 22% cash, 38% equity, 14% FIA, and 26% VIA.

[10] See Bernheim (1991), Hurd (1989), as well as Abel and Warshawsky (1988) for a discussion and estimates of the ‘strength of bequest’ parameters. We have taken 20% as an approximation.

5. Summary and Conclusions

Motivated by the recent interest on the topic of annuitization and payout annuities within the public debate about pension provision, this paper has investigated the theory and practice of constructing an optimal asset allocation during retirement. We have considered both financial market risk and longevity risk in the economic tradeoff.

Our main qualitative insight is as follows. The natural asset allocation spectrum consists of investments that go from safe (fixed) to risky (variable). In contrast, the product allocation spectrum ranges from conventional savings vehicles to annuitized payout (pension) instruments. The asset and product spaces are separate dimensions of a well-balanced financial portfolio; yet the product/asset allocation must be analyzed jointly.

Exhibit	Product Allocation	
	Conventional	Annuitized
Asset Allocation	**Fixed** (CDs, T-bills, bonds, bond mutual funds)	**Fixed** (Fixed payout annuity)
	Variable (Stocks, equity mutual funds)	**Variable** (Variable payout annuity)

More formally, we have presented a mathematical one-period model to analyze the optimal allocations within and between payout annuities. The numerical results confirm that the optimal allocations across assets are influenced by many factors, including age, risk aversion, subjective probability of survival, utility of bequest, and the expected risk and return tradeoffs of different investments. We also find that the global allocation between risky and risk-free assets is influenced only by the investor's risk tolerance; it is not significantly affected by the subjective probability of survival or the utility of consumption vs. bequest.

In some sense we are advocating a classical economic 'separation theorem' argument. We claim that the first step of a well-balanced retirement plan is to locate a suitable global mix of risky and risk-free assets independently of their mortality-contingent status. Then, once a comfortable balance has been struck between risk and return, the annuitization decision should be viewed as a second-step 'overlay' that is placed on top of the existing asset mix. And, depending on the strength of bequest motives and subjective health assessments, the optimal annuitized fraction will follow.

Of course, retirement is not just one point or period in time, and ongoing research by the authors is partitioning the golden years into various stages to examine the optimal allocation to payout annuities, as one moves towards the end of the life-cycle.

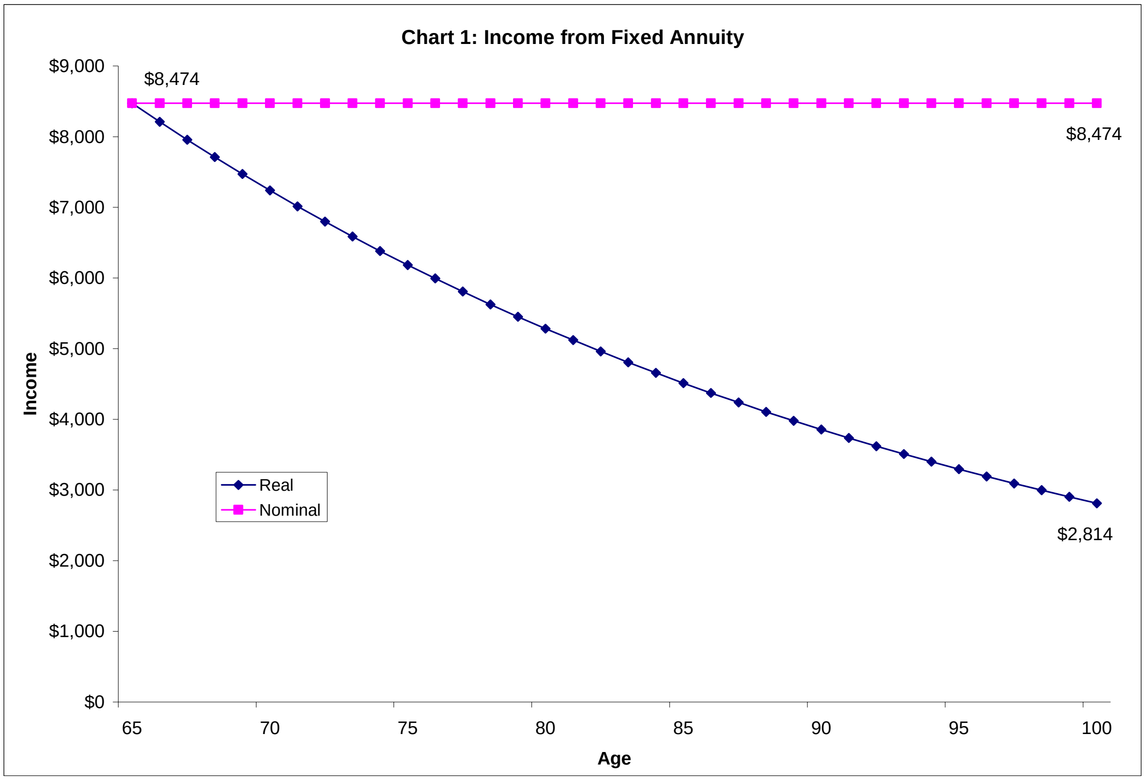


Chart 1: Income from Fixed Annuity
$9,000
$8,000
$7,000
$6,000
$5,000
$4,000
$3,000
$2,000
$1,000
$0
Income
$8,474
$8,474
$2,814
Real
Nominal
65
70
75
80
85
90
95
100
Age

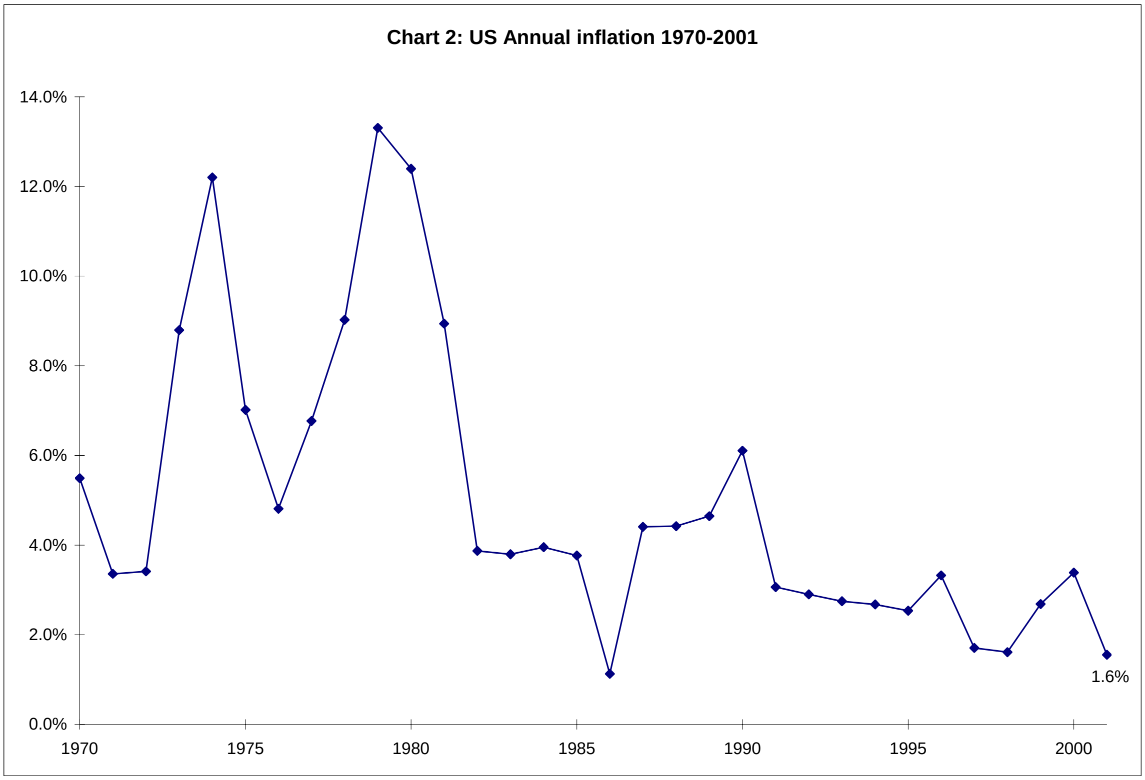


Chart 2: US Annual inflation 1970-2001
14.0%
12.0%
10.0%
8.0%
6.0%
4.0%
2.0%
0.0%
1.6%
1970
1975
1980
1985
1990
1995
2000

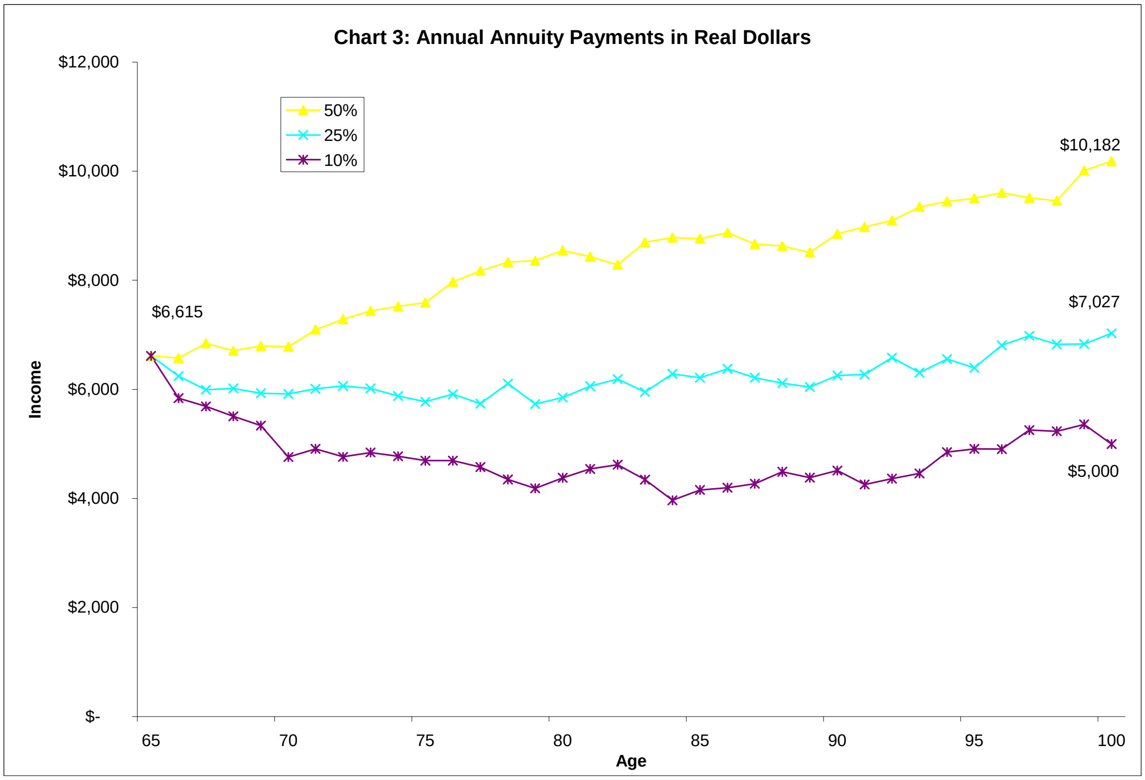


Chart 3: Annual Annuity Payments in Real Dollars
$12,000
$10,000
$8,000
$6,000
$4,000
$2,000
$-
Income
50%
25%
10%
$10,182
$7,027
$6,615
$5,000
65
70
75
80
85
90
95
100
Age

Chart 4: The Tradeoff between Bequest and Consumption

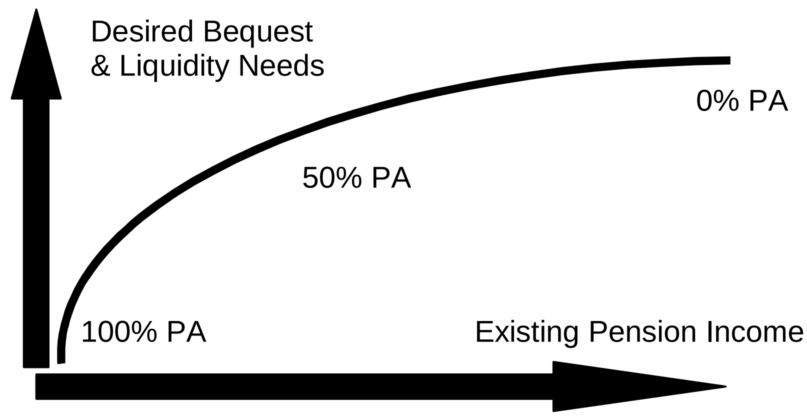


Table 2: Optimal Allocations:
Male age 60, 100% bequest, 20 year horizon, Rf=5%, R=10%, SD=20%

Risk Aversion	Money	Equity	FIA	VIA	Total Risk Free	Total Risky	Total Traditional	Total Annuity
1	2%	98%	0%	0%	2%	98%	100%	0%
1.5	16%	84%	0%	0%	16%	84%	100%	0%
2	36%	64%	0%	0%	36%	64%	100%	0%
2.5	50%	50%	0%	0%	50%	50%	100%	0%
3	56%	44%	0%	0%	56%	44%	100%	0%
3.5	62%	38%	0%	0%	62%	38%	100%	0%
4	68%	32%	0%	0%	68%	32%	100%	0%
4.5	72%	28%	0%	0%	72%	28%	100%	0%
5	74%	26%	0%	0%	74%	26%	100%	0%
5.5	78%	22%	0%	0%	78%	22%	100%	0%
6	80%	20%	0%	0%	80%	20%	100%	0%

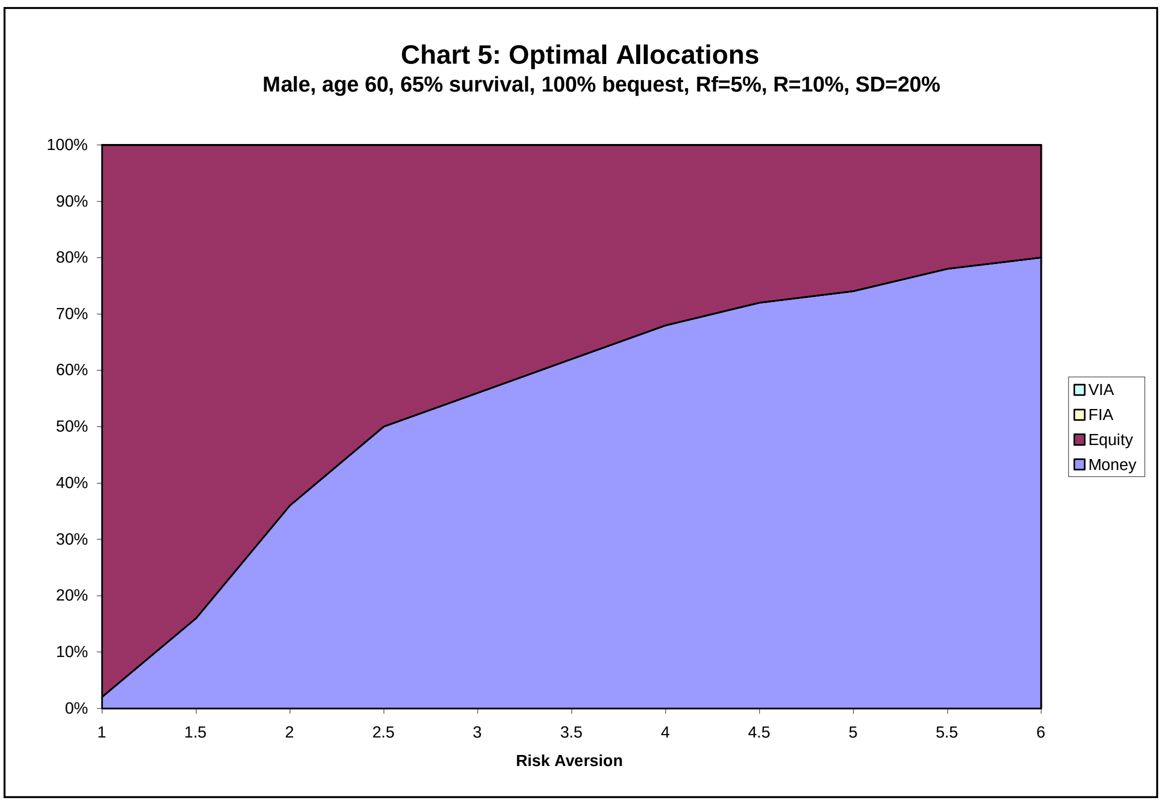


Table 2: Optimal Allocations:
Male age 60, 0% bequest, 20 year horizon, Rf=5%, R=10%, SD=20%

Risk Aversion	Money	Equity	FIA	VIA	Total Risk Free	Total Risky	Traditio nal	Total Annuity
1	0%	0%	2%	98%	2%	98%	0%	100%
1.5	0%	0%	16%	84%	16%	84%	0%	100%
2	0%	0%	36%	64%	36%	64%	0%	100%
2.5	0%	0%	50%	50%	50%	50%	0%	100%
3	0%	0%	58%	42%	58%	42%	0%	100%
3.5	0%	0%	64%	36%	64%	36%	0%	100%
4	0%	0%	68%	32%	68%	32%	0%	100%
4.5	0%	0%	70%	30%	70%	30%	0%	100%
5	0%	0%	76%	24%	76%	24%	0%	100%
5.5	0%	0%	78%	22%	78%	22%	0%	100%
6	0%	0%	80%	20%	80%	20%	0%	100%

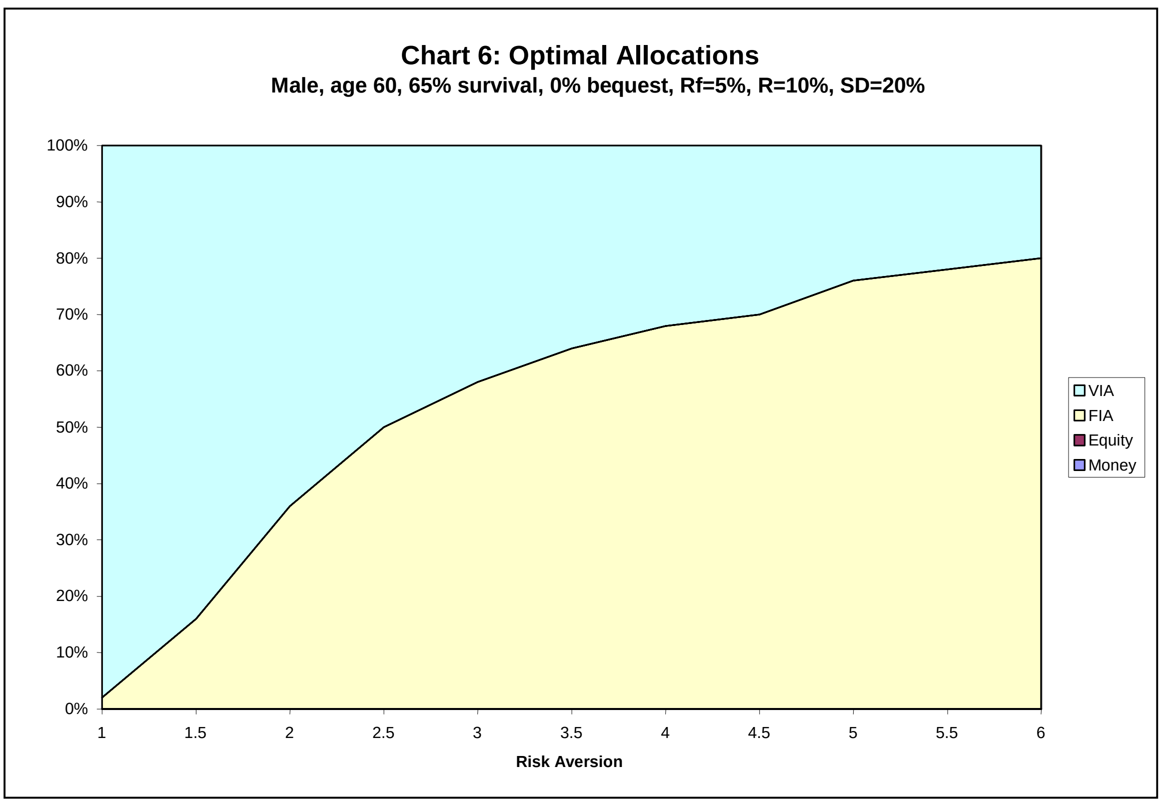
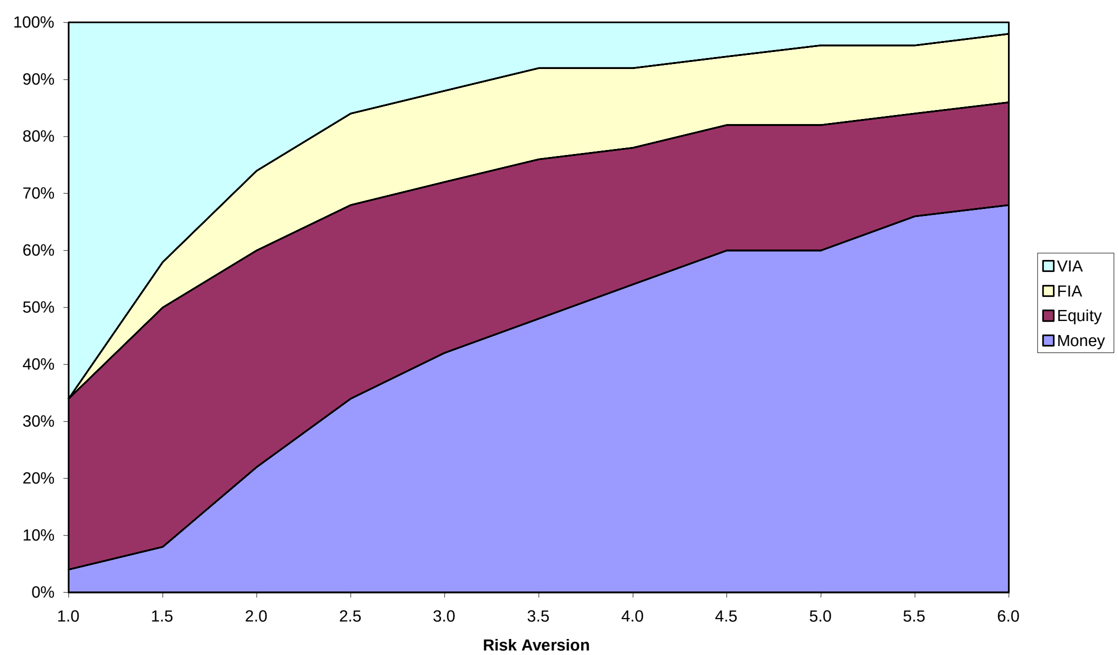


Table 4: Optimal Allocations:
Male age 60, 20% bequest, 20 year horizon, Rf=5%, R=10%, SD=20%

Risk Aversion	Money	Equity	FIA	VIA	Total Risk Free	Total Risky	Total Traditional	Total Annuity
1	4%	30%	0%	66%	4%	96%	34%	66%
1.5	8%	42%	8%	42%	16%	84%	50%	50%
2	22%	38%	14%	26%	36%	64%	60%	40%
2.5	34%	34%	16%	16%	50%	50%	68%	32%
3	42%	30%	16%	12%	58%	42%	72%	28%
3.5	48%	28%	16%	8%	64%	36%	76%	24%
4	54%	24%	14%	8%	68%	32%	78%	22%
4.5	60%	22%	12%	6%	72%	28%	82%	18%
5	60%	22%	14%	4%	74%	26%	82%	18%
5.5	66%	18%	12%	4%	78%	22%	84%	16%
6	68%	18%	12%	2%	80%	20%	86%	14%

Chart 7: Optimal Allocations
Male, age 60, 65% survival, 20% bequest, Rf=5%, R=10%, SD=20%



Appendix: Technical Model of Optimal Asset Allocation

In this technical appendix which we aim towards the braver readers, we present the formal mathematical model that underlies the numerical results in the body of paper. We extend the classical decision under uncertainty models to develop an optimal asset allocation with both conventional asset classes and payout annuities. We start by assuming that a rational utility maximizing investor is choosing the allocations of his or her retirement portfolio to maximize his or her utility. We also assume that there are only four different products to choose from: 1) risk-free asset; 2) risky asset; 3) immediate fixed annuity; and 4) immediate variable annuity. We can easily expand this model to incorporate more assets.

Of course, the model itself is formulated in one-period framework, which makes the life annuity more of a tontine, but the underlying idea is the same regardless of the number of periods within the model.

With four different categories to choose from, we focus on the classic asset allocation problem. How does a rational utility-maximizing individual go about selecting the right mix between the risky and risk-free categories and between traditional financial instruments and immediate annuity, or insurance category?

The category matrix presented in the following table summarizes the returns from the four possible investment products, conditional on being alive or dead.

The four basic investment products

	Alive	**Dead**
Risk-Free Asset (T-bills):	**R**	R
Risky Asset (Equity):	X	X
Immediate Fixed Annuity:	(1+R)/p –1	0
Immediate Variable Annuity:	(1+**X**)/p –1	0

From a mathematical point of view, we have the following problem. We are looking for asset allocation weights, denoted by $\{ a_1, a_2, a_3, a_4 \}$ that maximize the objective function:

$$
\begin{aligned}
&E[U(W)] = \overline{p} \times A \times E[u(a_1 wR + a_2 wX + a_3 wR / p + a_4 wX / p)] \\
&+ (1 - \overline{p}) \times D \times E[u(a_1 wR + a_2 wX)] \\
&S.T. \\
&a_1 + a_2 + a_3 + a_4 = 1 \\
&a_i > 0
\end{aligned}
$$

Where we use the following notation.

- The letter A denotes the relative strength placed on the utility of consumption.

- The letter D denotes the relative strength placed on the utility of bequest. The sum of A and D are assumed to be one, so there is only one free variable. Individuals with no utility of bequest will be assumed to have D = 0.
- The symbol p denotes the objective probability of survival, which is the probability that is used by the insurance company to price immediate annuities.
- The symbol $\bar{p}$ denotes the subjective probabilities of survival. The subjective probability of survival may not match the objective population (annuitant) probability. In other words, a person might believe he or she is healthier (or less healthy) than average. This would impact the expected utility but not the payout from the annuity, which is based on objective (annuitant) population survival rates.
- The letter X denotes the (one plus) random return from the risky asset and the letter R denotes the (one plus) risk-free rate.
- The expression $E[u(a_1wR + a_2wX + a_3wR/p + a_4wX/p)]$ denotes the utility from the live stage, while $E[u(a_1wR + a_2wX)]$ denotes utility from the dead state. Notice that the annuity term, which divides by the probability of survival, does not appear in the dead state. This is because the annuity does not payout.
- The function *u(.)* denotes the standard utility function of end-of-period wealth. Our model can handle cases of both constant relative risk aversion and decreasing relative risk aversion, as well as other functional forms that are consistent with loss aversion.

Since the weights $\{a_1, a_2, a_3, a_4\}$ sum up to one, we essentially have only three weights to solve. An important factor to consider in solving the utility maximization is that, as functions of (a_1, a_2, a_3), both E[U(W)] and its derivatives are defined by integrals that cannot be performed analytically, they must be performed numerically.

The technical problem to be solved is to maximize the expected utility E[U(W)] as a function of the weights a_1, a_2, and a_3, where $a_1 \geq 0$, $a_2 \geq 0$, $a_3 \geq 0$ and $a_1 + a_2 + a_3 \leq 1$. Now, although E[U(W)] is a non-linear function of the three free parameters (a_1, a_2, a_3), it is *strictly* concave, and hence one need only find a local maximum in order to find the global maximum.

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